Matrix ordered operator algebras.

Juschenko Kate

Mathematical Sciences, Chalmers University of Technology,

Mathematical Sciences, Göteborg University,

SE-412 96 Göteborg, Sweden

jushenko@math.chalmers.se

Popovych Stanislav

Mathematical Sciences, Chalmers University of Technology,

Mathematical Sciences, Göteborg University,

SE-412 96 Göteborg, Sweden

stanislp@math.chalmers.se

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Abstract

We study the question when for a given $*$-algebra $\mathcal{A}$ a sequence of cones $C_n \subseteq M_n(\mathcal{A})_{sa}$ can be realized as cones of positive operators in a faithful $*$-representation of $\mathcal{A}$ on a Hilbert space. We present a criterion analogous to Effros-Choi abstract characterization of operator systems. A characterization of operator algebras which are completely boundedly isomorphic to C^*-algebras is also presented.

KEYWORDS: $*$-algebra, faithful representation, Archimedean order, operator system.

1 Introduction.

An operator system S is a not necessarily closed subspace in $B(\mathcal{H})$ containing the identity operator $I_{\mathcal{H}}$, such that $x^* \in S$ for all $x \in S$.

In [3] Choi and Effros obtained an abstract characterization of operator systems among $*$-vector spaces. More precisely, a $*$-vector space V is a vector space over $\mathbb{C}$ with a given conjugate-linear map $x \to x^*$ such that $(x^*)^* = x$. A $*$-vector space is called *matrix ordered* if it possesses a sequence of cones C_n with the following properties:

1. For every $n \geq 1$ we have $C_n \subseteq M_n(S)_{sa}$.

2. $C_n \cap (-C_n) = \{0\}$.

3. For all $m, n \geq 1$ and every $A \in M_{n\times m}(\mathbb{C})$ we have $A^* C_n A \subseteq C_m$.

Here $M_n(S)_{sa}$ denotes the set of self-adjoint matrices $x^* = x$.

Two matrix ordered $*$-vector spaces S and S' are called *complete order isomorphic* if there exists a linear isomorphism $\phi\colon S \to S'$ such that $\phi^{(n)}(C_n) = C'_n$. Here $\phi^{(n)}((a_{ij})) = (\phi(a_{ij}))$ for every matrix $(a_{ij}) \in M_n(S)$.

An element $e \in S_{sa}$ is called a *matrix order unit* provided that for every $n \in \mathbb{N}$ and every $x \in M_n(S)_{sa}$ there exists $r > 0$ such that $re_n + x \in C_n$, where $e_n = e \otimes I_n$. A matrix order unit is called *Archimedean matrix order unit* if for all $n \in \mathbb{N}$ the inclusion $re_n + x \in C_n$ for all $r > 0$ implies that $x \in C_n$.

Theorem 1. (Choi-Effros'77) *If S is a matrix ordered $*$-vector space with an Archimedean matrix order unit e. Then there exist a Hilbert space $\mathcal{H}$, an operator system $S_1 \subseteq B(\mathcal{H})$ and a complete order isomorphism $\phi : S \to S_1$ such that $\phi(e) = I_{\mathcal{H}}$.*

We refer the reader to Section 2 for the definition of Archimedean matrix order unit.

A $*$-algebra $\mathcal{A}$ is matrix ordered if it is a matrix ordered $*$-vector space and for all n and m and all $A \in M_{n\times m}(\mathcal{A})$, we have that $A^*C_nA \subseteq C_m$. The main result of the paper is the following analog of the above theorem valid for matrix ordered $*$-algebras.

Theorem 2. *Let $\mathcal{A}$ be a matrix ordered unital $*$-algebra with unit e. If e is an Archimedean matrix order unit then there exist Hilbert space $\mathcal{H}$ and a unital $*$-subalgebra $\mathcal{A}_1 \subseteq B(\mathcal{H})$ such that $\mathcal{A}$ and $\mathcal{A}_1$ are complete order $*$-isomorphic.*

Here complete order $*$-isomorphism is a complete order isomorphism between $\mathcal{A}$ and $\mathcal{A}_1$ considered as matrix ordered $*$-vector spaces which is also a unital $*$-homomorphism. The $*$-algebra $\mathcal{A}_1$ is endowed with the matrix order consisting of the cones $M_n(A)_{sa} \cap B(\mathcal{H})^+$ of positive operators. The proof of Theorem 2 will be given in Section 3.

In other words Theorem 2 gives a characterization of the collections of cones $C_n \subseteq M_n(\mathcal{A})$ for which there exist a faithful $*$-representation π of $\mathcal{A}$ on a Hilbert space H such that C_n coincides with the cone of positive operators contained in $\pi^{(n)}(M_n(\mathcal{A}))$. Here $\pi^{(n)}((x_{i,j})) = (\pi(x_{i,j}))$ for every matrix $(x_{i,j}) \in M_n(\mathcal{A})$. Note that we do not assume that $\mathcal{A}$ has any faithful $*$-representation. This follows from the requirements imposed on the cones.

Recall that subspaces of $B(\mathcal{H})$ can be abstractly characterized as L^∞-matrix normed spaces (see [10]). Namely, a space V is called *L^∞-matrix normed space* if we are given norms $\|\cdot\|_{m,n}$ on $M_{m,n}(V)$ such that for all $A \in M_{p,m}(\mathbb{C})$, $X, Y \in M_{m,n}(V)$, $B \in M_{n,q}(\mathbb{C})$ we have

$$\|AXB\| \leq \|A\|\|X\|\|B\| \tag{1}$$

and

$$\|X \oplus Y\| = \max\{\|X\|, \|Y\|\} \tag{2}$$

It follows from the famous Blecher-Ruan-Sinclair theorem (see [1] and [2]) that in order to obtain an abstract characterization of subalgebras of $B(\mathcal{H})$

we need to allow matrices A and B in (1) to have coefficients in algebra V. The motivation of the present paper was to find similar modification of the axioms of matrix ordered $*$-vector space which works for $*$-algebras.

The proof of Ruan's theorem (see [10, 8]) uses reduction to the selfadjoint case and then Effros-Choi theorem. It looks attractive to deduce Blecher-Ruan-Sinclair theorem from Theorem 2.

The key ingredient of the proof of Theorem 2 is the case of one cone $C \subset \mathcal{A}_{sa}$ considered in Section 2. The cones C with property that $a^*Ca \subseteq C$ for all $a \in \mathcal{A}$ were introduced by R. Powers for the study of representations in unbounded operators in [9]. In Theorem 6 we prove that such cones C with the property that the unit of the algebra is an Archimedean order unit can be represented as a cone of positive operators. In Section 3 we prove the main result Theorem 2.

Based on the above characterization of $*$-subalgebras in $B(\mathcal{H})$ we study the question when an operator algebra is similar to a C^*-algebra.

