

Lecture 2: First definitions of amenability, elementary operations that preserve it

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0.1 Means and measures.

In this section we specify a connection between means and finitely additive probability measures. This connection will be important for proving equivalences of several definitions of amenability. More on means and measures can be found in classical books on functional analysis and measure theory, for example in [33].

Let X be a set and $\mathcal{P}(X)$ be the set of all subsets of X . For $E \in \mathcal{P}(X)$ denote by χ_E the characteristic function of the set E , i.e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise.

Definition 0.1.1. A map $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ is called finitely additive measure if it satisfies:

- (i) $\mu(X) = 1$
- (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$ and $A, B \in \mathcal{P}(X)$.

Denote the set of all finitely additive probability measures on X by $PM(X)$. One of the examples of finitely additive measures is the counting measure on supported on a finite set, i.e., $\mu(F) = |E \cap F|/|F|$ for some finite set F in X .

Definition 0.1.2. A mean on a set X is a functional $m \in l_\infty(X)^*$ which satisfies

- (i) $m(\chi_X) = 1$,
- (ii) $m(f) \geq 0$ for all $f \geq 0$, $f \in l_\infty(X)$.

Define $M(X)$ to be the set of all means on X ,

$$Proba(X) = \{f : X \rightarrow \mathbb{R}_+ : \sum_{x \in X} f(x) = 1\}$$

$$Proba_{fin}(X) = \{f : X \rightarrow \mathbb{R}_+ : \sum_{x \in X} f(x) = 1, f \text{ is finitely supported}\}$$

Then $Proba_{fin}(X) \subseteq Proba(X) \subseteq M(X)$. Indeed, for each $h \in Proba(X)$ we can define a mean $m_h \in M(X)$ by

$$m_h(f) = \sum_{x \in X} h(x)f(x)$$

Fact 0.1.3. We list the following classical properties of means:

- (i) For each $m \in M(X)$ we have $\|m\| = 1$.
- (ii) The set of all means $M(X) \subset l_\infty(X)^*$ is convex and closed in the weak*-topology.
- (iii) $Proba_{fin}(X)$ and $Proba(X)$ are convex in $M(X)$.
- (iv) The set $Proba_{fin}(X)$, and thus $Proba(X)$, is weak*-dense in $M(X)$.

Proof. To prove (i), observe that $\|m\| \geq 1$, since $m(\chi_X) = 1$. Now, if $f, h \in l_\infty(X)$ and $f \leq h$ then $m(h - f) \geq 0$ and thus $m(f) \leq m(h)$. Applying this to $f \leq \|f\|_\infty \cdot \chi_X$, we obtain $m(f) \leq \|f\|_\infty \cdot m(\chi_X) = \|f\|_\infty$, therefore $\|m\| = 1$.

The steps (ii) and (iii) are trivial.

To see (iv), assume that $m \in M(X)$ is not a weak*-limit of means in $Proba_{fin}(X)$. By Hahn-Banach theorem, we can find a function f in $l_\infty(X)$ and $\delta > 0$, such that $m(f) \geq \delta + \overline{m}(f)$ for all $\overline{m} \in Proba_{fin}(X)$. In particular, this holds for Dirac measures $\delta_x \in Proba_{fin}(X)$, for which we have $\delta_x(f) = f(x)$. Therefore, $m(f) > \sup_{x \in X} f(x)$ for all $f \in l_\infty(X)$, this is impossible. \square

For each mean $m \in M(X)$ we can associate a finitely additive measure on X :

$$\hat{m}(A) = m(\chi_A), \text{ for all } A \in \mathcal{P}(X).$$

Indeed, since $\chi_A \leq \chi_X$ we have that \hat{m} takes its values in $[0, 1]$. Moreover, the linearity of m implies $\hat{m}(A \cup B) = \hat{m}(A) + \hat{m}(B)$ for all $A, B \in \mathcal{P}(X)$ with $A \cap B = \emptyset$.

Theorem 0.1.4. The map between means and finitely additive probability measures given by $m \mapsto \hat{m}$ is bijective.

