Lecture 1: Paradoxical decompositions of groups and their actions.

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0.1 Paradoxical actions of groups

Definition 0.1.1. Let G be a group acting on a set X. A subset $E \subset X$ is **paradoxical** if there exist a pairwise disjoint subsets $A_1, \dots, A_n$ and $B_1, \dots, B_m$ in E and there exist $g_1 \dots g_n$, $h_1, \dots, h_m$ in G such that

$$E = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j.$$

An action of a group G on a set X is paradoxical if X is paradoxical. The group itself is called paradoxical if the action of the group on itself by left multiplication is paradoxical. Later on we will see that the group is paradoxical if and only if it is non-amenable.

The very first example and the only know explicit construction of paradoxical decomposition of a group is the free group on two generators $\mathbb{F}_2$. Let $a, b \in \mathbb{F}_2$ be the free generators of $\mathbb{F}_2$. Denote by $\omega(x)$ the set of all reduced words in $\mathbb{F}_2$ that start with x. Thus the group can be decomposed into pairwise disjoint sets as follows

$$\mathbb{F}_2 = \{e\} \cup \omega(a) \cup \omega(a^{-1}) \cup \omega(b) \cup \omega(b^{-1}).$$

Since $\mathbb{F}_2 \backslash \omega(x) = x\omega(x^{-1})$ for all x in $\{a, a^{-1}, b, b^{-1}\}$ we have a paradoxical decomposition:

$$\mathbb{F}_2 = \omega(a) \cup a\omega(a^{-1}) = \omega(b) \cup b\omega(b^{-1}).$$

In fact with few additional assumptions one can push a paradoxical decomposition of a group to a set on which it acts. This is exactly the place with the Axiom of Choice is needed.

Theorem 0.1.2. A group a G is paradoxical if and only if there exists a free action on a set X which is paradoxical.

Proof. Let $G = \bigcup\limits_{i=1}^{n} g_i A_i = \bigcup\limits_{j=1}^{m} h_j B_j$ be a paradoxical decomposition of G. By Axiom of Choice we can select a subset M of X which contains exactly one element from each orbit of G. We have that $\bigcup\limits_{g \in G} gM$ is disjoint partition of M. Indeed, if $gx = hy$ for some $g, h \in G$ and $x, y \in X$, then $x = y$ by the choice of M. Since the action is free, we have $g = h$. Define now

$$\widehat{A}_j = \bigcup_{g \in A_i} gM \text{ and } \widehat{B}_j = \bigcup_{g \in B_j} gM.$$

Obviously these sets remain disjoint, on the other hand we have

$$X = \bigcup_{i=1}^{n} g_i \widehat{A}_i = \bigcup_{j=1}^{m} h_j \widehat{B}_j.$$

To show the converse consider an obit $\mathcal{O}$ of a point in X. Then the action of G on $\mathcal{O}$ is exactly an action on the cosets space by some subgroup $H < G$. Thus paradoxical decomposition of $\mathcal{O}$ implies paradoxical decomposition of G. □

This implies, in particular, that all free actions of $\mathbb{F}_2$ admit paradoxical decomposition. Note that if the action of G on X is transitive then it is not necessary to use the axiom of choice.

0.2 Hausdorff paradox

Since all free actions of the free group on two generators give rise to paradoxical actions, the idea of paradoxical decomposition in Euclidean space is build up on the existence of free subgroups in the group of rotations in tree-dimensional Euclidean space $SO(3)$.

It is well known fact that $SO(3)$ contains a lot of copies of the free group on two generators. There many non-constructive proofs of this. For example, one can invoke Tit's alternative to show this. In fact, a stronger statement is true. If we consider $SO(3) \times SO(3)$ with product topology, then the set of all pairs that generate $\mathbb{F}_2$ is dense in $SO(3) \times SO(3)$.

The first explicit construction of the free subgroup in $SO(3)$ goes back to Hausdorff, [51]. Here we give a simplified construction of Świerczkowski, [95].

Theorem 0.2.1. There are two rotations in $SO(3)$ which generate the free group on two generators.