Let $\mathcal{B}$ be a unital (closed) operator algebra in $B(\mathcal{H})$. The algebra $M_n(B(\mathcal{H}))$ of $n \times n$ matrices with entries in $B(\mathcal{H})$ has a norm $\|\cdot\|_n$ via the identification of $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^n)$, where $\mathcal{H}^n$ is the direct sum of n copies of a Hilbert space $\mathcal{H}$. The algebra $M_n(\mathcal{B})$ inherits a norm $\|\cdot\|_n$ via natural inclusion into $M_n(B(\mathcal{H}))$. The norms $\|\cdot\|_n$ are called matrix norms on the operator algebra $\mathcal{B}$. If $\phi\colon \mathcal{B} \to \mathcal{B}_1$ is a linear bounded map between two operator algebras then $\phi^{(n)}$ maps $M_n(\mathcal{B})$ into $M_n(\mathcal{B}_1)$ and $\|\phi\|_{cb} = \sup_n \|\phi^{(n)}\|$ is called *the completely bounded norm* of ϕ. The map ϕ is called *completely bounded* if $\|\phi\|_{cb} < \infty$. The map ϕ is called *completely isometric* if $\phi^{(n)}$ is such for all n. Two operator algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are called *completely boundedly isomorphic* if there is a completely bounded isomorphism $\phi\colon \mathcal{B}_1 \to \mathcal{B}_2$ with completely bounded inverse.

In [6] C. Le Merdy presented necessary and sufficient conditions for $\mathcal{B}$ to be self-adjoint. These conditions involve all completely isometric representations of $\mathcal{B}$ on Hilbert spaces. Our characterization is different in the following respect. If S is a bounded invertible operator in $B(\mathcal{H})$ and $\mathcal{A}$ is a C^*-algebra in $B(\mathcal{H})$ then the operator algebra $S^{-1}\mathcal{A}S$ is not necessarily self-adjoint but only completely boundedly isomorphic to a C^*-algebra. By Haagerup's theorem every completely bounded isomorphism π from a C^*-algebra $\mathcal{A}$ to an operator algebra $\mathcal{B}$ has the form $\pi(a) = S^{-1}\rho(a)S$, $a \in \mathcal{A}$, for some $*$-isomorphism $\rho : \mathcal{A} \to B(\mathcal{H})$ and invertible $S \in B(\mathcal{H})$. Thus the question whether an operator algebra $\mathcal{B}$ is completely boundedly isomorphic to a C^*-algebra is equivalent to the question if there is bounded invertible operator S such that $S\mathcal{B}S^{-1}$ is a C^*-algebra.

We will present a criterion for an operator algebra $\mathcal{B}$ to be completely boundedly isomorphic to a C^*-algebra in terms of the existence of a collection of cones $C_n \in M_n(\mathcal{B})$ satisfying certain axioms (see def. 9). The axioms are derived from the properties of the cones of positive elements of a C^*-algebra preserved under completely bounded isomorphisms.

2 Faithful $*$-representation of $*$-algebras.

In this section, we let $\mathcal{A}$ be a unital $*$-algebra and we let e denote its unit. Let $\mathcal{A}_{sa}$ denote the set of self-adjoint elements in $\mathcal{A}$. A subset $C \subset \mathcal{A}_{sa}$ containing e is *algebraically admissible* cone (see [9]) provided that

(i) C is a cone in $\mathcal{A}_{sa}$, i.e. $\lambda x + \beta y \in C$ for all x, $y \in C$ and $\lambda \geq 0$, $\beta \geq 0$, $\lambda, \beta \in \mathbb{R}$;

(ii) $C \cap (-C) = \{0\}$;

(iii) $xCx^* \subseteq C$ for every $x \in \mathcal{A}$;

With a cone C we can associate a partial order $\geq_C$ on the real vector space $\mathcal{A}_{sa}$ given by the rule $a \geq_C b$ if $a - b \in C$. It is clear that $(\mathcal{A}_{sa}, \leq_C)$

is a preordered real vector space. Henceforth we will suppress subscript C if it will not lead to ambiguity. An element $e \in \mathcal{A}_{sa}$ is called a *order unit* provided that for every $x \in \mathcal{A}_{sa}$ there exists $r > 0$ such that $re + x \in C$. An order unit is called *Archimedean* provided that the inclusion $re + x \in C$ for all $r > 0$ implies that $x \in C$.

The following lemma is straightforward.

Lemma 3. *For every $x \in \mathcal{A}$, $x^*x \in C$. In particular $a^2 \in C$ for $a \in \mathcal{A}_{sa}$. If for $a, b \in \mathcal{A}_{sa}$, $a \geq b$ then for every $x \in \mathcal{A}$, $x^*ax \geq x^*bx$.*

The following lemma is a direct consequence of the above.

Lemma 4. *Let $\mathcal{A}$ be a $*$-algebra with algebraically admissible cone C and unit e which is an order unit. The function $\|\cdot\|$ defined as*

$$\|a\| = \inf\{r > 0 : re \pm a \in C\}$$

is a seminorm on the $\mathbb{R}$-space $\mathcal{A}_{sa}$. Moreover $\|x^*ax\| \leq \|x^*x\|\|a\|$ for every $x \in \mathcal{A}$ and $a \in \mathcal{A}_{sa}$.

Lemma 5. *Let $\mathcal{A}$ be a $*$-algebra with algebraically admissible cone C and with unit e which is an Archimedean order unit. For $x \in \mathcal{A}$ define $|x| = \sqrt{\|x^*x\|}$. Then*

1. *$|\lambda x| = (\lambda\overline{\lambda})^{1/2}|x|$ for every $\lambda \in \mathbb{C}$ and $x \in \mathcal{A}$;*
2. *$|xy| \leq |x||y|$ for every x, y in $\mathcal{A}$;*
3. *$\|a\| \leq |a|$ for every $a \in \mathcal{A}_{sa}$.*

Proof. The first statement is trivial. For x, y in $\mathcal{A}$, by Lemma 4, we have $\|(xy)^*xy\| = \|y^*(x^*x)y\| \leq \|y^*y\|\|x^*x\|$. Hence $|xy| \leq |x||y|$. By Lemma 3, $(\|a\|e \pm a)^2 \in C$. Thus $-(\|a\|^2e + a^2) \leq 2\|a\|a \leq \|a\|^2e + a^2$. If $a^2 \leq \varepsilon e$ then $-(\|a\|^2 + \varepsilon)e \leq 2\|a\|a \leq (\|a\|^2 + \varepsilon)e$. Consequently, $\|2 \cdot \|a\| \cdot a\| \leq \|a\|^2 + \varepsilon$. Thus, $\|a\|^2 \leq \varepsilon$. Letting $\varepsilon \searrow \|a^2\|$ we obtain that $\|a\|^2 \leq \|a^2\|$. Therefore, $\|a\| \leq |a|$. □

Theorem 6. *Let $\mathcal{A}$ be a $*$-algebra with unit e and $C \subseteq \mathcal{A}_{sa}$ be a cone containing e. If $xCx^* \subseteq C$ for every $x \in \mathcal{A}$ and e is an Archimedean order unit then there is a unital $*$-representation $\pi : \mathcal{A} \to B(H)$ such that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$. Moreover*

1. $\|\pi(x)\| = \inf\{r > 0 : r^2 - x^*x \in C\}$.

2. $\ker \pi = \{x : x^*x \in C \cap (-C)\}$.

