Algebraic reformulation of Connes embedding problem and the free group algebra.

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Abstract

We give a modification of I. Klep and M. Schweighofer algebraic reformulation of Connes' embedding problem by considering $*$-algebra of the countably generated free group. This allows to consider only quadratic polynomials in unitary generators instead of arbitrary polynomials in self-adjoint generators.

KEYWORDS: Connes' Embedding Problem, II_1-factor, sum of hermitian squares, positivity.

1 Introduction.

Let $\omega \in \beta(\mathbb{N}) \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$ and R be the hyperfinite II_1-factor with faithful tracial normal state τ. Then the subset I_ω in $l^\infty(\mathbb{N}, R)$ consisting of $(x_1, x_2, \ldots)$ with $\lim_{n\to\omega} \tau(x_n^* x_n) = 0$ is a closed ideal in $l^\infty(\mathbb{N}, R)$ and a quotient algebra $R^\omega = l^\infty(\mathbb{N}, R)/I_\omega$ is a von Neumann II_1-factor called *ultrapower* of R. It is naturally endowed with a faithful tracial normal state

$$\tau_\omega((x_n) + I_\omega) = \lim_{n\to\omega} \tau(x_n).$$

A. Connes' embedding problem asks whether every finite von Neumann algebra with fixed normal faithful tracial state can be embedded into R_ω in a trace-preserving way.

It is well know that Connes' embedding problem is equivalent to the problem whether every finite set $x_1, \ldots, x_n$ of self-adjoint contractions in

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arbitrary II_1-factor (M,τ) has *matricial microstates*, i.e whether for any $\varepsilon > 0$ and $t \geq 1$ there is $k \in \mathbb{N}$ and self-adjoint contractive $k \times k$-matrices $A_1, \ldots, A_n$ such that $|\operatorname{tr}(w(x_1, \ldots, x_n)) - \tau(w(A_1, \ldots, A_n))| < \varepsilon$ for all words w of length at most t.

In [3] D. Hadwin proved that solving Connes' embedding problem in affirmative is equivalent to proving that there is no polynomial $p(x_1, ..., x_n)$ in non-commutative variables such that

1. $tr_k(p(A_1, \ldots, A_n)) \geq 0$ for every k and self-adjoint contractions $A_1, ..., A_n \in M_k$.
2. $\tau(p(T_1, \ldots, T_n)) < 0$, where $T_1, \ldots, T_n$ are self-adjoint contractive elements in a finite factor with trace τ.

Recently I. Klep and M. Schweighofer established that Connes' embedding problem has the following equivalent algebraic reformulation.

Let $f(X_1, \ldots, X_m)$ be a self-adjoint element in a free associative algebra $\mathbb{K}\langle\overline{X}\rangle$ with countable family of self-adjoint generators $\overline{X} = \{X_1, X_2, \ldots\}$, where $\mathbb{K} = \mathbb{R}$ or $K = \mathbb{C}$. If $\operatorname{tr}(f(A_1, \ldots, A_m)) \geq 0$ for any n and family of self-adjoint contractive matrices $A_1, \ldots, A_m \in M_n(\mathbb{K})$ then f has the property that for every $\varepsilon > 0$ we have $\varepsilon e + f = g + c$ where c is a sum of commutators in $\mathbb{K}\langle\overline{X}\rangle$, g belongs to quadratic module generated by $1 - X_i^2$ and e is the unit in $\mathbb{K}\langle\overline{X}\rangle$. Recall that a *quadratic module* is the smallest subset of $\mathbb{K}\langle\overline{X}\rangle$ containing unit, closed under addition and conjugation $x \to g^* x g$ by arbitrary $g \in \mathbb{K}\langle\overline{X}\rangle$.

In the present paper we consider the group $*$-algebra $\mathcal{F}$ of the countably generated free group $\mathbb{F}_\infty = \langle u_1, u_2, \ldots\rangle$ instead of $\mathbb{K}\langle\overline{X}\rangle$. One reason is that we can use a more standard and well known set of hermitian squares $\{g^* g | g \in \mathcal{F}\}$ instead of quadratic module M and the second that we can bound the degree of polynomials f in the above reformulation by 2. This modification provides the following.

