

# Lecture 15: Grasshopper criteria. Gromov's doubling condition.

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## 0.1 Paradoxical decomposition. Tarski numbers.

Let  $\Gamma$  be a group acting on a set  $X$ . We start with the following basic lemma which will be useful in the later sections. We will follow the exposition of [87].

**Lemma 0.1.1.** Let a discrete group  $\Gamma$  act on a set  $X$ , then the following are equivalent:

- (i) there exist pairwise disjoint subsets  $A_1, \dots, A_n$  and  $B_1, \dots, B_m$  in  $X$  and there exist  $g_1 \dots g_n, h_1, \dots, h_m$  in  $\Gamma$  such that

$$X = \bigcup_{i=1}^n g_i A_i = \bigcup_{j=1}^m h_j B_j.$$

- (ii) there exist pairwise disjoint subsets  $A_1, \dots, A_n$  and  $B_1, \dots, B_m$  in  $X$  and there exist  $g_1 \dots g_n, h_1, \dots, h_m$  in  $\Gamma$  such that

$$X = \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^m h_j B_j.$$

- (iii) there exist pairwise disjoint subsets  $A_1, \dots, A_n$  and  $B_1, \dots, B_m$  in  $X$  and there exist  $g_1 \dots g_n, h_1, \dots, h_m$  in  $\Gamma$  such that

$$X = \bigsqcup_{i=1}^n A_i = \bigsqcup_{j=1}^m B_j = \bigsqcup_{i=1}^n g_i A_i = \bigsqcup_{j=1}^m h_j B_j.$$

*Proof.* The trivial implication is (iii)  $\implies$  (ii)  $\implies$  (i). Without loss of generality we can assume that  $g_1 = h_1 = e$ . Assume (i) and define inductively  $A'_1 = A_1$ ,  $A'_k = A_k \setminus g_k^{-1}(\bigsqcup_{i=1}^{k-1} g_i A'_i)$ , and  $B'_1 = B_1$ ,  $B'_k = B_k \setminus h_k^{-1}(\bigsqcup_{i=1}^{k-1} h_i B'_i)$ . It is easy to check that this sets satisfy (ii).

Assume (ii) and let  $A = \bigcup_{1 \leq i \leq n} A_i$ ,  $B = \bigcup_{1 \leq j \leq m} B_j$ . Note that  $A \cap B = \emptyset$ . Define a map  $f : X \rightarrow B$  by  $f(x) = b_x$ , where  $b_x \in B_j$  is a unique element such that  $x = h_j b_x$ .

Put  $D = \bigcup_{k=0}^{\infty} f^k(A)$ , with  $f^0(A) = A$ , and  $T = (X \setminus A) \setminus f(D)$ . Then  $A \cap f(D) = \emptyset$  and  $A \cup f(D) = D$ . Let  $D_j = B_j \cap h_j^{-1}D$ . Then we have

$$X = A \cup T \cup \left( \bigcup_{1 \leq j \leq m} D_j \right)$$

and

$$X = \bigsqcup_{i=1}^n g_i A_i = eT \cup \bigsqcup_{i=1}^m h_j D_j.$$

Since  $h_1 = e$ , we can gather  $eT$  and  $D_1$  together in order to keep  $m$  and  $n$  as in (ii) unchanged. Thus (iii) follows.  $\square$

The action is **paradoxical** if it satisfies one of the conditions of the lemma.

**Theorem 0.1.2** (Tarski). Let  $\Gamma$  be a discrete group. The following are equivalent:

- (i)  $\Gamma$  is amenable;
- (ii) The action of  $\Gamma$  on itself by left multiplication is not paradoxical;
- (iii)  $\Gamma$  does not admit a paradoxical action.

We start by proving a graph theoretical lemma, which will be also useful in the later sections.

**Theorem 0.1.3** (Hall's (1-1)-matching). Let  $G = (A, B, E)$  be a bipartite graph with the set of edges  $E$  between the sets  $A$  and  $B$ . Assume that the degree of every vertex in  $A$  is finite and for every finite set  $D \subseteq A$  we have

$$|N(D)| \geq |D|,$$

where  $N(D)$  is the set of all vertices in  $B$  connected with a vertex in  $D$ . Then there exists an injective map  $i$  with domain equals to  $A$ , such that

$$(a, i(a)) \in E \text{ for every } a \in A.$$

*Proof.* It is sufficient to prove the lemma for finite graphs. Indeed, assume that the statement is true for all finite subsets of  $A$ . Let  $A_n$  be an increasing to  $A$  sequence of finite subset of  $A$  and  $i_n, j_n$  be corresponding maps. Since these maps correspond to finite subgraphs of  $G$ , we can take a cluster point in the topology of  $\{0, 1\}^E$ . Then this cluster point satisfies all required properties.