3. *If* $C \cap (-C) = \{0\}$ *then* $\ker \pi = \{0\}$,

$$\|\pi(a)\| = \inf\{r > 0 : r \pm a \in C\} \text{ for all } a \in \mathcal{A}_{sa}$$

and $\pi(C) = \pi(\mathcal{A}) \cap B(H)^+$

Proof. By Lemma 4 we have that $\|\cdot\| : \mathcal{A}_{sa} \to \mathbb{R}_+$ is a seminorm on $\mathbb{R}$-space $\mathcal{A}_{sa}$. Let us prove that $|x| = \sqrt{\|x^*x\|}$ for $x \in \mathcal{A}$ defines a pre-C^*-norm on $\mathcal{A}$.

First we will prove that $|x^*| = |x|$ for every $x \in A$. For this it suffices to show that $|x^*| \leq |x|$. In fact, if this is true then $|x| = |(x^*)^*| \leq |x^*|$. By definition $|x^*|^2 = \|xx^*\|$. Since xx^* is self-adjoint, $\|xx^*\| \leq |xx^*|$ by Lemma 5. Thus $|x^*|^2 \leq |xx^*| \leq |x||x^*|$. If $|x^*| = 0$ then $0 \leq |x|$ and the required inequality holds, otherwise we have $|x^*| \leq |x|$.

For every $x \in A$ by Lemma 5 we have $|x^*x| \leq |x||x^*| = |x|^2$ and $|x|^2 = \|x^*x\| \leq |x^*x|$. Thus $|x|^2 = |x^*x|$.

Applying the previous equality to a self-adjoint element a we obtain $|a|^2 = |a^*a| = |a^2|$. Thus $|a^2| = |a|^2$.

We will prove that $|x+y| \leq |x|+|y|$. For every $x \in A$ one has $\|x^2+x^{*2}\| \leq 2\|x^*x\|$. Indeed, since $x + x^*$ is self-adjoint we have $(x + x^*)^2 \geq 0$, i.e

$$x^2 + x^{*2} + xx^* + x^*x \geq 0.$$

From this it follows that $x^2+x^{*2} \geq -\{x, x^*\}$ where $\{x, x^*\} = xx^*+x^*x$. Since $i(x - x^*)$ is also self-adjoint we have $-(x - x^*)^2 \geq 0$. Thus $\{x, x^*\} \geq x^2 + x^{*2}$

and therefore $-\{x, x^*\} \leq x^2 + x^{*2} \leq \{x, x^*\}$. Hence

$$\begin{aligned} \|x^2 + x^{*2}\| &\leq \|\{x, x^*\}\| = \|xx^* + x^*x\| \\ &\leq \|xx^*\| + \|x^*x\| = |x|^2 + |x^*|^2 \\ &= 2|x|^2 = 2\|xx^*\|. \end{aligned}$$

We will prove the following.

$$\|x^* + x\| \leq 2\|x^*x\|^{1/2} = 2|x|. \tag{3}$$

Indeed, for self-adjoint a by Lemma 5, $\|a\|^2 \leq \|a^2\|$ hence

$$\begin{aligned} \|x + x^*\|^2 &\leq \|x^2 + x^{*2} + xx^* + x^*x\| \\ &\leq \|x^2 + x^{*2}\| + \|xx^* + x^*x\| \\ &\leq 2\|x^*x\| + \|x^*x\| + \|xx^*\| \\ &= 4\|x^*x\|. \end{aligned}$$

Thus $\|x^* + x\| \leq 2|x|$. We will prove that $\|x^*y + y^*x\| \leq 2|x||y|$. Indeed, the substitution x^*y instead of x in (3) implies $\|x^*y + y^*x\| \leq 2|x^*y| \leq 2|x||y|$.

The inequality $|x + y| \leq |x| + |y|$ follows from the following estimates:

$$\begin{aligned} |x + y|^2 &\leq \|x^*x\| + \|y^*y\| + \|x^*y + y^*x\| \\ &\leq |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2. \end{aligned}$$

Thus $|\cdot|$ is pre-C^*-norm.

If N denotes the null-space of $|\cdot|$ then the completion $\mathcal{B} = \overline{\mathcal{A}/N}$ with respect to the resulting norm is a C^*-algebra and the canonical epimorphism $\pi : \mathcal{A} \to \mathcal{A}/N$ is a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$. We can assume without loss of generality that $\mathcal{B}$ is a concrete C^*-algebra in $B(H)$ for some

Hilbert space H. Thus $\pi : \mathcal{A} \to B(H)$ can be regarded as a unital $*$-representation. Clearly,

$$\|\pi(x)\| = |x| \text{ for all } x \in \mathcal{A}.$$

This implies (1).

To show (2) take $x \in \ker \pi$ then $\|\pi(x)\| = 0$ and $re \pm x^*x \in C$ for all $r > 0$. Since e is an Archimedean unit we have $x^*x \in C \cap (-C)$. Conversely if $x^*x \in C \cap (-C)$ then $re \pm x^*x \in C$, for all $r > 0$, hence $\|\pi(x)\| = 0$ and (2) holds.

Let us prove that $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$. Let $x \in \mathcal{A}_{sa}$ and $\pi(x) \geq 0$. Then there exists a constant $\lambda > 0$ such that $\|\lambda I_H - \pi(x)\| \leq \lambda$, hence $|\lambda e - x| \leq \lambda$. Since $\|a\| \leq |a|$ for all self-adjoint $a \in \mathcal{A}$, see Lemma 5, we have $\|\lambda e - x\| \leq \lambda$. Thus given $\varepsilon > 0$ we have $(\lambda + \varepsilon)e \pm (\lambda e - x) \in C$. Hence $\varepsilon e + x \in C$. Since e is Archimedean $x \in C$.

Conversely, let $x \in C$. To show that $\pi(x) \geq 0$ it is sufficient to find $\lambda > 0$ such that $\|\lambda I_H - \pi(x)\| \leq \lambda$. Since $\|\lambda I_H - \pi(x)\| = |\lambda e - x|$ we will prove that $|\lambda e - x| \leq \lambda$ for some $\lambda > 0$. From the definition of norm $|\cdot|$ we have the following equivalences:

$$|\lambda e - x| \leq \lambda \quad \Leftrightarrow \quad (\lambda + \varepsilon)^2 e - (\lambda e - x)^2 \in C \text{ for all } \varepsilon > 0 \tag{4}$$

$$\Leftrightarrow \quad \varepsilon_1 e + x(2\lambda e - x) \geq 0, \text{ for all } \varepsilon_1 > 0. \tag{5}$$

By condition (iii) in the definition of an algebraically admissible cone we have that $xyx \in C$ and $yxy \in C$ for every $x, y \in C$. If $xy = yx$ then $xy(x + y) \in C$. Since e is an order unit we can choose $r > 0$ such that $re - x \in C$. Put $y = re - x$ to obtain $rx(re - x) \in C$. Hence (5) is satisfied with $\lambda = \frac{r}{2}$. Thus $\|\lambda e - \pi(x)\| \leq \lambda$ and $\pi(x) \geq 0$, which proves $\pi(C) = \pi(\mathcal{A}_{sa}) \cap B(H)^+$.

In particular, for $a = a^*$ we have

$$\|\pi(a)\| = \inf\{r > 0 : rI_H \pm \pi(a) \in \pi(C)\}. \tag{6}$$

We now in a position to prove claim (3). Suppose that $C \cap (-C) = 0$. Then $\ker \pi$ is a $*$-ideal and $\ker \pi \neq 0$ implies that there exists a self-adjoint $0 \neq a \in \ker \pi$, i.e. $|a| = 0$. Inequality $\|a\| \leq |a|$ implies $re \pm a \in C$ for all $r > 0$. Since e is Archimedean $\pm a \in C$, i.e. $a \in C \cap (-C)$ and, consequently, $a = 0$.