Theorem. *Connes' embedding conjecture is true iff for any self-adjoint $f \in \mathcal{F}$ of the form $f(u_1, \ldots, u_n) = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ condition*

$$Tr(f(V_1, \ldots, V_n)) \geq 0 \tag{1}$$

for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \ldots, V_n \in U(m)$ implies that for every $\varepsilon > 0$, $\varepsilon e + f = g + c$ where c is a sum of commutators and g is a sum of Hermitian squares.

We will call f satisfying (1) a *trace-positive quadratic polynomial*. Elements of the form $g+c$ with c being a sum of commutators are called *cyclically equivalent* to g (see Section 2).

In Section 3 we study a subset of correlation matrices of the form $[\mathrm{tr}(U_i^* U_j)]_{ij}$ where $U_1, \ldots, U_n$ runs over n-tuple of unitary matrices and $\mathrm{tr}(U)$ denotes normalized trace of U. Using Clifford algebra methods we show that this set contains all correlation matrices with real coefficients. This implies that all trace-positive quadratic polynomials f with real coefficients do satisfy the property from the above theorem.

The description of the set $\{[\mathrm{tr}(U_i^* U_j)]_{ij} \mid U_1, \ldots, U_n \in U_m(\mathbb{C}), m \geq 1\}$ seems to be unknown even for $n = 3$. In this case it is equivalent to the problem of description of the set of triples $(\mathrm{tr}(U), \mathrm{tr}(V), \mathrm{tr}(UV))$ where U and V are unitary matrices. Note that the lists of possible eigenvalues of U, V and UV can be described by generalization of Horn's inequalities (see [2]) but little is know about possible traces $(\mathrm{tr}(U), \mathrm{tr}(V), \mathrm{tr}(UV))$. The only known connection between these traces seems to be the inequality $\sqrt{1 - |\mathrm{tr}(UV)|^2} \leq \sqrt{1 - |\mathrm{tr}(U)|^2} + \sqrt{1 - |\mathrm{tr}(V)|^2}$ established in [13].

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2 An algebraic reformulation of Connes' problem.

Let $\mathcal{F}$ be the $*$-algebra of the countably generated free group $\mathbb{F}_\infty$. Let K denote the $\mathbb{R}$-subspace in $\mathcal{F}_{sa}$ generated by the commutators $fg - gf$ $(f, g \in F)$. We will say that f and g in $\mathcal{F}$ are *cyclically equivalent* (denote $f \overset{cyc}{\sim} g$) if $f - g \in K$. Let $\Sigma^2(\mathcal{F})$ denote the set of positive elements of the $*$-algebra $\mathcal{F}$, i.e. elements of the form $\sum_{j=1}^m f_j^* f_j$ with $f_j \in \mathcal{F}$. An element of the form $f^* f$ is called Hermitian square and therefore the cone $\Sigma^2(\mathcal{F})$ is called the cone of Hermitian squares.

Definition 1. *Let C be a subset of the vector space V. An element $v \in C$ is called an algebraic interior point of C if for every $u \in V$ there is $\varepsilon > 0$ in $\mathbb{R}$ s.t. $v + \lambda u \in C$ for all $0 \leq \lambda \leq \varepsilon$.*

Definition 2. *Let A be a unital $*$-algebra with the unit e. Then*
1. An element $a \in \mathcal{A}_{sa}$ is called bounded if there is $\alpha \in \mathbb{R}_+$ such that $\alpha e \pm a \in \Sigma^2(\mathcal{A})$.
2. An element $x = a + ib$ with $a,\ b \in \mathcal{A}_{sa}$ is bounded if the elements a and b are such.
3. The algebra $\mathcal{A}$ is bounded if all its elements are bounded.

It is well known that the set of all bounded elements in $\mathcal{A}$ is a $*$-subalgebra in $\mathcal{A}$ and that an element $x \in A$ is bounded if and only if xx^* is such (see for example [9, 5]). In particular $\mathcal{F}$ is a bounded $*$-algebra. Obviously this implies that the unit of the algebra is an algebraic interior point of $\Sigma^2(\mathcal{F})$.

The following lemma is a modification of Theorem 3.12 in [7].