Proof. Let $\mathcal{E}(X)$ be the set of all \mathbb{R} -valued functions on X which take only finitely many values. Then $\mathcal{E}(X)$ is dense in $l_\infty(X)$. Indeed, for each positive function in $h \in l_\infty(X)$ define $\lambda_i = \frac{i}{n} \|h\|_\infty$ and

$$f_n(x) = \min\{\lambda_i : h(x) \leq \lambda_i\}.$$

Then $f_n \in \mathcal{E}(X)$ and $\|h - f_n\|_\infty \leq \|h\|_\infty/n$. Since every function $h \in l_\infty(X)$ can be decomposed as a difference of two positive function, we have that $\mathcal{E}(X)$ is dense in $l_\infty(X)$.

For $\mu \in PM(X)$ and $h \in \mathcal{E}(X)$ define

$$\bar{\mu}(h) = \sum_{t \in \mathbb{R}} \mu(h^{-1}(t))t.$$

Since $h = \sum_{i \in I} \lambda_i \chi_{A_i}$ for some finite set I and μ is finitely additive, we have

$$\bar{\mu}(h) = \sum_{i \in I} \lambda_i \mu(A_i).$$

Moreover the map $\bar{\mu}$ is linear. Indeed, for all $\lambda \in \mathbb{R}$ we have

$$\bar{\mu}(\lambda h) = \sum_{i \in I} \lambda \mu(A_i) = \lambda \bar{\mu}(h).$$

For $h, g \in \mathcal{E}(X)$ we have

$$\begin{aligned} \bar{\mu}(h + g) &= \sum_{(x,y) \in \text{Image}(h) \times \text{Image}(g)} \mu(h^{-1}(x) \cap g^{-1}(y))(x + y) \\ &= \sum_{x \in \text{Image}(h)} \mu(h^{-1}(x)) + \sum_{y \in \text{Image}(g)} \mu(h^{-1}(y)) \\ &= \mu(h) + \mu(g). \end{aligned}$$

Since $|\bar{\mu}(h)| \leq \sum_{i \in I} |\lambda_i| \mu(A_i) \leq \|h\|_\infty \cdot \mu(X) = \|h\|_\infty$, we can extend $\bar{\mu}$ to a linear functional m on $l_\infty(X)$ with $\|m\| \leq 1$. Moreover, $m(\chi_X) = \mu(X) = 1$. By the construction, if $f \in \mathcal{E}(X)$ is a positive function then $m(f) \geq 0$. As we showed above, each positive function in $l_\infty(X)$ can be approximated by positive functions from $\mathcal{E}(X)$, thus we have $m(f) \geq 0$ for all positive functions $f \in l_\infty(X)$.

Since for every $E \subset X$

$$m(\chi_E) = \mu(E)$$

we have that $\hat{m} = \mu$ and thus the statement of the theorem follows. \square

Consider now an action of a group G on a set X . It is straight forward to check that a mean m is G -invariant if and only if the finitely additive probability measure \hat{m} is G -invariant.

0.2 First definitions: invariant mean, Følner condition.

The classical definition of amenable group which goes back to von Neumann is the following.

Definition 0.2.1. A group Γ is amenable if there exists a finitely additive measure μ on all subsets of Γ into $[0, 1]$ with $\mu(\Gamma) = 1$ and satisfying

$$\mu(gE) = \mu(E)$$

for every $E \subseteq \Gamma$ and $g \in \Gamma$.

The left regular representation of Γ , $\lambda : \Gamma \rightarrow U(l^2(\Gamma))$, is the representation acting on a Hilbert space $l^2(\Gamma)$ by unitary operators as follows:

$$\lambda_g(\delta_t) = \delta_{st} \text{ for all } g, t \in \Gamma,$$

where $\{\delta_t\}_{t \in \Gamma}$ is the canonical orthonormal basis of $l^2(\Gamma)$. Analogously, the right regular representation is defined by $\rho_s(\delta_t) = \delta_{ts^{-1}}$.

Von Neumann's main novelty for studying amenability is to consider the space of functions and means on a group. Denote by $l^\infty(\Gamma)$ the space of bounded functions on Γ . The space $l^\infty(\Gamma)$ can be considered as multiplication operators on $l^2(\Gamma)$: $f\delta_t = f(t)\delta_t$ for $t \in \Gamma$ and $f \in l^\infty(\Gamma)$. This gives an embedding $l^\infty(\Gamma) \subseteq B(l^2(\Gamma))$. The action of Γ on $l^\infty(\Gamma)$ is defined by $g.f(t) = f(g^{-1}t)$. In terms of bounded on $l^2(\Gamma)$ operators this is nothing but the action by conjugation, i.e., $g.f = \lambda(g)f\lambda(g)^{-1}$.