We will prove the theorem by the induction of the size of the set  $A$ . For  $|A| = 1$  the statement is trivial, assume that it holds for all sets with size less or equal to  $n$ . Let then  $|A| = n + 1$ . Take  $v \in A$ , then there exists at least one edge going out of  $v$ .

Assume that for all  $X \subset A \setminus \{v\}$ , we have  $|N(X)| \geq |X| + 1$ , then we can connect  $v$  with any edge and apply induction to  $A \setminus \{v\}$ . Thus the only case we need to treat is when there exists a subset  $X \subseteq A \setminus \{v\}$  with  $|N(X)| = |X|$ . Choose a minimal set  $X$  with this property. By the induction and minimality of  $X$  we have that  $X$  can be matched to  $N(X)$ . Consider now the complement of  $X$ , we will show that it can be matched with complement of  $N(X)$ . Let  $Y \subseteq A \setminus X$ , then it is easy to check that  $Y$  must have  $|Y|$  neighbors outside  $N(X)$ . Therefore, by induction, we can match those points with  $N(Y) \setminus N(X)$ . Combining both matchings we obtain a matching of a set  $A$ .  $\square$

**Corollary 0.1.4** (Hall's  $(k, 1)$ -matching). Let  $G = (A, B, E)$  be a bipartite graph with the set of edges  $E$  between the sets  $A$  and  $B$ , and let  $k$  be a natural number. Assume that the degree of every vertex in  $A$  is finite and for every finite set  $D \subseteq A$  we have

$$|N(D)| \geq k|D|,$$

where  $N(D)$  is the set of all vertices in  $B$  connected with a vertex in  $D$ . Then there are injective maps  $i_1, \dots, i_k$  with domain equals to  $A$  and pairwise disjoint images such that

$$(a, i_t(a)) \in E \text{ for every } a \in A \text{ and } 1 \leq t \leq k.$$

*Proof.* Let  $G$  satisfy the statement then taking a disjoint union of  $k$  copies of  $A$  and adding edges between each copy and  $B$  in the case there was an edge between  $A$  and  $B$ , we obtain a graph bipartite graph  $(A', B)$ , which satisfies the conditions of Hall's  $(1, 1)$ . Thus we can match the disjoint union of  $k$  copies of  $A$  with  $B$ , which defines us the maps  $i_t$ ,  $1 \leq t \leq k$ .  $\square$

Now we are ready to prove Theorem 0.1.2.

*Proof of the Theorem 0.1.2.* Assume that  $\Gamma$  is not amenable. Then there are a finite set  $S \subset \Gamma$  and  $\varepsilon > 0$  such that for every  $E \subset \Gamma$  there is  $g \in S$  such that

$$|gE \setminus E| \geq \varepsilon|E|.$$

Without loss of generality we can assume that  $S$  contains  $e$ . Thus  $E \subset SE$  and we have

$$|SE| \geq (1 + \varepsilon)|E|$$

Applying this inequality to the set  $S^{k-1}E$  we obtain

$$|S^k E| \geq (1 + \varepsilon)|S^{k-1}E| \geq \dots \geq (1 + \varepsilon)^k|E|$$

Choose  $k$  such that  $(1 + \varepsilon)^k \geq 2$  and denote  $S^k$  again by  $S$ . Thus we obtain

$$|SE| \geq 2|E|$$

Consider a bipartite graph  $G = (\Gamma, \Gamma, E)$  with edges between two copies of  $\Gamma$  defined by  $(g, h) \in E$  if and only if  $h = sg$  for some  $s \in S$ . Then for every finite set  $F$  we automatically have  $|N(F)| \geq 2|F|$ . Thus by Hall's (1,2)-matching theorem there are injective on  $\Gamma$  maps  $i$  and  $j$  with disjoint images such that for each  $g \in \Gamma$  there exists  $s, t \in S$  with  $i(g) = sg$  and  $j(g) = tg$ . For  $s, t \in S$ , define

$$\begin{aligned} A_s &= s\{g \in \Gamma : i(g) = sg\}, \\ B_t &= t\{g \in \Gamma : j(g) = tg\}. \end{aligned}$$

