

Lecture 12: Weak containment of representations

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0.1 Weak containment of representations

Let Γ be a discrete group with two representation $\pi : \Gamma \rightarrow B(\mathcal{H})$ and $\rho : \Gamma \rightarrow B(\mathcal{K})$ by unitary operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ correspondingly. The representation π is **weakly contained** in ρ , denoted by $\pi \prec \rho$, if for every $\xi \in \mathcal{H}$, finite set E of Γ and $\varepsilon > 0$ there are $\eta_1, \dots, \eta_n \in \mathcal{K}$ such that for all $g \in E$ we have

$$\left| \langle \pi(g)\xi, \xi \rangle - \sum_{i=1}^n \langle \rho(g)\eta_i, \eta_i \rangle \right| < \varepsilon.$$

Denote by $1_\Gamma : \Gamma \rightarrow \mathbb{C}$ the trivial representation, i.e., $1_\Gamma(g) = 1$ for every $g \in \Gamma$.

Theorem 0.1.1. A discrete group Γ is amenable if and only if $1_\Gamma \prec \lambda$.

Proof. We will show that $1_\Gamma \prec \lambda$ is equivalent to the existence of almost invariant vector for λ , which by Theorem ?? is equivalent to the amenability of Γ .

Assume that λ admits an almost invariant vector. It is sufficient to show that for every $\varepsilon > 0$ and a finite set E of Γ there are $\eta_1, \dots, \eta_n \in l^2(\Gamma)$ such that

$$|1 - \sum_{i=1}^n \langle \lambda(t)\eta_i, \eta_i \rangle| < \varepsilon, \text{ for every } t \in E.$$

This follows if we take $n = 1$ and $\eta_1 = \xi$, where ξ is almost invariant vector, i.e., $\|\lambda(g)\xi - \xi\| < \varepsilon$ for every $g \in E$.

Conversely, assume that $1_\Gamma \prec \lambda$, we will deduce that λ has an almost invariant vector. By the definition, for every $\varepsilon > 0$ and a finite subset E of Γ there are $\eta_1, \dots, \eta_n \in l^2(\Gamma)$, such that

$$|1 - \sum_{i=1}^n \langle \lambda(t)\eta_i, \eta_i \rangle| < \varepsilon, \text{ for every } t \in E. \quad (1)$$

Assuming that E contains the identity element e , we obtain

$$|1 - \sum_{i=1}^n \|\eta_i\|^2| < \varepsilon.$$

Rescaling the norm we may assume that $\sum_{i=1}^n \|\eta_i\|^2 = 1$ and (1) is still satisfied.

To reach a contradiction assume λ does not have an almost invariant vector. Then there exists $C > 0$ and a finite set $S \subset \Gamma$ such that for every $\xi \in l^2(\Gamma)$ we have

$$\|\xi\|^2|S| - \sum_{\gamma \in S} \langle \lambda(\gamma)\xi, \xi \rangle > C\|\xi\|^2$$

Now applying this to the vectors $\eta_1, \dots, \eta_n$ and summing up, we obtain

$$|S| - \sum_{\sigma \in S} \sum_{i=1}^n \langle \lambda(\sigma)\eta_i, \eta_i \rangle > C$$

This implies that there exists $\gamma \in S$ such that

$$1 - \sum_{i=1}^n \langle \lambda(\gamma)\eta_i, \eta_i \rangle > C/|S| > 0,$$

which contradicts to (1), thus λ admits an almost invariant vector. $\square$

More on weak containment of representations can be found in the book [8].

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