UNIFORMLY BOUNDED REPRESENTATIONS AND EXACT GROUPS

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ABSTRACT. We characterize groups with Guoliang Yu's property A (i.e., exact groups) by the existence of a family of uniformly bounded representations which approximate the trivial representation.

Property A is a large scale geometric property that can be viewed as a weak counterpart of amenability. It was shown in [12], that for a finitely generated group property A implies the Novikov conjecture. It was also quickly realized that this notion has many other applications and interesting connections, see [9, 10].

A well-known characterization of amenability states that the constant function 1 on G, as a coefficient of the trivial representation, can be approximated by diagonal, finitely supported coefficients of the left regular representation of G on $\ell_2(G)$. In this note we prove a counterpart of this result for groups with property A in terms of uniformly bounded representations. A representation π of a group G on a Hilbert space H is said to be uniformly bounded if $\sup_{g\in G} \|\pi_g\|_{B(H)} < \infty$.

Theorem 1. *Let G be a finitely generated group equipped with a word length function. G has property A (i.e., G is exact) if and only if for every $\varepsilon > 0$ there exists a uniformly bounded representation π of G on a Hilbert space H, a vector $v \in H$ and a constant $S > 0$ such that*

(1) $\|\pi_g v\| = 1$ *for all* $g \in G$,
(2) $|1 - \langle \pi_g v, \pi_h v\rangle| \le \varepsilon$ *if* $|g^{-1}h| \le 1$,
(3) $\langle \pi_g v, \pi_h v\rangle = 0$ *if* $|g^{-1}h| \ge S$.

Alternatively, the second condition can be replaced by an almost-invariance condition: $\|\pi_g v - \pi_h v\| \le \varepsilon$ if $|g^{-1}h| \le 1$. Another characterization of property A in this spirit, involving convergence for isometric representations on Hilbert C^*-modules was studied in [4].

Recall that the Fell topology on the unitary dual is defined using convergence of coefficients of unitary representations. Theorem 1 states that the trivial representation can be approximated by uniformly bounded representations, in a fashion similar to Fell's topology.

Similar phenomena were considered by M. Cowling [2, 3] in the case of the Lie group $\mathrm{Sp}(n,1)$. Recall that $\mathrm{Sp}(n,1)$ has property (T), and thus the trivial representation is an isolated point among the equivalence classes of unitary

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representations in the Fell topology. Cowling showed that nevertheless, for $\mathrm{Sp}(n,1)$ the trivial representation can be approximated by uniformly bounded representations in a certain sense. Theorem 1 gives a similar statement for all discrete groups with property A. Recall that almost all known groups with property (T) are known to have property A. In particular, the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, satisfy property A [5].

Moreover, under a stronger assumption that the group has Hilbert space compression strictly greater than 1/2 in the sense of [6], we obtain a path of uniformly bounded representations, whose coefficients continuously interpolate between the trivial and the left regular representation.

Theorem 1 suggests the possibility of negating property A using strengthened forms of Kazhdan's property that applies to uniformly bounded representations.

Question 1. *Are there finitely generated groups satisfying a sufficiently strong version of property (T) for uniformly bounded representations, so that these groups cannot have property A?*

Certain versions of such a property (T) for uniformly bounded representations were considered by Cowling [2, 3], but they would not apply directly in our case. Construction of new examples of finitely generated groups without property A is a major open problem in coarse geometry, with possible applications in operator algebras, index theory and topology of manifolds.

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1. UNIFORMLY BOUNDED REPRESENTATIONS AND PROPERTY A

Let H_0 be a Hilbert space with scalar product $\langle \cdot,\cdot \rangle_0$, and let T be a bounded, positive, self-adjoint operator on H_0. We additionally assume that T has a spectral gap; that is, there exists $\lambda > 0$ such that

$$\langle v, Tv\rangle_0 \geq \lambda \langle v, v\rangle_0 \tag{1}$$

for every $v \in H_0$.

The operator T induces a new inner product on V, the vector space underlying H_0, by the formula

$$\langle v, w\rangle_T = \langle v, Tw\rangle_0.$$

The norm $\|v\|_T$ induced by $\langle \cdot,\cdot\rangle_T$ is equivalent to the original norm on H_0, since

$$\lambda \|v\|_0^2 \leq \|v\|_T^2 \leq \|T\|_{B(H_0)}\ \|v\|_0^2.$$

Thus we obtain a new Hilbert space H_T by equipping V with the norm induced by T. A unitary representation π on H_0 naturally becomes a uniformly bounded representation on H_T. More precisely, the norm of the representation satisfies

$$\|\pi\| = \sup_{g\in G} \|\pi_g\|_{B(H_T)} \leq \frac{\|T\|_{B(H_0)}}{\lambda}.$$

In the Hilbert space H_T, the representation π satisfies

$$\pi_g^* = T^{-1}\pi_{g^{-1}}T,$$

for every $g \in G$.