A mean on the group Γ is a linear functional μ on $l^\infty(\Gamma)$ such that $\mu(\chi_\Gamma) = 1$, $\mu(f) \geq 0$ for all $f \geq 0$, $f \in l^\infty(\Gamma)$. It is called Γ -invariant if $\mu(t.f) = \mu(f)$ for all $f \in l^\infty(\Gamma)$ and $t \in \Gamma$.

We denote **the space of probability measures on Γ** by

$$Prob(\Gamma) = \{\mu \in l^1(\Gamma) : \|\mu\|_1 = 1 \text{ and } \mu \geq 0\}.$$

A subset $S \subset \Gamma$ is called symmetric if $S = S^{-1} = \{s^{-1} : s \in S\}$.

From the previous section we know that there is one-to-one correspondence between means and finitely additive measures. Moreover, it is straightforward to check that to Γ -invariant finitely additive measure this correspondence associates Γ -invariant mean. Thus we immediately obtain the following definition which is equivalent to the Definition 0.2.1.

Definition 0.2.2. A group Γ is amenable if it admits an invariant mean.

As our tools will develop we will present more and more sophisticated definitions of amenability. The main purpose of this is to cover all known examples of amenable groups. The following is the first set of equivalent definitions.

Theorem 0.2.3. For a discrete group Γ the following are equivalent:

1. Γ is amenable.
2. *Approximately invariant mean.* For any finite subset $E \subset \Gamma$ there is $\mu \in Prob(\Gamma)$ such that $\|s.\mu - \mu\|_1 \leq \varepsilon$ for all $s \in E$.
3. *Følner condition.* For any finite subset $E \subset \Gamma$ and $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that

$$|gF \Delta F| \leq \varepsilon |F| \text{ for all } g \in E.$$

Proof. (1) \implies (2). Let E be a finite set and $\mu \in l^\infty(\Gamma)^*$ be an invariant mean. Since $l^1(\Gamma)$ is dense in $l^\infty(\Gamma)^*$ in weak*-topology, let $\mu_i \in Prob(\Gamma)$ weak*-converges to μ . This implies that $s.\mu_i - \mu_i$ converges in weak*-topology to zero. Since weak*-convergence for functions in $l^\infty(\Gamma)$ and thus weakly in $l^1(\Gamma)$ for all $s \in \Gamma$. Consider the weak closure of the convex set

$$\left\{ \bigoplus_{s \in E} s.\mu - \mu : \mu \in Prob(\Gamma) \right\}.$$

We have that this closure contains zero. By Hahn-Banach theorem, it is also norm closed. Thus (2) follows.

(2) \implies (3). Given $E \subset \Gamma$ and $\varepsilon > 0$, let μ satisfy (2). Let $f \in l^1(\Gamma)$ and $r \geq 0$. Define $F(f, r) = \{t \in \Gamma : f(t) > r\}$.

For positive functions f, h in $l^1(\Gamma)$ with $\|f\|_1 = \|h\|_1 = 1$, we have $|\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| = 1$ if and only if $f(t) \leq r \leq h(t)$ or $h(t) \leq r \leq f(t)$. Hence for two functions bounded above by 1 we have

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| dr.$$

Thus we can apply this to $\mu(t)$ and $s.\mu(t)$:

$$\begin{aligned} \|s.\mu - \mu\|_1 &= \sum_{t \in \Gamma} |s.\mu(t) - \mu(t)| \\ &= \sum_{t \in \Gamma} \int_0^1 |\chi_{F(s.\mu,r)}(t) - \chi_{F(\mu,r)}(t)| dr \\ &= \int_0^1 \left(\sum_{t \in \Gamma} |\chi_{s.F(\mu,r)}(t) - \chi_{F(\mu,r)}(t)| \right) dr \\ &= \int_0^1 |s.F(\mu, r) \Delta F(\mu, r)| dr \end{aligned}$$