Lemma 3. *Let $f \in \mathcal{F}$ be self-adjoint. If for any II_1 factor M with faithful normal tracial state τ and separable predual and every n-tuple of unitary elements $U_1, \dots, U_n$ in the unitary group $\mathcal{U}(M)$ of M we have that*

$$\tau(f(U_1, \dots, U_n)) \geq 0$$

then for every $\varepsilon > 0$, $\varepsilon e + f \sim g$ for some $g \in \Sigma^2(\mathcal{F})$.

Proof. Clearly $\Sigma^2(\mathcal{F}) + K$ is a convex cone in $\mathbb{R}$-space $\mathcal{F}_{sa}$. Since e is an an algebraic internal point of $\Sigma^2(\mathcal{F})$ it is also an algebraic internal point of $\Sigma^2(\mathcal{F}) + K$.

Assume that there is $\varepsilon > 0$ such that $\varepsilon e + f \not\sim g$ for any $g \in \Sigma^2(\mathcal{F})$, i.e. $\varepsilon e + f \notin \Sigma^2(\mathcal{F}) + K$. By Eidelheit-Kakutani separation theorem there is $\mathbb{R}$-linear unital functional $L_0 : \mathcal{F}_{sa} \to \mathbb{R}$ s.t. $L_0(\Sigma^2(\mathcal{F}) + K) \subseteq \mathbb{R}_{\geq 0}$ and $L_0(\varepsilon e + f) \in \mathbb{R}_{\leq 0}$. Since $-K \subset \Sigma^2(\mathcal{F}) + K$ we have that $L_0(K) = 0$. In particular extending L_0 to $\mathbb{C}$-linear functional on $\mathcal{F}$ we get a tracial functional L. Since L maps $\Sigma^2(\mathcal{F})$ into the non-negative reals it defines a pre-Hilbert space structure on $\mathcal{F}$ by means of sesquilinear for $\langle p, q \rangle = L(q^*p)$, $p, q \in \mathcal{F}$. Let $N = \{p : \langle p, p \rangle = 0\}$. By Cauchy-Schwarz inequality $N = \{p : L(q^*p) = 0 \text{ for all } q \in \mathcal{F}\}$ and hence is a left ideal. Let H_0 be the pre-Hilbert space $\mathcal{F}/N$. Consider the left regular representation $\pi : \mathcal{F} \to L(H_0)$. Since π is a $*$-homomorphism for every $f \in \mathcal{F}$ operator $\pi(f)$ is bounded as a linear combination of unitary operators. Thus $\pi(f)$ can be extended to the bounded operator acting on the Hilbert space H which is the completion of H_0. Thus we have a representation $\pi : \mathcal{F} \to B(H)$ with a cyclic vector $\xi = e + N$ and such that $L(p) = \langle \pi(p)\xi, \xi \rangle$. In particular L is a contractive tracial state on $\mathcal{F}$ and thus defines a tracial state of the universal enveloping C^*-algebra $C^*(\mathcal{F})$. By Banach-Alaoglu and Krein-Milman theorem we can assume that L is an extreme point in the set of all tracial states and thus $\pi(\mathcal{F})$ generates a factor von Neumann algebra M (see [3]). Clearly M is a finite factor. If it is type I then it should be $\mathbb{C}$ (since ξ is a trace vector) and thus can be embedded into any II_1-factor in trace preserving way. Thus we can assume that M is a type II_1-factor. But then condition $L(f) < 0$ is impossible. □

Corollary 4. *If self-adjoint $f \in \mathcal{F}$ has real coefficients and for any real type II_1 von Neumann algebra (M, τ) with normal faithful tracial state τ and every n-tuple of unitary elements $U_1, \dots, U_n$ in M we have that*

$$\tau(f(U_1, \dots, U_n)) \geq 0$$

then the same holds for the complex II_1 von Neumann algebras.

Proof. Element f can be written as $f = \alpha + \sum_{w_j} \alpha_{w_j}(w_j + w_j^*)$ with $\alpha_{w_j} \in \mathbb{R}$ and for complex trace τ and $U_1, \dots, U_n \in U(M)$ we will have $\tau(f) = \alpha + 2\sum_{w_j} \alpha_{w_j} \operatorname{Re}\tau(w_j)$, i.e. $\tau(f) = (\operatorname{Re}\tau)(f)$. To finish the proof note that M can be regarded as a real finite von Neumann algebra with faithful trace $\operatorname{Re}\tau$. □