Proof. We define this rotation explicitly, by matrices

$$T^{\pm 1} = \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}; \qquad R^{\pm 1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} \\ 0 & \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}.$$

These are rotations by angle $\arccos(\frac{1}{3})$ around z-axis and x-axis. Let ω be a non-trivial reduced word in the free group on two generators. We will show that $q(\omega)$ is a non-trivial rotation, where q is a homomorphism of $\mathbb{F}_2$ that sends generators to T and R. To simplify notations we will denote $q(\omega)$ again by ω. Conjugating ω by T we may assume that ω ends by $T^{\pm 1}$ on the right. To prove the theorem it would suffice to show that

$$\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix},$$

where a, b, c are integers, b is not divisible by 3 and k is the length of ω.

In order to show this we will proceed by induction on the length of ω. If $|\omega| = 1$, then $\omega = T^{\pm 1}$ and

$$\omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ \pm 2\sqrt{2} \\ 0 \end{pmatrix}.$$

Now let ω be equal to $T^{\pm 1}\omega'$ or $R^{\pm 1}\omega'$, where ω' satisfies

$$\omega' \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}.$$

Applying matrices we see that either $a = a' \mp 4b'$, $b = b' \pm 2a'$, $c = 3c'$ or $a = 3a'$, $b = b' \mp 2c'$, $c = c' \pm 4b'$ and thus a, b and c are integers. It is left to show that b is not divisible by 3. Now ω can be written as $T^{\pm 1}R^{\pm 1}v$, $R^{\pm 1}T^{\pm 1}v$, $T^{\pm 1}T^{\pm 1}v$ or $R^{\pm 1}R^{\pm 1}v$, for some possibly empty word v. Thus we have 4 cases to consider:

1. $\omega = T^{\pm 1}R^{\pm 1}v$. In this case $b = b' \mp 2a'$ and a' is divisible by 3. By assumptions, b' is not divisible by 3 and thus b is not divisible by 3 as well.

2. $\omega = R^{\pm 1}T^{\pm 1}v$. In this case $b = b' \mp 2c'$ and c' is divisible by 3. But b' is not divisible by 3, thus b is not divisible by 3 as well.

3. $\omega = T^{\pm 1}T^{\pm 1}v$. By assumptions $v(1,0,0) = \frac{1}{3^{k-2}}(a'', \sqrt{2}b'', c'')$. It follows that $b = b' \pm 2a' = b \pm 2(a'' \mp 4b'') = b' + b'' \pm 2a'' - 9b'' = 2b' - 9b''$. Therefore, b is not divisible by 3.

4. $\omega = R^{\pm 1}R^{\pm 1}v$. This case is similar to the previous one.

□

Now the proof of Hausdorff paradox is almost straightforward.

Theorem 0.2.2 (Hausdorff paradox)**.** There exists a countable subset in a sphere S^2 such that its complement in S^2 is $SO(3)$-paradoxical.

Proof. We can not apply Theorem 0.1.2 right away, since each non-trivial rotation fixes two points on the sphere. For a fixed free subgroup of $SO(3)$, let M be the set of all points in S^2, which are fixed by some elements of this group. Obviously, M is countable and $S^2 \backslash M$ is invariant under the action of our free group. Thus by Theorem 0.1.2 the statement follows. □

In the next section we will show that the sphere S^2 itself has a $SO(3)$-paradoxical decomposition.

0.3 Banach-Tarski paradox

The classical Banach-Tarski paradox amounts to a decomposition of a unit ball into finitely many pieces, rearranging this pieces into unit balls. Since the group of rotations preserves the origin, it would not be sufficient to obtain Banach-Tarski paradox. For this purpose we add translations into the picture.

Definition 0.3.1. Let G be a group acting on a set X. Two subsets $A, B \subset X$ are equidecomposable if there exist a pairwise disjoint subsets $A_1, \ldots, A_n \subset A$ and a pairwise disjoint subsets $B_1, \ldots, B_n \subset B$ and $g_1 \ldots g_n$ in G such that

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{j=1}^{n} B_j$$

and $g_i(A_i) = B_i$ for all $1 \leq i \leq n$.