Since $\ker \pi = 0$ the inclusion $rI_H \pm \pi(a) \in \pi(C)$ is equivalent to $re \pm a \in C$, and by (6), $\|\pi(a)\| = \inf\{r > 0 : re \pm a \in C\}$. Moreover if $\pi(a) = \pi(a)^*$ then $a = a^*$. Thus we have $\pi(C) = \pi(\mathcal{A}) \cap B(H)^+$. □

Remark 7. *Note that J. Kelley and R. Vaught in 1953 proved that*

$$\sup\|\pi(x)\| = \inf\left\{t \in \mathbb{R}_+ | t^2 - x^*x \in \mathcal{A}_+\right\} \qquad (*)$$

where $\mathcal{A}_+ = \left\{\sum_{j=1}^n a_j^* a_j, n \in \mathbb{N}, a_j \in \mathcal{A}\right\}$, π *runs over all $*$-representations for Banach $*$-algebras $\mathcal{A}$ with isometric involution (see [5]). This is a particular case of claim (1) of Theorem 6 for a special choice of algebraically admissible cone* $C = \mathcal{A}_+$. *The proof of formula $(*)$ based on the Hahn-Banach theorem for any T^*-algebra (every $x \in \mathcal{A}_{sa}$ is bounded) presented in monograph [7].*

3 Operator realizations of matrix-ordered $*$-algebras.

The aim of this section is to give necessary and sufficient conditions on a sequences of cones $C_n \subseteq M_n(\mathcal{A})_{sa}$ for a unital $*$-algebra $\mathcal{A}$ such that C_n coincides with the cone $M_n(\mathcal{A}) \cap M_n(B(H))^+$ for some realization of $\mathcal{A}$ as a $*$-subalgebra of $B(H)$, where $M_n(B(H))^+$ denotes the set of positive operators acting on $H^n = H \oplus \ldots \oplus H$.

We say that a $*$-algebra $\mathcal{A}$ with unit e is *matrix ordered* if the following conditions hold:

(a) for each $n \geq 1$ we are given a cone C_n in $M_n(\mathcal{A})_{sa}$ and $e \in C_1$,

(b) $C_n \cap (-C_n) = \{0\}$ for all n,

(c) for all n and m and all $A \in M_{n\times m}(\mathcal{A})$, we have that $A^* C_n A \subseteq C_m$,

Let $\pi : \mathcal{A} \to B(H)$ be a $*$-representation. Define $\pi^{(n)} : M_n(\mathcal{A}) \to M_n(B(H))$ by $\pi^{(n)}((a_{ij})) = (\pi(a_{ij}))$.

Theorem 8. *If $\mathcal{A}$ is a matrix-ordered $*$-algebra with a unit e which is Archimedean matrix order unit then there exists a Hilbert space H and a faithful unital $*$-representation $\tau : \mathcal{A} \to B(H)$, such that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$ for all n. Conversely, every unital $*$-subalgebra $\mathcal{D}$ of $B(H)$ is matrix-ordered by cones $M_n(\mathcal{D})^+ = M_n(\mathcal{D}) \cap B(H)^+$ and the unit of this algebra is an Archimedean order unit.*

Proof. Consider an inductive system of $*$-algebras and unital injective $*$-homomorphisms $\phi_n : M_{2^n}(\mathcal{A}) \to M_{2^{n+1}}(\mathcal{A})$:

$$\phi_n(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \text{ for all } n \geq 0, a \in M_{2^n}(\mathcal{A}).$$

Let $\mathcal{B} = \varinjlim M_{2^n}(\mathcal{A})$ be the inductive limit of this system. By (c) in the definition of the matrix ordered algebra we have $\phi_n(C_{2^n}) \subseteq C_{2^{n+1}}$. We will identify $M_{2^n}(\mathcal{A})$ with a subalgebra of $\mathcal{B}$ via canonical inclusions. Let $C = \bigcup\limits_{n\geq 1} C_{2^n} \subseteq \mathcal{B}_{sa}$ and let e_∞ be the unit of $\mathcal{B}$.

Let us prove that C is an algebraically admissible cone. Clearly, C satisfies conditions (i) and (ii) of the definition of an algebraically admissible cone. To prove (iii) suppose that $x \in \mathcal{B}$ and $a \in C$, then for some n we have $a \in C_{2^n}$ and $x \in M_{2^n}(\mathcal{A})$. Therefore, by (c), $x^* a x \in C$. Thus (iii) is proved. Since e is an Archimedean matrix order unit we obviously have that e_∞ is also an Archimedean order unit. Thus the $*$-algebra $\mathcal{B}$ satisfies the assumptions of

Theorem 6 and therefore there is a faithful $*$-representation $\pi : \mathcal{B} \to B(H)$ such that $\pi(C) = \pi(\mathcal{B}) \cap B(H)^+$.

Let $\xi_n : M_{2^n}(\mathcal{A}) \to \mathcal{B}$ be the canonical injections ($n \geq 0$). Then $\tau = \pi \circ \xi_0 : \mathcal{A} \to B(H)$ is an injective $*$-homomorphism.

We claim that $\tau^{(2^n)}$ is unitary equivalent to $\pi \circ \xi_n$. By replacing π with π^α, where α is an infinite cardinal, we can assume that π^α is unitary equivalent to π. Since $\pi \circ \xi_n : M_{2^n}(\mathcal{A}) \to B(H)$ is a $*$-homomorphism there exist Hilbert space K_n, $*$-homomorphism $\rho_n : \mathcal{A} \to B(K_n)$ and unitary operator $U_n : K_n \otimes \mathbb{C}^{2^n} \to H$ such that

$$\pi \circ \xi_n = U_n(\rho_n \otimes id_{M_{2^n}})U_n^*.$$

For $a \in \mathcal{A}$, we have

$$\begin{aligned} \pi \circ \xi_0(a) &= \pi \circ \xi_n(a \otimes E_{2^n}) \\ &= U_n(\rho_n(a) \otimes E_{2^n})U_n^*, \end{aligned}$$

where E_{2^n} is the identity matrix in $M_{2^n}(\mathbb{C})$. Thus $\tau(a) = U_0\rho_0(a)U_0^* = U_n(\rho_n(a) \otimes E_{2^n})U_n^*$. Let $\sim$ stands for the unitary equivalence of representations. Since $\pi \circ \xi_n \sim \rho_n \otimes id_{M_{2^n}}$ and $\pi^\alpha \sim \pi$ we have that $\rho_n^\alpha \otimes id_{M_{2^n}} \sim \pi^\alpha \circ \xi_n \sim \rho_n \otimes id_{M_{2^n}}$. Hence $\rho_n^\alpha \sim \rho_n$. Thus $\rho_n \otimes E_{2^n} \sim \rho_n^{2^n\alpha} \sim \rho_n$. Consequently $\rho_0 \sim \rho_n$ and $\pi \circ \xi_n \sim \rho_0 \otimes id_{M_{2^n}} \sim \tau \otimes id_{M_{2^n}}$. Therefore $\tau^{(2^n)} = \tau \otimes id_{M_{2^n}}$ is unitary equivalent to $\pi \circ \xi_n$.