Lemma 5. *If $f \in \mathbb{R}[\mathbb{F}_\infty]$, $f = f^*$ and for every real type II_1 von Neumann algebra (M, τ) we have that $\tau(f) \geq 0$ then for every $\varepsilon > 0$, $\varepsilon + f \overset{cyc}{\sim} g$ for some $g \in \left\{\sum_{j=1}^m g_j^* g_j \mid m \in \mathbb{N}, g_j \in \mathbb{R}\langle \mathbb{F}_\infty \rangle\right\}$.*

Proof. The proof of this statement can be obtained by obvious modification of the proof of lemma 3. The only nontrivial part is that the unit e is an algebraic internal point but this is equivalent to $\mathbb{R}\langle \mathbb{F}_\infty \rangle$ being bounded $*$-algebra. The proof of the last fact can be found in [12]. □

This lemma gives another proof of corollary 4. In sequel we will need the following lemma.

Lemma 6. *If (M, τ) is a II_1 factor which can be embedded into R^ω and $f \in \mathcal{F}$ is self-adjoint then the condition* $\operatorname{tr}(f(V_1, \dots, V_n)) \geq 0$ *for all $m \geq 0$ and all unitary $V_1, \dots, V_n$ in $M_{m\times m}(\mathbb{C})$ implies that $\tau(f(U_1, \dots, U_n)) \geq 0$ for all unitary $U_1, \dots, U_n$ in M.*

Proof. Considering M as a subalgebra in R^ω and τ as a restriction of the trace on R^ω we can find a representing sequences $\left\{u_j^{(k)}\right\}_{j=1}^{\infty}$ for U_k, $k = 1,\ldots,n$ in $l^\infty(\mathbb{N},R)$ which are unitary elements in von Neumann algebra $l^\infty(\mathbb{N},R)$. This can be done since every unitary in von Neumann algebra R^ω can be lifted to a unitary in von Neumann algebra $l^\infty(\mathbb{N},R)$ with respect to canonical morphism $\pi : l^\infty(\mathbb{N},R) \to R^\omega$. Taking j sufficiently large we can approximate mixed moments of $U_1,\ldots,U_k$ up to order m, i.e. $\tau(U_{s_1}\ldots U_{s_t})$ with $t \le m$ and $s_1,\ldots,s_t \in \{1,\ldots,n\}$, by the mixed moments of unitary matrices $u_1^{(k)},\ldots,u_n^{(k)}$. □

The following theorem is Proposition 4.6 in [6]

Theorem 7. (E. Kirchberg) *Let $(M,\ \tau)$ be von Neumann algebra with separable predual and faithful normal tracial state τ. If for all $n \ge 1$ and for all unitaries $u_1,\ldots,u_n$ in M and for arbitrary $\varepsilon > 0$ there exists $m \ge 1$ and unitary $m \times m$ matrices $V_1,\ldots,V_n \in U(m)$ s.t. for all i,j:*

$$|\tau(u_i^* u_j) - \frac{1}{m}\operatorname{Tr}(V_i^* V_j)| \ < \ \varepsilon, \tag{2}$$

$$|\tau(u_j) - \frac{1}{m}\operatorname{Tr}(V_j)| \ < \ \varepsilon \tag{3}$$

then M can be embedded into R^ω.

Remark 8. *We may drop condition* (3) *since we may take $u_0 = 1, u_1,\ldots,u_n$ and by* (2) *find matrices $W_0,\ldots,W_n$ such that $|\tau(u_i^* u_j) - \frac{1}{m}\operatorname{Tr}(W_i^* W_j)| < \varepsilon$ for all i and j. Thus* (2) *and* (3) *will be satisfied if we take $V_j = W_0^* W_j$.*

The proof of the following theorem is an adaptation of the proof of Proposition 3.17 from [7].

Theorem 9. *Let (M,τ) be II_1-factor with separable predual. If for every self-adjoint element $f \in \mathcal{F}$ of the form $f = \alpha + \sum_{i\neq j} \alpha_{ij} u_i^* u_j$ the condition*

$$\operatorname{Tr}(f(V_1,\ldots,V_n)) \ge 0$$

for all $m \ge 1$ and every n-tuple of unitary matrices $V_1,\ldots,V_n \in U(m)$ implies that $\tau(f(U_1,\ldots,U_n)) \ge 0$ for all unitaries $U_1,\ldots,U_n$ in M then M can be embedded into R^ω.