It is straightforward to check that equidecomposibility is an equivalence relation. We denote it by $A \sim_G B$ or by $A \sim B$ when the ambient group is clear.

Proposition 0.3.2. Let G be a group that acts on a set X. If F is paradoxical and F is equidecomposable to E, then E is paradoxical.

Proof. Let $F = \bigcup_{i=1}^{n} g_i A_i = \bigcup_{j=1}^{m} h_j B_j$ be a paradoxical decomposition of E. Thus, by transitivity, E is equidecomposable to both $\bigcup_{i=1}^{n} A_i$ and $\bigcup_{j=1}^{m} B_j$. This implies that E is paradoxical. □

It turns out that for countable sets of the sphere the situation is even more paradoxical then we might expect.

Proposition 0.3.3. Let D be a countable subset of S^2. Then S^2 and $S^2 \backslash D$ are $SO(3)$-equidecomposable.

Proof. Since D is countable we can find a line L which does not intersect D. Let Λ be a collection of all $\alpha \in [0, 2\pi]$ for which there exist a natural number n and a point x in D such that both x and $\rho(x)$ are in D, where ρ is the rotation around L by an angle $n\alpha$. Obviously, Λ is countable and we can find $\theta \in [0, 2\pi) \backslash \Lambda$. Let ρ be a rotation by θ around L, then $\rho^n(D) \cap \rho^k(D) = \emptyset$ for all natural numbers $k \neq n$. Thus for $D' = \bigcup_{n \geq 0} \rho^n(D)$ we have

$$S^2 = D' \cup (S^2 \backslash D') \sim \rho(D') \cup (S^2 \backslash D') = S^2 \backslash D,$$

which proves the claim. □

The direct corollary of this proposition and Hausdorff paradox is the following.

Corollary 0.3.4. S^2 is $SO(3)$-paradoxical.

Now we are ready to prove Banach-Tarski paradox, [4].

Theorem 0.3.5 (Banach-Tarski Paradox)**.** Every ball in $\mathbb{R}^3$ can be paradoxical decomposed by rotations and translations.

Proof. Let B_1 be a unit ball around the origin. By Corollary 0.3.4, we can find $A_1, \dots, A_n$, $B_1, \dots, B_n \subset S^2$ and $g_1, \dots, g_n, h_1, \dots, h_m \in SO(3)$ which satisfy Definition 0.1.1. Define

$$\widehat{A}_i = \{tx : t \in (0,1], x \in A_i\} \text{ and } \widehat{B}_j = \{tx : t \in (0,1], x \in B_j\}.$$

Then we have $\widehat{A}_1, \dots, \widehat{A}_n$, $\widehat{B}_1, \dots, \widehat{B}_n \subset B_1 \backslash \{0\}$ are pairwise disjoint and $B_1 = \bigcup_{1 \le j \le n} g_i \widehat{A}_i = \bigcup_{1 \le j \le m} h_j \widehat{A}_j$. Thus $B_1 \backslash \{0\}$ is paradoxical. We will show that it is equidecomposable to B_1.

Let $x = (0, 0, \frac{1}{2})$. Let L be a line which does not contain $\{0\}$ with $x \in L$. Fix any rotation ρ of infinite order around this line. Let $D = \{\rho^n(0) : n \ge 1\}$, then $0 \notin D$ and

$$B_1 = D \cup (B_1 \backslash D) \sim \rho(D) \cup (B_1 \backslash D) = B_1 \backslash \{0\}.$$

Thus by Proposition 0.3.2 we have the statement. $\square$

Bibliography

[1] Amir, G., Angel, O., Virág, B., *Amenability of linear-activity automaton groups*, Journal of the European Mathematical Society, 15 (2013), no. 3, 705–730.

[2] Amir, G., Virág, B., *Positive speed for high-degree automaton groups*, (preprint, arXiv:1102.4979), 2011.