Then it is trivial to check that these sets are pairwise disjoint and we have the following paradoxical decomposition:

$$\Gamma = \bigcup_{s \in S} s^{-1}A_s = \bigcup_{t \in S} t^{-1}B_t.$$

Conversely, if  $\Gamma$  is amenable then for every action of  $\Gamma$  on a set  $X$ , there exists a  $\Gamma$ -invariant finitely additive probability measure  $\mu$  on  $X$ . To reach a contradiction assume that the action has a paradoxical decomposition, i.e., there are pairwise disjoint sets  $A_1, \dots, A_n, B_1, \dots, B_m$  in  $X$  and  $g_1, \dots, g_n, h_1, \dots, h_m$  in  $\Gamma$  such that

$$X = \bigcup_{i=1}^n g_i A_i = \bigcup_{j=1}^m h_j B_j.$$

Applying  $\mu$  to this equality we obtain

$$\begin{aligned} 1 = \mu(X) &\geq \sum_{i=1}^n \mu(A_i) + \sum_{j=1}^m \mu(B_j) \\ &= \sum_{i=1}^n \mu(g_i A_i) + \sum_{j=1}^m \mu(h_j B_j) \\ &\geq 2\mu(X) = 2, \end{aligned}$$

which is a contradiction. Thus the action is not paradoxical.  $\square$

Sometimes non-amenable groups are also called *paradoxical*. By Tarski's Theorem, each non-amenable group admits a paradoxical decomposition. The minimal number of sets required in this decomposition, i.e., the number  $n + m$ , is called **Tarski number** of the group.

## 0.2 Gromov's doubling condition. Grasshopper criteria.

Through this section we assume that  $\Gamma$  is generated by a finite set  $S$ . Denote by  $d : \Gamma \times \Gamma \in \mathbb{N}$  the distance in the Cayley graph of  $\Gamma$  with respect to  $S$ .

Assume that  $\Gamma$  acts on a set  $X$  and let  $Y$  be a subset of  $X$ . A injective map  $\tau : Y \rightarrow X$  is called *wobbling* if there exists a finite set  $E \subset \Gamma$  such that for each  $y \in Y$  we have that  $\tau(y) = gy$  for some  $g \in E$ . The set  $E$  will be called the set that defines the wobbling  $\tau$ .

The proof of the following equivalences is very similar to the proof of Tarski's theorem. The name "Grasshopper criteria" came from the following visual explanation. We put one grasshopper in each vertex of the Cayley graph and let them jump by connecting edges. Then, if grasshoppers will be smart enough, after finite number of jumps each of them will find itself at a vertex with one mate.

**Theorem 0.2.1.** The following are equivalent

- (i)  $\Gamma$  is not amenable;

- (ii) **Gromov's doubling condition.** There exists a finite set  $S$ , for every finite set  $E \subset \Gamma$

$$|N(E)| \geq 2|E|,$$

where  $N(E) = \{g \in \Gamma : g = sf \text{ for some } f \in E \text{ and } s \in S\}$ .

- (iii) **Grasshopper criteria.** There exists a map  $\phi : \Gamma \rightarrow \Gamma$  such that

$$\sup(d(g, \phi(g)) : g \in \Gamma) < \infty$$

and for every  $g \in \Gamma$  we have  $|\phi^{-1}(g)| = 2$ ;

- (iv) There are two wobbling maps  $i, j : \Gamma \rightarrow \Gamma$  with  $i(\Gamma) \cap j(\Gamma) = \emptyset$  and  $i(\Gamma) \cup j(\Gamma) = \Gamma$ ;

*Proof.* Because of the Følner condition, amenable group can not satisfy the inequality from (ii). Thus (i)  $\implies$  (ii).

Let us prove that (ii) implies (i), assume that for every finite set  $E \subset \Gamma$

$$|N_S(E)| \geq 2|E|.$$

Consider a bipartite graph  $G = (\Gamma, \Gamma, E)$  with edges between two copies of  $\Gamma$  defined by  $(g, h) \in E$  if and only if  $h = sg$  for some  $s \in S$ . Then for every finite set  $F$  we have  $|N(F)| \geq 2|F|$ . Thus by Hall's (1,2)-matching theorem there are injective on  $\Gamma$  maps  $i$  and  $j$ ,  $i(\Gamma) \cap j(\Gamma) = \emptyset$  and for each  $g \in \Gamma$  there exists  $s, t \in S$  with  $i(g) = sg$  and  $j(g) = tg$ .

For  $s, t \in E$ , define

$$A_s = s\{g \in \Gamma : i(g) = sg\}, \quad B_t = t\{g \in \Gamma : j(g) = tg\}.$$

It is trivial to check that these sets are pairwise disjoint and we have the following paradoxical decomposition:

$$\Gamma = \bigcup_{s \in E} s^{-1}A_s = \bigcup_{t \in E} t^{-1}B_t,$$

which implies that  $\Gamma$  is not amenable.

The equivalence of (iii) and (iv) with (i) follows from the Lemma 0.1.1 and Theorem 0.1.2  $\square$





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