Proof. Take $n \geq 1$. Consider the finite dimensional vector space $W = \{\alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j | \alpha_{ij} \in \mathbb{C}\}$. Denote by C the convex hull of the set F of the functionals $T \in W^*$ of the form $T(p) = \frac{1}{m} \operatorname{Tr}(p(V_1, \ldots, V_n))$ where $m \geq 1$ and $V_1, \ldots, V_n \in U(m)$. Take arbitrary n-tuple of unitary elements $U_1, \ldots, U_n$ in M and put $L(p) = \tau(p(U_1, \ldots, U_n))$ for $p \in W$. Assume that $L \notin C$. By Hahn-Banach theorem there is $f \in W^{**} = W$ and $c \in \mathbb{R}$ s.t. $Re(L(f)) < c < Re(T(f))$ for all $T \in C$. Since $e \in W$ we can substitute $f - c$ instead of f and thus assume that $c = 0$. Since $T(f^*) = \overline{T(f)}$ for every $T \in C$ and $L(f^*) = \overline{L(f)}$ we have that $L(f + f^*) = 2Re(L(f)) < 0 < 2Re(T(f)) = T(f + f^*)$ which is a contradiction. Thus $L \in C$. Let T be a rational convex combination of elements $T_1, \ldots, T_s$ from F and T_k corresponds to n-tuples $V_{j,k}$. Then $T = \frac{1}{q}(p_1 T_1 + \ldots + p_s T_s)$ for some positive integers $p_1, \ldots, p_s, q$. Taking block-diagonal $V_j = (V_{j,1}^{\otimes p_1} \oplus \ldots \oplus V_{j,s}^{\otimes p_s})$ we see that $T \in F$. Thus each element of C, in particular element L can be approximated by elements of F. By the Kirchberg's Theorem we have that M can be embedded into R^ω.

□

Theorem 10. *Connes' embedding conjecture problem has affirmative solution iff for any self-adjoint $f \in \mathcal{F}$ of the form $f = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ condition*

$$\operatorname{Tr}(f(V_1, \ldots, V_n)) \geq 0$$

for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \ldots, V_n \in U(m)$ implies that for every $\varepsilon > 0$, $\varepsilon e + f \sim g$ with $g \in \Sigma^2(\mathcal{F})$.

Proof. If Connes' embedding problem has affirmative solution and quadratic $f \in \mathcal{F}_{sa}$ is such that $\operatorname{Tr}(f(V_1, \ldots, V_n)) \geq 0$ for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \ldots, V_n \in U(m)$ then by lemma 6 we have $\tau(f(U_1, \ldots, U_n)) \geq 0$ for any unitary $U_1, \ldots, U_n$ in M. Hence by lemma 3, $\varepsilon e + f$ is cyclically equivalent to a sum of Hermitian squares. This proves that the conditions of the theorem are necessary.

If $\varepsilon e + f$ is cyclically equivalent to an element in $\Sigma^2(\mathcal{F})$ for every $\varepsilon > 0$ then clearly $\tau(f(U_1, \ldots, U_n)) \geq 0$ for any unitary $U_1, \ldots, U_n$ in M. Hence the sufficiency of the theorem conditions follows from Theorem 9.

□

3 The trace-positive quadratic polynomials.

The results of the preceding section motivate the study of trace-positive self-adjoint quadratic polynomials $f = \alpha e + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ in unitary generators $u_1, \dots, u_n$, i.e. polynomials having the property that $\mathrm{Tr}(f(V_1, \dots, V_n)) \geq 0$ for every $m \geq 1$ and every n-tuple of unitary matrices $V_1, \dots, V_n \in U(m)$. If A denotes the matrix

$$\begin{pmatrix} \alpha/n & \alpha_{12} & \dots & \alpha_{1n} \\ \overline{\alpha_{12}} & \alpha/n & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \overline{\alpha_{1n}} & \overline{\alpha_{2n}} & \dots & \alpha/n \end{pmatrix}$$

then $\mathrm{Tr}\, f(U_1, \dots, U_n) \geq 0$ can be expressed as positivity of the sum of all entries of the Schur product $A \circ X$ where $X = [\mathrm{tr}(U_i^* U_j)]_{ij}$.