Since $\|f\|_1 = 1$ and μ satisfies (2), it follows

$$\begin{aligned} \int_0^1 \sum_{s \in E} |s.F(\mu, r) \Delta F(\mu, r)| dr &\leq \varepsilon |E| \\ &= \varepsilon |E| \sum_{t \in \Gamma} \mu(t) \\ &= \varepsilon |E| \sum_{t \in \Gamma} \int_0^{\mu(t)} dr \\ &= \varepsilon |E| \int_0^1 |\{t \in \Gamma : \mu(t) > r\}| dr \\ &= \varepsilon |E| \int_0^1 |F(\mu, r)| dr. \end{aligned}$$

Thus there exists r such that

$$\sum_{s \in E} |s.F(\mu, r) \Delta F(\mu, r)| \leq \varepsilon |E| |F(\mu, r)|.$$

(3) \implies (1). Let E_i be an increasing to Γ sequence of finite subsets and $\{\varepsilon_i\}$ be a converging to zero sequence. By (3) we can find F_i that satisfy

$$|gF_i \Delta F_i| \leq \varepsilon |F_i| \text{ for all } g \in E_i.$$

Denote by $\mu_i = \frac{1}{|F_i|} \chi_{F_i} \in \text{Prob}(\Gamma)$, then

$$\|s.\mu_i - \mu_i\|_1 = \frac{1}{|F_i|} |gF_i \Delta F_i|.$$

Let $\mu \in l^\infty(\Gamma)^*$ be a cluster point in the weak*-topology of the sequence μ_i , then μ is an invariant mean. \square

Let S be a generating set of the group Γ . A sequence $\{F_i\}_{i \in \mathbb{N}}$ is called a *Følner sequence* for S if there exists a sequence $\{\varepsilon_i\}_{i \in \mathbb{N}}$ that converges to zero that

$$|gF_i \Delta F_i| \leq \varepsilon_i |F_i| \text{ for all } g \in S \text{ and all } i \in \mathbb{N}.$$

Let $s, h \in \Gamma$ and F be a finite subset of Γ , then

$$|shF \Delta F| = |(shF \Delta sF) \cup (sF \Delta F)| \leq |hF \Delta F| + |sF \Delta F|.$$

Since any finite set is contained in a ball of a large enough radius of the Cayley graph, by inequality above we have that a finitely generated group Γ is amenable if and only if it admits a Følner sequence for some generating set.

0.3 First examples

Below we present first examples of amenable and non-amenable groups that can be constructed using the basic definitions.

Finite groups. Finite groups are amenable because they are Følner sets themselves.

Groups of subexponential growth. A group Γ has *subexponential growth* if $\limsup_n |S^n|^{1/n} = 1$ for any finite subset $S \subset \Gamma$, where $S^n = \{s_1 \dots s_n : s_1, \dots, s_n \in S\}$. Obviously, for a finitely generated group it is sufficient to verify the above condition on a symmetric generating set.

The fact that all non-amenable groups have exponential growth can be deduced from the Følner condition.

Indeed, let Γ be a group of subexponential growth. Let E be a finite symmetric subset of Γ and denote $B_n = E^n$. By the definition of subexponential growth we have that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$|B_{k+1}|/|B_k| \leq (1 + \varepsilon).$$

Indeed, to reach a contradiction assume that for there exists $\varepsilon > 0$ such that for all k

$$|B_{k+1}| > (1 + \varepsilon)|B_k|.$$

Hence $|B_{k+1}| > (1 + \varepsilon)^k |B_1|$ and therefore $\limsup_n |B_n|^{1/n} \geq 1 + \varepsilon$, which is a contradiction.

Now for any $g \in E$ we have

$$\frac{|gB_k \Delta B_k|}{|B_k|} \leq \frac{|B_{k+1}| - |B_k|}{|B_k|} \leq \varepsilon.$$

Thus Γ is amenable.

Abelian groups. There are many ways to see that abelian groups are amenable. One of them is to notice that finitely generated abelian groups are of subexponential growth.