We will now relate property A to the existence of uniformly bounded representations with the desired properties. From the discussion on renormings via positive operators we derive the following fact.

Lemma 2. *Let G be a finitely generated group equipped with a word length function and $\varepsilon > 0$. If there exists a Hilbert space H_0, a positive self-adjoint bounded operator T of H satisfying (1) , a unitary representation π, a unit vector $v \in H_0$ and $S > 0$ such that*

(1) $\langle \pi_g v, T\pi_g v\rangle_H = 1$ *for every* $g \in G$,
(2) $|1-\langle \pi_g v, T\pi_h v\rangle_H| \leq \theta$ *whenever* $|g^{-1}h| \leq 1$,
(3) $\langle \pi_g v, T\pi_h v\rangle_H = 0$ *whenever* $|g^{-1}h| \geq S$,

then there exists a uniformly bounded representation π of G on a Hilbert space H_T and $v \in H_T$, satisfying the conditions listed in Theorem 1.

Proof. Let V denote the vector space underlying H. We equip V with a scalar product $\langle v,w\rangle_T = \langle v, Tw\rangle_0$ and obtain the space H_T as explained in the previous section. Viewing π and v with respect to this new norm gives the required properties. □

Recall that a Hermitian kernel on a set X is a function $K : X \times X \to \mathbb{C}$ such that $K(x,y) = \overline{K(y,x)}$. K is said to be positive definite if for every finitely supported function $f : X \to \mathbb{C}$ we have

$$\sum_{x,y\in X} K(x,y)f(x)f(y) \geq 0.$$

Positive definite kernels can be used to characterize property A, we use this description as the definition. We refer to [9–11] for more details and other characterizations of property A.

Theorem 3 (see [11])**.** *A discrete metric space X has property A if and only if for every $R,\varepsilon > 0$ there exists a Hermitian positive definite kernel $K : X \times X \to [0,1]$ and $S > 0$, satisfying*

(1) $K(x,x) = 1$ *for every* $x \in X$,
(2) $|1-K(x,y)| \leq \varepsilon$ *if* $d(x,y) \leq R$,
(3) $K(x,y) = 0$ *if* $d(x,y) \geq S$.

For a finitely generated group G we take X to be G with the word length metric. In that case it suffices to consider only $R = 1$. A Hermitian kernel K on X induces a self-adjoint linear operator on $\ell_2(X)$, denoted also by K, by viewing K as a matrix over X. We will identify the operator with the kernel representing it.

Lemma 4. *Let G be a finitely generated group with Yu's property A. Then for every $\varepsilon > 0$ there exists an operator T of a Hilbert space H, a unitary representation π of G on H and a unit vector $v \in H$, satisfying the conditions of lemma 2.*

Proof. Let $\varepsilon > 0$. Given K as in Theorem 3, define an operator

$$T = \frac{1}{1+\varepsilon}(K + \varepsilon I),$$

where I is the identity on H.

It is clear that since K is a positive operator, T is also positive. It is easy to check that since T is represented by a kernel, which takes values in the interval $[0,1]$ and vanishes outside of a neighborhood of the diagonal, T is a bounded operator on $\ell_2(G)$. Finally,

$$\langle v, Tv\rangle = \langle v, Kv\rangle + \varepsilon\langle v, v\rangle \geq \varepsilon\langle v, v\rangle.$$

Thus T is a positive, self-adjoint, bounded operator of $H_0 = \ell_2(G)$ and it satisfies (1). Consequently we can construct a new Hilbert space H_T, isomorphic to $\ell_2(G)$, as explained earlier.

Consider now π, the left regular representation of G on $\ell_2(G)$, viewed as a representation on H_T. By the previous discussion, π is a uniformly bounded representation on H_T.

Denote by δ_g the Dirac mass at $g \in G$ and let $v = \delta_e$. Whenever $g \neq h$ we have

$$\langle \pi_g v, T\pi_h v\rangle = \frac{1}{1+\varepsilon}\langle \pi_g v, K\pi_h v\rangle = \frac{1}{1+\varepsilon}\langle \delta_g, K\delta_h\rangle = \frac{1}{1+\varepsilon}K(g,h), \tag{2}$$

and

$$\|\pi_g v\|_T = \langle \delta_g, T\delta_g\rangle = 1.$$

For $g, h \in G$ such that $|g^{-1}h| = 1$ we can estimate

$$\begin{aligned} |1 - \langle \pi_g v, T\pi_h v\rangle| &= |1 - \langle \delta_g, T\delta_h\rangle| \\ &= |1 - T(g,h)| \\ &= |1 - \frac{1}{1+\varepsilon}K(g,h)| \\ &\leq \varepsilon + \frac{\varepsilon}{1+\varepsilon}, \end{aligned}$$

by (2). Also,

$$\langle \pi_g v, T\pi_h v\rangle = \frac{1}{1+\varepsilon}\langle \pi_g v, K\pi_h v\rangle = 0,$$

whenever $|g^{-1}h| \geq S$. Thus T, π and v satisfy the required conditions with S and $\varepsilon' = \varepsilon + \dfrac{\varepsilon}{1+\varepsilon} \leq 2\varepsilon$. ☐

We are now in the position to prove the main theorem.