Thus the trace-positive polynomials f can be characterized as those for which the sum of all entries of $A \circ X$ for all $X \in K_n := \{[\mathrm{tr}(U_i^* U_j)]_{ij} \mid m \geq 1, U_1, \dots, U_n \in U(m)\}$. Thus our primary objective is to describe the sets $K_n \subseteq M_n(\mathbb{C})$. Note that in the case A is positive semidefinite we have $f \in \Sigma^2(\mathcal{F})$. Indeed in this case A is a sum of rank one positive semidefinite matrices $A = \sum_s (\beta_{s,1}, \dots, \beta_{sn})^T (\beta_{s,1}, \dots, \beta_{sn})$ and hence $f = \sum_s (\sum_j \beta_{s,j} u_j)^* (\sum_j \beta_{s,j} u_j)$. We will also be interested in real analog of the sets K_n, i.e. the sets $K_n(\mathbb{R}) = K_n \cap M_n(\mathbb{R})$. Note that the sets of the traces of monomials of unitary operators and their asymptotic properties in the context of Connes' embedding problem also studied in [10] and [11].

A self-adjoint matrix A such that $f = (u_1^{-1}, \dots, u_n^{-1}) A (u_1, \dots, u_n)^T$ is defined uniquely except for the diagonal entries. This motivates the following definition. We will call A and B *diagonally equivalent* and write $A \stackrel{d}{\sim} B$ if $A - B$ is a diagonal matrix with vanishing trace.

Definition 11. *Let $S \subseteq M_n(\mathbb{C})$ and $A \in M_n(\mathbb{C})$ be self-adjoint. We say that A is S-positive and denote $A \geq_S 0$ if there is self-adjoint B such that $A \stackrel{d}{\sim} B$ and*

$$\sum_{ij} b_{ij} s_{ij} \geq 0$$

for all $s \in S$.

The three natural choices for S will be

$$F_n = \{(t_{ij}) | t_{jj} = 1 \text{ and } |t_{ij}| \leq 1 \text{ for all } i, j\},$$

$P_n \subset F_n$ consisting of positive matrices and the set $K_n \subset F_n$. Clearly, an self-adjoint matrix $A = [a_{ij}]$ is K_n-positive iff $f = \sum_i a_{ii}e + \sum_{i\neq j} a_{ij}u_i^* u_j$ is a trace positive quadratic polynomial. Note that if

$$A \geq_{F_n} 0$$

then

$$\operatorname{Tr} A \geq \sum_{i\neq j} |a_{ij}|$$

and hence $A \overset{d}{\sim} B$ for some diagonally dominant matrix B. In this case polynomial $f = (u_1^{-1}, \ldots, u_n^{-1})A(u_1, \ldots, u_n)^T$ is a sum of hermitian squares. However if $A \geq_{P_n} 0$ then A need not be diagonally equivalent to positive matrix. Note that for the three choices of S mentioned above one can use equality instead of diagonal equivalence since diagonal entries of elements in S equal to 1.

The following lemma gives a description of cyclically equivalent quadratic polynomials.

Lemma 12. *For every matrix A the element $(u_1^{-1}, \ldots, u_n^{-1})A(u_1, \ldots, u_n)^T$ is cyclically equivalent to*

$$\sum_k g_k^{-1}(u_1^{-1}, \ldots, u_n^{-1})A_{g_k}(u_1, \ldots, u_n)^T g_k \tag{4}$$

for any finite collection $g_1, \ldots, g_k \in \mathbb{F}_\infty$ and any matrices A_g such that

$$\sum_k A_{g_k} \overset{d}{\sim} A. \tag{5}$$

Any element $g \in \mathcal{F}$ such that $g \overset{cyc}{\sim} f$ is of the form (4) *for some matrices satisfying* (5)*. Moreover for self-adjoint g matrices A_g can also be chosen to be self-adjoint.*

Proof. The lemma follows from the following easy observation. For any w_1 and w_2 in $\mathbb{F}_\infty$ the element $w_1 - w_2$ is a commutator $ab - ba$ for some a, $b \in \mathbb{F}_\infty$ if and only if w_1 and w_2 are conjugated. Hence K consists of finitely supported sums of the form

$$\sum_j \sum_k \alpha_{jk} g_k^{-1} w_j g_k$$

where w_j, g_k belong to $\mathbb{F}_\infty$ and $\sum_k \alpha_{jk} = 0$ for all j. □