What is left to show is that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$. Note that $\pi \circ \xi_n(M_{2^n}(\mathcal{A})) \cap B(H)^+ = \pi(C_{2^n})$. Indeed, the inclusion $\pi \circ \xi(C_{2^n}) \subseteq M_{2^n}(\mathcal{A}) \cap B(H)^+$ is obvious. To show the converse take $x \in M_{2^n}(\mathcal{A})$ such that $\pi(x) \geq 0$. Then $x \in C \cap M_{2^n}(\mathcal{A})$. Using (c) one can easily show that $C \cap M_{2^n}(\mathcal{A}) = C_{2^n}$. Hence $\pi \circ \xi_n(M_{2^n}(\mathcal{A})) \cap B(H)^+ = \pi(C_{2^n})$. Since $\tau^{(2^n)}$ is unitary equivalent to $\pi \circ \xi_n$ we have that $\tau^{(2^n)}(C_{2^n}) = M_{2^n}(\tau(\mathcal{A})) \cap B(H^{2^n})^+$.

Let us now show that $\tau^{(n)}(C_n) = M_n(\tau(\mathcal{A}))^+$. For $X \in M_n(\mathcal{A})$ denote

$$\widetilde{X} = \begin{pmatrix} X & 0_{n\times(2^n-n)} \\ 0_{(2^n-n)\times n} & 0_{(2^n-n)\times(2^n-n)} \end{pmatrix} \in M_{2^n}(\mathcal{A}).$$

Then, clearly, $\tau^{(n)}(X) \geq 0$ if and only if $\tau^{(2^n)}(\widetilde{X}) \geq 0$. Thus $\tau^{(n)}(X) \geq 0$ is equivalent to $\widetilde{X} \in C_{2^n}$ which in turn is equivalent to $X \in C_n$ by (c). □

Theorem 2 is a direct corollary of the above theorem.

4 Operator Algebras completely boundedly isomorphic to C^*-algebras.

In the sequel all operator algebras will be assumed to be norm closed.

Operator algebras $\mathcal{A}$ and $\mathcal{B}$ are called completely boundedly isomorphic if there is a completely bounded isomorphism $\tau : \mathcal{A} \to \mathcal{B}$ with completely bounded inverse. The aim of this section is to give necessary and sufficient conditions for an operator algebra to be completely boundedly isomorphic to a C^*-algebra. To do this we introduce a concept of $*$-admissible cones which reflect the properties of the cones of positive elements of a C^*-algebra preserved under completely bounded isomorphism.

Definition 9. *Let $\mathcal{B}$ be an operator algebra with unit e. A sequence $C_n \subseteq M_n(\mathcal{B})$ of closed (in the norm $\|\cdot\|_n$) cones will be called $*$-admissible if it satisfies the following conditions:*

1. *$e \in C_1$;*
2. *(i) $M_n(\mathcal{B}) = (C_n - C_n) + i(C_n - C_n)$, for all $n \in \mathbb{N}$,*

   (ii) $C_n \cap (-C_n) = \{0\}$, for all $n \in \mathbb{N}$,

   (iii) $(C_n - C_n) \cap i(C_n - C_n) = \{0\}$, for all $n \in \mathbb{N}$;
3. *(i) for all c_1, $c_2 \in C_n$ and $c \in C_n$, we have that $(c_1 - c_2)c(c_1 - c_2) \in C_n$,*

   (ii) for all n, m and $B \in M_{n\times m}(\mathbb{C})$ we have that $B^* C_n B \subseteq C_m$;

4. there is $r > 0$ such that for every positive integer n and $c \in C_n - C_n$ we have $r\|c\|e_n + c \in C_n$,

5. there exists a constant $K > 0$ such that for all $n \in \mathbb{N}$ and $a, b \in C_n - C_n$ we have $\|a\|_n \leq K \cdot \|a + ib\|_n$.

Theorem 10. *If an operator algebra $\mathcal{B}$ has a $*$-admissible sequence of cones then there is a completely bounded isomorphism τ from $\mathcal{B}$ onto a C^*-algebra $\mathcal{A}$. If, in addition, one of the following conditions holds*

(1) there exists $r > 0$ such that for every $n \geq 1$ and $c, d \in C_n$ we have $\|c + d\| \geq r\|c\|$.

(2) there exists $\alpha > 0$ such that

$$\|(x - iy)(x + iy)\| \geq \alpha\|x - iy\|\|x + iy\|$$

for all $x, y \in C_n - C_n$

then the inverse $\tau^{-1} : \mathcal{A} \to \mathcal{B}$ is also completely bounded.

Conversely, if such an isomorphism τ exists then $\mathcal{B}$ possesses a $*$-admissible sequence of cones and conditions (1) *and* (2) *are satisfied.*

The proof will be divided into 4 lemmas.

Let $\{C_n\}_{n\geq 1}$ be a $*$-admissible sequence of cones of $\mathcal{B}$. Let $\mathcal{B}_{2^n} = M_{2^n}(\mathcal{B})$, $\phi_n : \mathcal{B}_{2^n} \to \mathcal{B}_{2^{n+1}}$ be unital homomorphisms given by $\phi_n(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, $x \in \mathcal{B}_{2^n}$. Denote by $\mathcal{B}_\infty = \underrightarrow{\lim}\mathcal{B}_{2^n}$ the inductive limit of the system $(\mathcal{B}_{2^n}, \phi_n)$. As all inclusions ϕ_n are unital $\mathcal{B}_\infty$ has a unit, denoted by e_∞. Since $\mathcal{B}_\infty$ can be considered as a subalgebra of the corresponding inductive limit of $M_{2^n}(B(\mathcal{H}))$ we can define the closure of $\mathcal{B}_\infty$ in this C^*-algebra denoted by $\overline{\mathcal{B}}_\infty$.

Now we will define an involution on $\mathcal{B}_\infty$. Let $\xi_n : M_{2^n}(\mathcal{B}) \to \mathcal{B}_\infty$ be the canonical morphisms. By $(3ii)$, $\phi_n(C_{2^n}) \subseteq C_{2^{n+1}}$. Hence $C = \bigcup_n \xi_n(C_{2^n})$ is a well defined cone in $\mathcal{B}_\infty$. Denote by $\overline{C}$ its completion. By $(2i)$ and $(2iii)$, for every $x \in \mathcal{B}_{2^n}$, we have $x = x_1 + ix_2$ with unique $x_1, x_2 \in C_{2^n} - C_{2^n}$. By $(3ii)$ we have $\begin{pmatrix} x_i & 0 \\ 0 & x_i \end{pmatrix} \in C_{2^{n+1}} - C_{2^{n+1}}$, $i = 1, 2$. Thus for every $x \in \mathcal{B}_\infty$ we have unique decomposition $x = x_1 + ix_2$, $x_1 \in C - C$, $x_2 \in C - C$. Hence the mapping $x \mapsto x^\sharp = x_1 - ix_2$ is a well defined involution on $\mathcal{B}_\infty$. In particular, we have an involution on $\mathcal{B}$ which depends only on the cone C_1.