[3] Banach, St., *Théorie des opérations linéaires.* Chelsea Publishing Co., New York, 1955. vii+254 pp.

[4] Banach, St., Tarski, A., *Sur la decomposition des ensembles de points en parties respectivement congruents,* Fund. Math., 14 (1929), 127–131.

[5] Bartholdi, L., Kaimanovich, V., Nekrashevych, V., *On amenability of automata groups*, Duke Mathematical Journal, 154 (2010), no. 3, 575–598.

[6] Bartholdi, L., Virág, B., *Amenability via random walks*, Duke Math. J., 130 (2005), no. 1, 39–56.

[7] Bartholdi, L., Grigorchuk, R., Nekrashevych, V., *From fractal groups to fractal sets*, Fractals in Graz 2001. Analysis – Dynamics – Geometry – Stochastics (Peter Grabner and Wolfgang Woess, eds.), Birkhäuser Verlag, Basel, Boston, Berlin, 2003, pp. 25–118.

[8] Bekka, B., de la Harpe, P., Valette, A., *Kazhdan Property (T).* Cambridge University Press, 2008.

[9] Bell, G., Dranishnikov, A., *Asymptotic Dimension.* Topology Appl., 12 (2008) 1265–1296.

[10] Benjamini, I., Hoffman, C., *ω-periodic graphs*, Electron. J. Combin., 12 (2005), Research Paper 46, 12 pp. (electronic).

[11] Benjamini, I., Schramm, O., *Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant.* Geometric and Functional Analysis 7 (1997), 3, 403–419

[12] Bellissard, J., Julien, A., Savinien, J., *Tiling groupoids and Bratteli diagrams*, Ann. Henri Poincaré 11 (2010), no. 1-2, 69–99.

[13] Bezuglyi, S., Medynets, K., *Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems*, Colloq. Math., 110 (2), (2008), 409–429.

[14] Blackadar, B., *K-theory for operator algebras.* Vol. 5. Cambridge University Press, (1998).

[15] Bleak, C., Juschenko, K., Ideal structure of the C*-algebra of Thompson group T. arXiv preprint arXiv:1409.8099.

[16] Bogoliubov, N., Krylov, N., *La theorie generalie de la mesure dans son application a l'etude de systemes dynamiques de la mecanique nonlineaire*, Ann. Math. II (in French), 38 (1), (1937), 65–113.

[17] Bondarenko, I., *Groups generated by bounded automata and their Schreier graphs*, PhD dissertation, Texas A& M University, 2007.

[18] Bondarenko, I., *Finite generation of iterated wreath products*, Arch. Math. (Basel), 95 (2010), no. 4, 301–308.

[19] Bondarenko, I., Ceccherini-Silberstein, T., Donno, A., Nekrashevych, V., *On a family of Schreier graphs of intermediate growth associated with a self-similar group*, European J. Combin., 33 (2012), no. 7, 1408–1421.

[20] Bratteli, O., *Inductive limits of finite-dimensional C^*-algebras*, Transactions of the American Mathematical Society, 171 (1972), 195–234.

[21] Brieussel, J., *Amenability and non-uniform growth of some directed automorphism groups of a rooted tree*, Math. Z., 263 (2009), no. 2, 265–293.

[22] Brieussel, J., *Folner sets of alternate directed groups*, to appear in Annales de l'Institut Fourier.

[23] Ceccherini-Silberstein, T., Grigorchuk, R., de la Harpe, P., *Amenability and paradoxical decompositions for pseudogroups and discrete metric spaces*, Proc. Steklov Inst. Math. 1999, no. 1 (224), 57–97.

[24] Chou, C., *Elementary amenable groups* Illinois J. Math. 24 (1980), 3, 396–407.

[25] de Cornulier, Yves, *Groupes pleins-topologiques, d'aprs Matui, Juschenko, Monod,...*, (written exposition of the Bourbaki Seminar of January 19th, 2013, available at `http://www.normalesup.org/~cornulier/plein.pdf`)

[26] Dahmani, F., Fujiwara, K., Guirardel, V., *Free groups of the interval exchange transformation are rare.* Preprint, arXiv:1111.7048

[27] Day, M., *Amenable semigroups*, Illinois J. Math., 1 (1957), 509–544.