4 The Clifford Algebras and positive polynomials with real coefficients.

For a real Hilbert space V there is a unique associative algebra $\mathcal{C}(V)$ with a linear embedding $J : V \to \mathcal{C}(V)$ with generating range and such that for all $x, y \in V$

$$J(x)J(y) + J(y)J(x) = 2\langle x, y\rangle . \tag{6}$$

The algebra $\mathcal{C}(V)$ is called Clifford algebras associated to V. Clifford algebra can be realized on a Hilbert space such that for every $x \in V$ with $\|x\| = 1$ operator $J(x)$ is symmetry, i.e. $J(x)^* = J(x)$ and $J(x)^2 = I$. To see this consider Pauli matrices

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Clearly U and Q are self-adjoint unitary matrices and $U^2 = I, Q^2 = I$, $QU + UQ = 0$. Then matrices $Q_j = U \otimes \ldots \otimes U \otimes Q \otimes I \otimes I \ldots$ are symmetries and $\{Q_i, Q_j\} = 2\delta_{ij} I$. Hence operator $J(x) = \sum_j x_j Q_j$ is also a symmetry for unit real vector x. For further properties of Clifford algebras we refer to the books [1] and [8].

Theorem 13. *For every real correlation matrix $P \in M_n(\mathbb{R})$ there is n-tuple of symmetries $S_1, \ldots, S_n$ in finite dimensional real Hilbert space s.t. $P = [\mathrm{tr}(S_i^* S_j)]_{ij}$.*

Proof. Every correlation $n \times n$-matrix P is a Gram matrix for a system of unit vectors $x_1, \ldots, x_n$, i.e. $P = [\langle x_i, x_j\rangle]_{ij}$. Taking Clifford symmetries $S_j = J(x_j)$ as in the paragraph preceding the theorem we see that $P = [\mathrm{tr}(S_i^* S_j)]_{ij}$. $\square$

Proposition 14. *For every $n \geq 1$ the closure $T_n(\mathbb{R})$ of the set of matrices*

$$\{[\tau(U_i^* U_j)]_{ij} | U_1, \ldots, U_n \in \mathcal{U}(M)\}$$

does not depend on real type II_1 von Neumann algebra (M, τ).

If self-adjoint $f(u_1, \ldots, u_n) \in \mathcal{F}$ has real coefficients and possess property that for every n-tuple of unitary matrices $U_1, \ldots, U_n \in U(m)$ we have $tr(f(U_1, \ldots, U_n)) \geq 0$ then for every $\varepsilon > 0$, $\varepsilon e + f \overset{cyc}{\sim} g$ for some $g \in \left\{\sum_{j=1}^m g_j^* g_j | m \in \mathbb{N}, g_j \in \mathbb{R}[\mathbb{F}_\infty]\right\}$.

Proof. Since every II_1 factor contains matrix algebras of arbitrary size we see that $T_n(\mathbb{R})$ coincides with the set of correlation matrices. The last statement follows from Lemma 5. □

Corollary 15. *If quadratic $f \in \mathcal{F}$, $f(u_1, \dots, u_n) = \alpha + \sum_{i \neq j} \alpha_{ij} u_i^* u_j$ is such that*

$$\mathrm{Tr}(f(U_1, \dots, U_n)) = 0$$

for all unitary matrices $U_1, \dots, U_n$ then $f = 0$.

Proof. For every $k \neq j$ and $t \in [0, 1]$ the matrix $P_1 = I + (E_{kj} + E_{jk})t$ is a real correlation matrix. Hence by the theorem there are unitary matrices $U_1, \dots, U_n$ such that $P_1 = [\mathrm{tr}(U_t^* U_s)]_{ts}$. Then the matrix $P_2 = I + (iE_{kj} - iE_{jk})t$ is equal to $[\mathrm{tr}(V_t^* V_s)]_{ts}$ where $V_t = U_t$ for $t \neq j$ and $V_j = iU_j$ are unitary matrices. Hence $\alpha + (\alpha_{kj} + \alpha_{jk})t = 0$ and $\alpha + (\alpha_{kj} - \alpha_{jk})it = 0$. From which follows that $\alpha = \alpha_{kj} = 0$ and hence $f = 0$. □

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