Lemma 11. *Involution on $\mathcal{B}_\infty$ is defined by the involution on $\mathcal{B}$, i.e. for all $A = (a_{ij})_{i,j} \in M_{2^n}(\mathcal{B})$*

$$A^\sharp = (a_{ji}^\sharp)_{i,j}.$$

Proof. Assignment $A^\circ = (a_{ji}^\sharp)_{i,j}$, clearly, defines an involution on $M_{2^n}(\mathcal{B})$. We need to prove that $A^\sharp = A^\circ$.

Let $A = (a_{ij})_{i,j} \in M_{2^n}(\mathcal{B})$ be self-adjoint $A^\circ = A$. Then $A = \sum_i a_{ii} \otimes E_{ii} + \sum_{i<j} (a_{ij} \otimes E_{ij} + a_{ij}^\sharp \otimes E_{ji})$ and $a_{ii}^\sharp = a_{ii}$, for all i. By $(3ii)$ we have $\sum_i a_{ii} \otimes E_{ii} \in C_{2^n} - C_{2^n}$. Since $a_{ij} = a'_{ij} + ia''_{ij}$ for some $a'_{ij}, a''_{ij} \in C_{2^n} - C_{2^n}$ we have

$$\begin{aligned}
a_{ij} \otimes E_{ij} + a_{ij}^\sharp \otimes E_{ji} &= (a'_{ij} + ia''_{ij}) \otimes E_{ij} + (a'_{ij} - ia''_{ij}) \otimes E_{ji} \\
&= (a'_{ij} \otimes E_{ij} + a'_{ij} \otimes E_{ji}) + (ia''_{ij} \otimes E_{ij} - ia''_{ij} \otimes E_{ji}) \\
&= (E_{ii} + E_{ji})(a'_{ij} \otimes E_{ii} + a'_{ij} \otimes E_{jj})(E_{ii} + E_{ij}) \\
&- (a'_{ij} \otimes E_{ii} + a'_{ij} \otimes E_{jj}) \\
&+ (E_{ii} - iE_{ji})(a''_{ij} \otimes E_{ii} + a''_{ij} \otimes E_{jj})(E_{ii} + iE_{ij}) \\
&- (a''_{ij} \otimes E_{ii} + a''_{ij} \otimes E_{jj}) \in C_{2^n} - C_{2^n}.
\end{aligned}$$

Thus $A \in C_{2^n} - C_{2^n}$ and $A^\sharp = A$. Since for every $x \in M_{2^n}(\mathcal{B})$ there exist unique $x_1 = x_1^\circ$ and $x_2 = x_2^\circ$ in $M_{2^n}(\mathcal{B})$, such that $x = x_1 + ix_2$, and unique

$x_1' = x_1'^\sharp$ and $x_2' = x_2'^\sharp$, such that $x = x_1' + ix_2'$, we have that $x_1 = x_1^\sharp = x_1'$, $x_2 = x_2^\sharp = x_2'$ and involutions $\sharp$ and $\circ$ coincide. $\square$

Lemma 12. *Involution $x \to x^\sharp$ is continuous on $\mathcal{B}_\infty$ and extends to an involution on $\overline{\mathcal{B}}_\infty$. With respect to this involution $\overline{C} \subseteq (\overline{\mathcal{B}}_\infty)_{sa}$ and $x^\sharp \overline{C} x \subseteq \overline{C}$ for every $x \in \overline{\mathcal{B}}_\infty$.*

Proof. Consider a convergent net $\{x_i\} \subseteq \mathcal{B}_\infty$ with the limit $x \in \mathcal{B}_\infty$. Decompose $x_i = x_i' + ix_i''$ with $x_i', x_i'' \in C - C$. By (5), the nets $\{x_i'\}$ and $\{x_i''\}$ are also convergent. Thus $x = a + ib$, where $a = \lim x_i' \in \overline{C - C}$, $b = \lim x_i'' \in \overline{C - C}$ and $\lim x_i^\sharp = a - ib$. Therefore the involution defined on $\mathcal{B}_\infty$ can be extended by continuity to $\overline{\mathcal{B}}_\infty$ by setting $x^\sharp = a - ib$.

Under this involution $\overline{C} \subseteq (\overline{\mathcal{B}}_\infty)_{sa} = \{x \in \overline{\mathcal{B}}_\infty : x = x^\sharp\}$.

Let us show that $x^\sharp c x \in \overline{C}$ for every $x \in \overline{\mathcal{B}}_\infty$ and $c \in \overline{C}$. Take firstly $c \in C_{2^n}$ and $x \in \mathcal{B}_{2^n}$. Then $x = x_1 + ix_2$ for some x_1, $x_2 \in C_{2^n} - C_{2^n}$ and

$$(x_1 + ix_2)^\sharp c(x_1 + ix_2) = (x_1 - ix_2)c(x_1 + ix_2)$$
$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -x_1 & -ix_2 \\ ix_2 & x_1 \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} -x_1 & -ix_2 \\ ix_2 & x_1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

By (3i), Lemma 11 and (3ii) $x^\sharp c x \in C_{2^n}$.

Let now $c \in \overline{C}$ and $x \in \overline{\mathcal{B}}_\infty$. Suppose that $c_i \to c$ and $x_i \to x$, where $c_i \in C$, $x_i \in \mathcal{B}_\infty$. We can assume that c_i, $x_i \in B_{2^{n_i}}$. Then $x_i^\sharp c_i x_i \in C_{2^{n_i}}$ for all i and since it is convergent we have $x^\sharp c x \in \overline{C}$. $\square$

Lemma 13. *The unit of $\overline{\mathcal{B}}_\infty$ is an Archimedean order unit and $(\overline{\mathcal{B}}_\infty)_{sa} = \overline{C} - \overline{C}$.*

Proof. Firstly let us show that e_∞ is an order unit. Clearly, $(\overline{\mathcal{B}}_\infty)_{sa} = \overline{C - C}$. For every $a \in \overline{C - C}$, there is a net $a_i \in C_{2^{n_i}} - C_{2^{n_i}}$ convergent to a. Since $\sup_i \|a_i\| < \infty$ there exists $r_1 > 0$ such that $r_1 e_{n_i} - a_i \in C_{2^{n_i}}$, i.e. $r_1 e_\infty - a_i \in C$. Passing to the limit we get $r_1 e_\infty - a \in \overline{C}$. Replacing a by $-a$ we can find $r_2 > 0$ such that $r_2 e_\infty + a \in \overline{C}$. If $r = \max(r_1, r_2)$ then $re_\infty \pm a \in \overline{C}$. This proves that e_∞ is an order unit and that for all $a \in \overline{C - C}$ we have $a = re_\infty - c$ for some $c \in \overline{C}$. Thus $\overline{C - C} \in \overline{C} - \overline{C}$. The converse inclusion, clearly, holds. Thus $\overline{C - C} = \overline{C} - \overline{C}$.