Proof of Theorem 1. If G is a finitely generated group with property A then we apply lemma 4 and lemma 2 and the claim follows.

Conversely, given $\varepsilon > 0$, the corresponding representation π and a vector v define $K(g,h) = \langle \pi_g v, \pi_h v\rangle$. Then K is positive definite and it is easy to check that it satisfies the conditions required by Theorem 3. □

A path of representations. Let G be a finitely generated group. G coarsely embeds into the Hilbert space H if there exists a map $f : G \to H$, two non-decreasing functions $\rho_-, \rho_+ : [0,\infty) \to [0,\infty)$ such that

$$\rho_-(d(g,h)) \le \|f(g) - f(h)\|_H \le \rho_+(d(g,h)),$$

and $\lim_{t\to\infty} \rho_-(t) = \infty$. Such an f is called a coarse embedding.

It is shown in [6] that if there exists $\theta > 0$ such that $\rho_-(t) \ge Ct^{1/2+\theta} + D$ for $t \ge E$, for some constants $C, D, E > 0$, then the positive definite kernel

$$K_\alpha(g,h) = e^{-\alpha\|f(g)-f(h)\|_H^2},$$

induces a bounded positive operator on $\ell_2(G)$. The proof relies on the Schur test. The existence of θ as above is strictly stronger than property A. Similarly as above we can use these kernels to construct uniformly bounded representations.

Let $f : [0,\infty) \to [0,\infty)$ be a smooth function such that

(1) $\lim_{t\to 0} f(\alpha) = 0$,
(2) $\lim_{t\to\infty} f(\alpha)$ exists.

Applying the previous construction to the operators

$$T_\alpha = K_\alpha + f(\alpha)I,$$

we obtain a family of representations $\{\pi_\alpha\}_{\alpha=0}^{\infty}$, that interpolates between the coefficients of the trivial representation at $\alpha = 0$ and the left regular representation at $\alpha = \infty$.

2. Concluding remarks: Norms and strong property (T)

It is natural to ask how the norm $\|\pi\|$ of the representations in Theorem 1 behaves when $\varepsilon \to 0$. The norm of the uniformly bounded representation π induced by renorming of a Hilbert space H_0 via a positive self-adjoint operator T is the number

$$\|\pi\| = \inf\left\{ c \in [1,\infty) \;\middle|\; \begin{array}{l} c^2 T - \pi_{g^{-1}} T \pi_g \text{ is a positive operator} \\ \text{on } H_0 \text{ for every } g \in G \end{array} \right\}.$$

Estimating the above norm does not seem to be an easy task. Since the bottom of the spectrum λ of T converges toward zero as $\varepsilon \to 0$, the right hand side of the estimate $\|\pi\| \le \frac{\|T\|_{B(\ell_2(G))}}{\lambda}$ tends to infinity and it is natural to expect that the norms of π will blow up to infinity as our coefficients of π approach the trivial representation. For some groups this cannot be improved.

Consider the following strong version of property (T): $H^1(G,\pi) = 0$ *for any uniformly bounded representation* π *of* G *on a Hilbert space*. Equivalently, every affine action with linear part π given by a uniformly bounded representation on a Hilbert space, has a fixed point. This property is possessed by higher rank lattices [Shalom, unpublished], universal lattices [7] and Gromov monsters [8]. As a consequence we have

Proposition 5. *Let G have the above strong property (T) for uniformly bounded representations. Then for any family of uniformly bounded representations π satisfying Theorem 1, $\|\pi\| \to \infty$ as $\varepsilon \to 0$.*

Proof. Assume the contrary. Then for every $\varepsilon > 0$ there exists a uniformly bounded representation $\pi = \pi_\varepsilon$ and vectors v_ε, satisfying the conditions of Theorem 1, with the additional property that $\sup \|\pi_\varepsilon\| \leq C$ for some constant $C > 0$.

Choosing a summable sequence of ε we construct a Hilbert space $H = \bigoplus_\varepsilon H_\varepsilon$ and a representation $\rho = \bigoplus \pi_\varepsilon$. By the assumption on the uniform bound on norms of π_ε the representation ρ is uniformly bounded on H. Now construct a cocycle $b_g = \oplus(\pi_\varepsilon)_g v_\varepsilon - v_\varepsilon$. Following the proof of [1] we conclude that b is a proper cocycle, in particular b is not a coboundary. □

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