[28] Day, M., *Semigroups and amenability*, Semigroups, K. Folley, ed., Academic Press, New York, (1969), 5–53

[29] Deuber, W., Simonovits, W., Sós, V., *A note on paradoxical metric spaces*, Studia Sci. Math. Hungar. 30 (1995), no. 1-2, 17–23.

[30] Dixmier, J., *Les C^*-algebres et leurs representations.* Editions Jacques Gabay, (1969).

[31] Dixmier, J., *Les algbres d'opérateurs dans l'espace hilbertien: algébres de von Neumann*, Gauthier-Villars, (1957).

[32] van Douwen, E., *Measures invariant under actions of* $\mathbb{F}_2$, Topology Appl. 34(1) (1990), 53-68.

[33] Dunford, N., Schwartz, J., *Linear Operators. I. General Theory.* With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7 Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London 1958 xiv+858 pp.

[34] Elek, G., Monod, N.,, *On the topological full group of minimal* $\mathbb{Z}^2$*-systems*, to appear in Proc. AMS.

[35] Exel, R., Renault, J., *AF-algebras and the tail-equivalence relation on Bratteli diagrams*, Proc. Amer. Math. Soc., 134 (2006), no. 1, 193–206 (electronic).

[36] Fink, E., *A finitely generated branch group of exponential growth without free subgroups*, (preprint arXiv:1207.6548), 2012.

[37] Giordano, Th., Putnam, I., Skau, Ch., *Full groups of Cantor minimal systems*, Israel J. Math., 111 (1999), 285–320.

[38] Glasner, E., Weiss, B., *Weak orbit equivalence of Cantor minimal systems*, Internat. J. Math., 6 (4), (1995), 559–579.

[39] Greenleaf, F., *Amenable actions of locally compact groups*, Journal of functional analysis, 4, 1969.

[40] Grigorchuk, R., Nekrashevich, V., Sushchanskii, V., *Automata, dynamical systems and groups*, Proceedings of the Steklov Institute of Mathematics, 231 (2000), 128–203.

[41] Grigorchuk, R., *On Burnside's problem on periodic groups*, Functional Anal. Appl., 14 (1980), no. 1, 41–43.

[42] Grigorchuk, R., *Symmetric random walks on discrete groups*, "Multi-component Random Systems", pp. 132–152, Nauk, Moscow, 1978.

[43] Grigorchuk, R., *Milnor's problem on the growth of groups*, Sov. Math., Dokl, 28 (1983), 23–26.

[44] Grigorchuk, R., *Degrees of growth of finitely generated groups and the theory of invariant means*, Math. USSR Izv., 25 (1985), no. 2, 259–300.

[45] Grigorchuk, R., *An example of a finitely presented amenable group that does not belong to the class EG*, Mat. Sb., 189 (1998), no. 1, 79–100.

[46] Grigorchuk. R., *Superamenability and the occurrence problem of free semigroups.* (Russian) Funktsional. Anal. i Prilozhen. 21 (1987), no. 1, 74–75.

[47] Grigorchuk, R., Medynets, K.,*Topological full groups are locally embeddable into finite groups*, Preprint, `http://arxiv.org/abs/math/1105.0719v3`.

[48] Grigorchuk, R., Żuk, A., *On a torsion-free weakly branch group defined by a three state automaton*, Internat. J. Algebra Comput., 12 (2002), no. 1, 223–246.

[49] Gromov, M., *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2, London Math. Soc. Lecture Note Ser. 182 (1993).

[50] Grünbaum, B., Shephard, G., *Tilings and patterns*, W. H. Freeman and Company, New York, 1987.

[51] Hausdorff, F., *Bemerkung über den Inhalt von Punktmengen.* (German) Math. Ann. 75 (1914), no. 3, 428–433.