If $x \in (\overline{\mathcal{B}}_\infty)_{sa}$ such that for every $r > 0$ we have $r + x \in \overline{C}$ then $x \in \overline{C}$ since $\overline{C}$ is closed. Hence e_∞ is an Archimedean order unit. $\square$

Lemma 14. *$\mathcal{B}_\infty \cap \overline{C} = C$.*

Proof. Denote by $\mathcal{D} = \varinjlim M_{2^n}(B(\mathcal{H}))$ the C^*-algebra inductive limit corresponding to the inductive system ϕ_n and denote $\phi_{n,m} = \phi_{m-1} \circ \ldots \circ \phi_n : M_{2^n}(B(\mathcal{H})) \to M_{2^m}(B(\mathcal{H}))$. For $n < m$ we identify $M_{2^{m-n}}(M_{2^n}(B(\mathcal{H})))$ with $M_{2^m}(B(\mathcal{H}))$ by omitting superfluous parentheses in a block matrix $B = [B_{ij}]_{ij}$ with $B_{ij} \in M_{2^n}(B(\mathcal{H}))$.

Denote by $P_{n,m}$ the operator $diag(I, 0, \ldots, 0) \in M_{2^{m-n}}(M_{2^n}(B(\mathcal{H})))$ and set $V_{n,m} = \sum_{k=1}^{2^{m-n}} E_{k,k-1}$. Here I is the identity matrix in $M_{2^n}(B(\mathcal{H}))$ and $E_{k,k-1}$ is $2^n \times 2^n$ block matrix with identity operator at $(k, k-1)$-entry and all other entries being zero. Define an operator $\psi_{n,m}([B_{ij}]) = diag(B_{11}, \ldots, B_{11})$. It is easy to see that

$$\psi_{n,m}([B_{ij}]) = \sum_{k=0}^{2^{m-n}-1} (V_{n,m}^k P_{n,m}) B (V_{n,m}^k P_{n,m})^*.$$

Hence by $(3ii)$

$$\psi_{n,m}(C_{2^m}) \subseteq \phi(C_{2^n}) \subseteq C_{2^m}. \tag{7}$$

Clearly, $\psi_{n,m}$ is a linear contraction and

$$\psi_{n,m+k} \circ \phi_{m,m+k} = \phi_{m,m+k} \circ \psi_{n,m}$$

Hence there is a well defined contraction $\psi_n = \lim_m \psi_{n,m} : \mathcal{D} \to \mathcal{D}$ such that

$$\psi_n|_{M_{2^n}(B(H))} = id_{M_{2^n}(B(\mathcal{H}))},$$

where $M_{2^n}(B(\mathcal{H}))$ is considered as a subalgebra in $\mathcal{D}$. Clearly, $\psi_n(\overline{\mathcal{B}}_\infty) \subseteq \overline{\mathcal{B}}_\infty$ and $\psi_n|_{\mathcal{B}_{2^n}} = id$. Consider C and C_{2^n} as subalgebras in $\mathcal{B}_\infty$. By (7) we have $\psi_n : C \to C_{2^n}$.

To prove that $\mathcal{B}_\infty \cap \overline{C} = C$ take $c \in \mathcal{B}_\infty \cap \overline{C}$. Then there is a net c_j in C such that $\|c_j - c\| \to 0$. Since $c \in \mathcal{B}_\infty$, $c \in \mathcal{B}_{2^n}$ for some n, and consequently $\psi_n(c) = c$. Thus

$$\|\psi_n(c_j) - c\| = \|\psi_n(c_j - c)\| \leq \|c_j - c\|.$$

Hence $\psi_n(c_j) \to c$. But $\psi_n(c_j) \in C_{2^n}$ and the latter is closed. Thus $c \in C$. The converse inclusion is obvious. $\square$

Remark 15. *Note that for every $x \in \mathcal{D}$*

$$\lim_n \psi_n(x) = x. \tag{8}$$

Indeed, for every $\varepsilon > 0$ there is $x \in M_{2^n}(B(H))$ such that $\|x - x_n\| < \varepsilon$. Since ψ_n is a contraction and $\psi_n(x_n) = x_n$ we have

$$\begin{aligned} \|\psi_n(x) - x\| &\leq \|\psi_n(x) - x_n\| + \|x_n - x\| \\ &= \|\psi_n(x - x_n)\| + \|x_n - x\| \leq 2\varepsilon. \end{aligned}$$

Since $x_n \in M_{2^n}(B(\mathcal{H}))$ also belong to $M_{2^m}(B(\mathcal{H}))$ for all $m \geq n$, we have that $\|\psi_m(x) - x\| \leq 2\varepsilon$. Thus $\lim_n \psi_n(x) = x$.

Proof of Theorem 10. By Lemma 12 and 13 the cone $\overline{C}$ and the unit e_∞ satisfies all assumptions of Theorem 6. Thus there is a homomorphism $\tau : \overline{\mathcal{B}}_\infty \to B(\widetilde{H})$ such that $\tau(a^\sharp) = \tau(a)^*$ for all $a \in \overline{\mathcal{B}}_\infty$. Since the image of τ is a $*$-subalgebra of $B(\widetilde{H})$ we have that τ is bounded by [4, (23.11), p. 81]. The arguments at the end of the proof of Theorem 8 show that the restriction of τ to $\mathcal{B}_{2^n}$ is unitary equivalent to the 2^n-amplification of $\tau|_{\mathcal{B}}$. Thus $\tau|_{\mathcal{B}}$ is completely bounded.

Let us prove that $\ker(\tau) = \{0\}$. By item 3 in Theorem 8 it is sufficient to show that $\overline{C} \cap (-\overline{C}) = 0$. If $c, d \in \overline{C}$ such that $c + d = 0$ then $c = d = 0$. Indeed, for every $n \geq 1$, $\psi_n(c) + \psi_n(d) = 0$. By Lemma 14, we have

$$\psi_n(\overline{C}) \subseteq \overline{C} \cap \mathcal{B}_{2^n} = C_{2^n}.$$

Therefore $\psi_n(c)$, $\psi_n(d) \in C_{2^n}$. Hence $\psi_n(c) = -\psi_n(d) \in C_{2^n} \cap (-C_{2^n})$ and, consequently, $\psi_n(c) = \psi_n(d) = 0$. Since $\|\psi_n(c) - c\| \to 0$ and $\|\psi_n(d) - d\| \to 0$ by Remark 15, we have that $c = d = 0$. If $x \in \overline{C} \cap (-\overline{C})$ then $x + (-x) = 0$, $x, -x \in \overline{C}$ and $x = 0$. Thus τ is injective.

We will show that the image of τ is closed if one of the conditions (1) or (2) of the statement holds.

Assume firstly that operator algebra $\mathcal{B}$ satisfies the first condition. Since $\tau(\overline{\mathcal{B}}_\infty) = \tau(\overline{C}) - \tau(\overline{C}) + i(\tau(\overline{C}) - \tau(\overline{C}))$ and $\tau(\overline{C})$ is exactly the set of positive operators in the image of τ, it is suffices to prove that $\tau(\overline{C})$ is closed. By item 3 in Theorem 6, for self-adjoint (under involution $\sharp$) $x \in \overline{\mathcal{B}}_\infty$ we have

$$\|\tau(x)\|_{B(\widetilde{H})} = \inf\{r > 0 : re_\infty \pm x \in \overline{C}\}.$$