Free groups. The free group \mathbb{F}_2 of rank 2 is the typical example of non-amenable group. Let $a, b \in \mathbb{F}_2$ be the free generators of \mathbb{F}_2 . Denote by $\omega(x)$ the set of all reduced words in \mathbb{F}_2 that start with x . Thus the group can be decomposed as follows

$$\mathbb{F}_2 = \{e\} \cup \omega(a) \cup \omega(a^{-1}) \cup \omega(b) \cup \omega(b^{-1}).$$

To reach a contradiction assume that \mathbb{F}_2 has an invariant mean μ . Since the group is infinite and μ is invariant we have $\mu(\chi_{\{t\}}) = 0$ for all $t \in \mathbb{F}_2$. Moreover, applying the fact that $\omega(x) = x(\mathbb{F}_2 \setminus \omega(x^{-1}))$ for $x \in \{a, b, a^{-1}, b^{-1}\}$ we obtain:

$$\begin{aligned}
1 &= \mu(\chi_{\mathbb{F}_2}) = \mu(\chi_{\{e\}}) + \mu(\chi_{\omega(a)}) \\
&\quad + \mu(\chi_{\omega(a^{-1})}) + \mu(\chi_{\omega(b)}) + \mu(\chi_{\omega(b^{-1})}) \\
&= \mu(\chi_{\{e\}}) + [1 - \mu(\chi_{\omega(a^{-1})})] + [1 - \mu(\chi_{\omega(a)})] \\
&\quad + [1 - \mu(\chi_{\omega(b^{-1})})] + [1 - \mu(\chi_{\omega(b)})] = 3.
\end{aligned}$$

which is a contradiction.

0.4 Operations that preserve amenability

In this section we list basic operations that preserve amenability.

Subgroups. Amenability passes to subgroups. Indeed, let H be a subgroup of an amenable group Γ and \mathcal{R} be a complete set of representatives of the right cosets of H .

Given $\varepsilon > 0$ and a finite set $F \subset H$, let $\mu \in \text{Prob}(\Gamma)$ be an approximately invariant mean that satisfies $\|s.\mu - \mu\|_1 \leq \varepsilon$ for all $s \in F$. Define $\tilde{\mu} \in \text{Prob}(H)$ by

$$\tilde{\mu}(h) = \sum_{r \in \mathcal{R}} \mu(hr).$$

Since for all $s \in F$ we have

$$\begin{aligned}
\|s.\tilde{\mu} - \tilde{\mu}\|_1 &= \sum_{h \in H} |s.\tilde{\mu}(h) - \tilde{\mu}(h)| \\
&= \sum_{h \in H} \left| \sum_{r \in \mathcal{R}} s.\mu(hr) - \mu(hr) \right| \\
&\leq \|s.\mu - \mu\|_1 \leq \varepsilon,
\end{aligned}$$

it follows that $\tilde{\mu}$ is an approximate mean for F , thus H is amenable.

Quotients. Let Γ be an amenable group with normal subgroup H , then Γ/H is amenable. Define a map $\phi : l^\infty(\Gamma/H) \rightarrow l^\infty(\Gamma)$ by $\phi(f)(t) = f(tH)$. Then if μ is an invariant mean of Γ , we have that $\mu \circ \phi$ is an invariant mean of Γ/H . Indeed, for every $f \in l^\infty(\Gamma/H)$ we have

$$\mu(\phi(g.f)) = \mu(t \mapsto f(g^{-1}tH)) = \mu(t \mapsto f(tH))$$

Extensions. Let Γ be a group with normal subgroup H such that both H and Γ/H are amenable, then Γ is amenable.

Let μ_H and $\mu_{\Gamma/H}$ be invariant means of H and Γ/H correspondingly. For $\phi \in l^\infty(\Gamma)$, let $\phi_\mu \in l^\infty(\Gamma/H)$ be defined by $\phi_\mu(gH) = \mu_H((g \cdot \phi)|_H)$, since μ_H is H -invariant we have that ϕ_μ is well-defined. Then the functional $\phi \mapsto \mu_{\Gamma/H}(\phi_\mu)$ is an invariant mean of Γ .

Direct limits. The direct limit of groups $\Gamma = \varinjlim \Gamma_i$ have the property that for each finite set in the limit group Γ , the group generated by this set belongs to one of Γ_i . Thus if all Γ_i are amenable we can apply Følner's definition of amenability to conclude that Γ is also amenable.

In particular, a group is amenable if and only if all its finitely generated subgroups are amenable.

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