[52] Herman, R., Putnam, I., Skau, Ch., *Ordered Bratteli diagrams, dimension groups, and topological dynamics*, Intern.J̃. Math., 3 (1992), 827–864.

[53] Ishii, Y., *Hyperbolic polynomial diffeomorphisms of* $\mathbb{C}^2$*. I. A non-planar map*, Adv. Math., 218 (2008), no. 2, 417–464.

[54] Ishii, Y., *Hyperbolic polynomial diffeomorphisms of* $\mathbb{C}^2$*. II. Hubbard trees*, Adv. Math., 220 (2009), no. 4, 985–1022.

[55] Ishii, Y., *Hyperbolic polynomial diffeomorphisms of* $\mathbb{C}^2$*. III: Iterated monodromy groups*, (preprint), 2013.

[56] Juschenko, K., Monod, N., *Cantor systems, piecewise translations and simple amenable groups.* To appear in Annals of Math, 2013.

[57] Juschenko, K., Nagnibeda, T., *Small spectral radius and percolation constants on non-amenable Cayley graphs.*, arXiv preprint arXiv:1206.2183.

[58] Juschenko, K., Nekrashevych, V., de la Salle, M., *Extensions of amenable groups by recurrent groupoids.* arXiv:1305.2637.

[59] Juschenko, K., de la Salle, M., *Invariant means of the wobbling groups.* arXiv preprint arXiv:1301.4736 (2013).

[60] Kaimanovich, V., *Boundary behaviour of Thompson's group.* Preprint.

[61] Katok, A., Hasselblatt, B., *Introduction to the modern theory of dynamical systems.* volume 54 of Encyclopedia of Mathematics and its Appli- cations. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[62] Katok, A., Stepin, A., *Approximations in ergodic theory.* Uspehi Mat. Nauk 22 (1967), no. 5(137), 81–106 (in Russian).

[63] Keane, K., *Interval exchange transformations.* Math. Z. 141 (1975), 25–31.

[64] Kesten, H., *Symmetric random walks on groups.* Trans. Amer. Math. Soc. 92 1959 336354.

[65] Lavrenyuk, Y., Nekrashevych, V., *On classification of inductive limits of direct products of alternating groups*, Journal of the London Mathematical Society 75 (2007), no. 1, 146–162.

[66] Laczkovich, M.,, *Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem*, J. Reine Angew. Math. 404 (1990) 77–117.

[67] Lebesgue, H., *Sur l'intégration et la recherche des fonctions primitive*, professées au au Collége de France (1904)

[68] Leinen, F., Puglisi, O., *Some results concerning simple locally finite groups of 1-type*, Journal of Algebra, 287 (2005), 32–51.

[69] Matui, H., *Some remarks on topological full groups of Cantor minimal systems,* Internat. J. Math. 17 (2006), no. 2, 231–251.

[70] Medynets, K., *Cantor aperiodic systems and Bratteli diagrams*, C. R. Math. Acad. Sci. Paris, 342 (2006), no. 1, 43–46.

[71] Milnor, J., *Pasting together Julia sets: a worked out example of mating*, Experiment. Math.,13 (2004), no. 1, 55–92.

[72] Milnor, J., *A note on curvature and fundamental group,* J. Differential Geometry, 2 (1968) 1–7.

[73] Milnor, J., *Growth of finitely generated solvable groups,* J. Differential Geometry, 2 (1968) 447-449.

[74] Mohar, B. *Isoperimetric inequalities, growth, and the spectrum of graphs.*Linear Algebra Appl. 103 (1988), 119131.

[75] Nekrashevych, V., *Self-similar inverse semigroups and groupoids*, Ukrainian Congress of Mathematicians: Functional Analysis, 2002, pp. 176–192.

[76] Nekrashevych, V., *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, Amer. Math. Soc., Providence, RI, 2005.

[77] Nekrashevych, V., *Self-similar inverse semigroups and Smale spaces*, International Journal of Algebra and Computation, 16 (2006), no. 5, 849–874.