If $\tau(c_\alpha) \in \tau(C)$ is a Cauchy net in $B(\widetilde{H})$ then for every $\varepsilon > 0$ there is γ such that $\varepsilon \pm (c_\alpha - c_\beta) \in \overline{C}$ when $\alpha \geq \gamma$ and $\beta \geq \gamma$. Since $\overline{C} \cap \mathcal{B}_\infty = C$, $\varepsilon \pm (c_\alpha - c_\beta) \in C$. Denote $c_{\alpha\beta} = \varepsilon + (c_\alpha - c_\beta)$ and $d_{\alpha\beta} = \varepsilon - (c_\alpha - c_\beta)$. The set of pairs (α, β) is directed if $(\alpha, \beta) \geq (\alpha_1, \beta_1)$ iff $\alpha \geq \alpha_1$ and $\beta \geq \beta_1$. Since $c_{\alpha\beta} + d_{\alpha\beta} = 2\varepsilon$ this net converges to zero in the norm of $\overline{\mathcal{B}}_\infty$. Thus by assumption 4 in the definition of $*$-admissible sequence of cones, $\|c_{\alpha\beta}\|_{\overline{\mathcal{B}}_\infty} \to 0$. This implies that c_α is a Cauchy net in $\overline{\mathcal{B}}_\infty$. Let $c = \lim c_\alpha$. Clearly, $c \in \overline{C}$. Since τ is continuous $\|\tau(c_\alpha) - \tau(c)\|_{\overline{\mathcal{B}}_\infty} \to 0$. Hence the closure $\overline{\tau(C)}$ is contained in $\tau(\overline{C})$. By continuity of τ we have $\tau(\overline{C}) \subseteq \overline{\tau(C)}$. Hence $\tau(\overline{C}) = \overline{\tau(C)}$, $\tau(\overline{C})$ is closed.

Let now $\mathcal{B}$ satisfy condition (2) of the theorem. Then for every $x \in \overline{\mathcal{B}}_\infty$ we have $\|x^\sharp x\| \geq \alpha \|x\| \|x^\sharp\|$. By [4, theorem 34.3] $\overline{\mathcal{B}}_\infty$ admits an equivalent

C^*-norm $|\cdot|$. Since τ is a faithful $*$-representation of the C^*-algebra $(\overline{\mathcal{B}}_\infty, |\cdot|)$ it is isometric. Therefore $\tau(\overline{\mathcal{B}}_\infty)$ is closed.

Let us show that $(\tau|_{\mathcal{B}})^{-1} : \tau(\mathcal{B}) \to \mathcal{B}$ is completely bounded. The image $\mathcal{A} = \tau(\overline{\mathcal{B}}_\infty)$ is a C^*-algebra in $B(\widetilde{H})$ isomorphic to $\overline{\mathcal{B}}_\infty$. By Johnson's theorem two Banach algebra norms on a semi-simple algebra are equivalent, hence, $\tau^{-1} : \mathcal{A} \to \overline{\mathcal{B}}_\infty$ is a bounded homomorphism. Let $R = \|\tau^{-1}\|$. Let us show that $\|(\tau|_{\mathcal{B}})^{-1}\|_{cb} = R$. Since

$$\tau|_{\mathcal{B}_{2^n}} = U_n(\tau|_{\mathcal{B}} \otimes id_{M_{2^n}})U_n^*,$$

for some unitary $U_n : K \otimes \mathbb{C}^{2^n} \to \widetilde{H}$ we have for any $B = [b_{ij}] \in M_{2^n}(\mathcal{B})$

$$\begin{aligned} \|\sum b_{ij} \otimes E_{ij}\| &\leq R\|\tau(\sum b_{ij} \otimes E_{ij})\| \\ &= R\|U_n(\sum \tau(b_{ij}) \otimes E_{ij})U_n^*\| \\ &= R\|\sum \tau(b_{ij}) \otimes E_{ij}\|. \end{aligned}$$

This is equivalent to

$$\|\sum \tau^{-1}(b_{ij}) \otimes E_{ij}\| \leq R\|\sum b_{ij} \otimes E_{ij}\|,$$

hence $\|(\tau^{-1})^{(2^n)}(B)\| \leq R\|B\|$. This proves that $\|(\tau|_{\mathcal{B}})^{-1}\|_{cb} = R$.

The converse statement evidently holds with $*$-admissible sequence of cones given by $(\tau^{(n)})^{-1}(M_n(\mathcal{A})^+)$.$\square$

Conditions (1) and (2) were used to prove that the image of isomorphism τ is closed. The natural question one can ask is wether there exists a Banach operator algebra isomorphic to a non-closed self-adjoint operator algebra via bounded isomorphism. The following example gives the affirmative answer to this question.

Example 16. *Consider the algebra $\mathcal{B} = C^1([0,1])$ as an operator algebra in C^*-algebra $\bigoplus_{q\in\mathbb{Q}\cap[0,1]} M_2(C([0,1]))$ via inclusion*

$$f(\cdot) \mapsto \oplus_{q\in\mathbb{Q}\cap[0,1]} \begin{pmatrix} f(q) & f'(q) \\ 0 & f(q) \end{pmatrix}.$$

The induced norm

$$\|f\| = \sup_{q\in\mathbb{Q}\cap[0,1]} \left[\frac{1}{2}(2|f(q)|^2 + |f'(q)|^2 + |f'(q)|\sqrt{4|f(q)|^2 + |f'(q)|^2})\right]^{\frac{1}{2}}$$

satisfies the inequality $\|f\| \geq \frac{1}{\sqrt{2}}\max\{\|f\|_\infty, \|f'\|_\infty\} \geq \frac{1}{2\sqrt{2}}\|f\|_1$ *where* $\|f\|_1 = \|f\|_\infty + \|f'\|_\infty$ *is the standard Banach norm on* $C^1([0,1])$*. Thus* $\mathcal{B}$ *is a closed operator algebra with isometric involution* $f^\sharp(x) = \overline{f(x)}$*,* $x \in [0,1]$*. The identity map* $C^1([0,1]) \to C([0,1])$*,* $f \mapsto f$ *is a* $*$*-isomorphism of* $\mathcal{B}$ *with non-closed self-adjoint subalgebra of* $C([0,1])$*.*

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References

[1] D. Blecher, Z-J. Ruan, A. M. Sinclair, *A characterization of operator algebras.* Journal of Functional Analysis **89** (1990), 288-301

[2] D. Blecher, *A completely bounded characterization of operator algebras.* Math. Annalen **303** (1995), 227-239.

[3] M.D. Choi, E.G. Effros, Injectivity and operator spaces. *J. Functional Analysis* **24** (1977), no. 2, 156–209.

[4] R.S. Doran, V.A. Belfi, Characterizations of C^*-algebras. The Gelfand-Naĭmark theorems. Monographs and Textbooks in Pure and Applied Mathematics, 101. Marcel Dekker, Inc., New York, 1986.

[5] J. L. Kelley, R. L. Vaught, *The positive cone in Banach algebras.* Trans. Amer. Math. Soc. **74**, (1953). 44–55.

[6] C. Le Merdy, *Self adjointness criteria for operator algebras*, Arch. Math. 74 (2000), p. 212- 220.

[7] T. Palmer, *Banach algebras and the general theory of $*$-algebras.* Vol. 2. $*$-algebras. Encyclopedia of Mathematics and its Applications, 79. Cambridge University Press, Cambridge, 2001.

[8] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78** Cambridge University Press, Cambridge, 2002.

[9] R. Powers, *Selfadjoint algebras of unbounded operators II*, Trans. Amer. Math. Soc. **187** (1974), 261–293.

[10] Z.J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76** (1988), 218-230