[78] Nekrashevych, V., *A minimal Cantor set in the space of 3-generated groups*, Geometriae Dedicata, 124 (2007), no. 2, 153–190.

[79] Nekrashevych, V., *Symbolic dynamics and self-similar groups*, Holomorphic dynamics and renormalization. A volume in honour of John Milnor's 75th birthday (Mikhail Lyubich and Michael Yampolsky, eds.), Fields Institute Communications, vol. 53, A.M.S., 2008, pp. 25–73.

[80] Nekrashevych, V., *Combinatorics of polynomial iterations*, Complex Dynamics – Families and Friends (D. Schleicher, ed.), A K Peters, 2009, pp. 169–214.

[81] Nekrashevych, V., *Free subgroups in groups acting on rooted trees*, Groups, Geometry, and Dynamics **4** (2010), no. 4, 847–862.

[82] Neumann, P., *Some questions of Edjvet and Pride about infinite groups*, Illinois J. Math., 30 (1986), no. 2, 301–316.

[83] von Neumann, J., *Zur allgemeinen Theorie des Masses*, Fund. Math., vol 13 (1929), 73-116.

[84] Nash-Williams, C. St. J. A., *Random walk and electric currents in networks*, Proc. Cambridge Philos. Soc., 55 (1959), 181–194.

[85] Oliva, R., *On the combinatorics of external rays in the dynamics of the complex Hénon map*, PhD dissertation, Cornell University, 1998.

[86] Osin, D., *Elementary classes of groups*, (in Russian) Mat. Zametki 72 (2002), no. 1, 84–93; English translation in Math. Notes 72 (2002), no. 1-2, 75–82.

[87] Rejali, A., Yousofzadeh, A., *Configuration of groups and paradoxical decompositions*, Bull. Belg. Math. Soc. Simon Stevin 18 (2011), no. 1, 157–172.

[88] Rosenblatt, J., *A generalization of Følner's condition*, Math. Scand., 33 (1973), 153–170.

[89] Rudin, W., *Functional analysis*, New York, McGraw-Hill, (1973)

[90] Sakai, S., *C^*-algebras and W^*-algebras* (Vol. 60). Springer. (1971).

[91] Segal, D., *The finite images of finitely generated groups*, Proc. London Math. Soc. (3), 82 (2001), no. 3, 597–613.

[92] Sidki, S., *Automorphisms of one-rooted trees: growth, circuit structure and acyclicity*, J. of Mathematical Sciences (New York), 100 (2000), no. 1, 1925–1943.

[93] Sidki, S., *Finite automata of polynomial growth do not generate a free group*, Geom. Dedicata, 108 (2004), 193–204.

[94] Schreier, O., *Die Utregruppen der freien Gruppen*, Abhandlungen Math. Hamburg 5 (1927), 161–183.

[95] Świerczkowski, S.,*On a free group of rotations of the Euclidean space.* Nederl. Akad. Wetensch. Proc. Ser. A 61 = Indag. Math. 20 1958 376–378.

[96] Tarski, A., *Algebraische Fassung de Massproblems*, Fund. Math. 31 (1938), 47–66

[97] Takesaki, M., *Theory of operator algebras I, II, III.* Vol. 2. Springer, 2003.

[98] Viana, M., *Ergodic theory of interval exchange maps.* Rev. Mat. Complut. 19 (2006), no. 1, 7–100.

[99] Vital, G., *Sul problema della misura dei gruppi di punti di una retta*, Bologna,Tip. Camberini e Parmeggiani (1905).

[100] Wagon, S., *Banach-Tarski paradox*, Cambridge: Cambridge University Press. ISBN: 0-521-45704-1

[101] Woess, W., *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, 2000.

[102] Wolf, J., *Growth of finitely generated solvable groups and curvature of Riemanniann manifolds*, J. Differential Geometry 2 (1968), 421–446.

[103] Woryna, A., *The rank and generating set for iterated wreath products of cyclic groups*, Comm. Algebra, 39 (2011), no. 7, 2622–2631.

[104] Zimmer, R., *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.