Lecture 20: Extensive amenability

by Kate Juschenko

Assume that a discrete group G is acting on a set X. In this chapter we will assume that the action is transitive, thus the Schreier graph of the action is connected. This assumption is natural, since in order to show that the action of G on X is amenable it is enough to show that the action of G on some orbit is amenable.

Consider the direct sum $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$, i.e., the group of all finitely supported sequences with values in $\{0,1\}$ with addition mod 2. Another interpretation of $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ is as the set of all finite subsets of X, denoted by $\mathcal{P}_f(X)$, with multiplication given by the symmetric difference. We will use all this interpretations of $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ depending on which is more appropriate to a context.

The action of an element $g \in G$ on X induces an automorphism of $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ given by the action on the indexes of a sequence $(w_x)_{x \in X} \in \bigoplus_X \mathbb{Z}/2\mathbb{Z}$ by

$$g(w_x)_{x \in X} = (w_{g^{-1}x})_{x \in X}.$$

Thus we can form a semidirect product $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$, which is also called *wreath product* of $\mathbb{Z}/2\mathbb{Z}$ and G, denoted by $\mathbb{Z}/2\mathbb{Z} \wr_X G$. The multiplication is given as follows:

$$(E,g) \cdot (F,h) = (E \Delta g(F), gh) \text{ for } g,h \in G \text{ and } E, F \in \mathcal{P}_f(X).$$

The lamplighter $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$ acts on the cosets $\bigoplus_X (\mathbb{Z}/2\mathbb{Z} \rtimes G)/G$ by multiplication on the left, which can be viewed as an action on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ by the following rule:

$$(E,g)(F) = g(F) \Delta E \text{ for } g \in G \text{ and } E \in \mathcal{P}_f(X).$$

Definition 0.0.1. An action of G on X is **extensively amenable** if the affine action of the semidirect product $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$ on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ is amenable.

In fact, as we will see in Theorem 0.0.3, the $\mathbb{Z}/2\mathbb{Z}$ can be replaced by any amenable group in the definition above.

Note that if the group G is amenable then so is $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$, thus any action of G is extensively amenable. However, as we will see later, the action

of the wobbling group $W(\mathbb{Z})$ on $\mathbb{Z}$ is extensively amenable. Never the less $W(\mathbb{Z})$ contains the free non-abelian group on two generators.

Consider the map $\phi : \mathcal{P}_f(X) \backslash \emptyset \to Proba(X)$, which sends a finite subset to the uniform measure on it. It is a G-map. Then the extensive amenability of the action of G implies, in particular, that there exists G-invariant mean μ on $\mathcal{P}_f(X)$. The push-forward of μ along ϕ is a G-invariant mean on X. Thus we have just showed the following.

Lemma 0.0.2. If the action of G on X is extensively amenable, then this action is amenable.

Note that the converse is not true, we provide examples in the next section.

In the case of non-amenable groups it is always a difficult question to decide whether a particular amenable action is extensively amenable. A good example of this difficulty will be illustrated on the wobbling groups. Before proceeding to examples, we will provide more equivalent definitions of extensive amenability in the theorem below.

Since the definitions we are aiming to present have many different flavors, let us first discuss some notation we are going to use. We consider a measure space $(\{0,1\}^X, \mu)$, where the set of all sequences in $\{0,1\}^X$ is equipped with measure μ which is equal to the product measure of the uniform measures on $\{0,1\}$. This defines the Hilbert space $L^2(\{0,1\}^X, \mu)$ of all square integrable functions on $\{0,1\}^X$ with respect to the measure μ and the inner product

$$\langle f, g \rangle = \int_{\{0,1\}^X} f\overline{g} \ d\mu$$

Denote by χ_A the characteristic function of a set $A \subset \{0,1\}^X$.

Theorem 0.0.3. Let G act transitively on X and fix a point p in X. The following are equivalent:

(i) There exists a net of unit vectors $f_n \in L^2(\{0,1\}^X, \mu)$ such that for every $g \in G$

$$\|gf_n - f_n\|_2 \to 0 \text{ and } \|f_n \cdot \chi_{\{(w_x) \in \{0,1\}^X : w_p = 0\}}\|_2 \to 1;$$

3

(ii) The action of G on X is extensively amenable;

(iii) There exists a constant $C > 0$ such that the action of G on $\mathcal{P}_f(X)$ admits an invariant mean giving weight C to the collection of sets containing p;

(iv) The action of G on $\mathcal{P}_f(X)$ admits an invariant mean giving the full weight to the collection of sets containing p;

(v) The action of G on $\mathcal{P}_f(X)$ admits an invariant mean such that for all $E \in \mathcal{P}_f(X)$ it gives the full weight to the collection of sets containing E;

Proof. (i) $\implies$ (ii). Let $f_n \in L^2(\{0,1\}^X, \mu)$ be a net of functions which satisfy (i). Note that, replacing f_n with $f_n \cdot \chi_{\{w_p=0\}}$ and normalizing we obtain a net of unit vectors in $L^2(\{0,1\}^X, \mu)$ such that $f_n \cdot \chi_{\{w_p=0\}} = f_n$ and

$$\|gf_n - f_n\|_2 \to 0 \text{ for all } g \in G.$$

The group $\{0,1\}^X$ is the Pontriagin dual of $\mathcal{P}_f(X)$, with the pairing function $\{0,1\}^X \times \mathcal{P}_f(X)$ given by

$$\langle w, E \rangle = exp(i\pi \sum_{x \in E} w_x)$$

Thus we can define the Fourier transform $\hat{f} \in l^2(\mathcal{P}_f(X))$ of $f \in L^2(\{0,1\}^X, \mu)$ by

$$\hat{f}(E) = \int_{\{0,1\}^X} f(w)\langle w, E \rangle d\mu(w).$$

We will show that the net $\hat{f}_n$ is approximately invariant under the action of $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$.

From the properties of Fourier transform and an easy fact that it is G-equivariant, we have

$$\|g\hat{f}_n - \hat{f}_n\|_2 = \|(gf_n - f_n)\hat{}\|_2 = \|gf_n - f_n\|_2 \to 0.$$

Thus $\hat{f}_n$ remains G-invariant.

Since the action of G on X is transitive it is sufficient to show that $\hat{f}_n$ is $\{p\}$-almost invariant, where $\{p\} \in \mathcal{P}_f(X)$:

$$\{p\}\hat{f}_n(E) = \hat{f}_n(E\Delta\{p\})$$

$$= \int_{\{0,1\}^X} f_n\langle w, E\Delta\{p\}\rangle d\mu(w)$$

$$= \int_{\{0,1\}^X} f_n\chi_{\{w_p=0\}}(w)\langle w, E\Delta\{p\}\rangle d\mu(w)$$

$$= \int_{\{0,1\}^X} f_n\chi_{\{w_p=0\}}(w)exp(i\pi \sum_{x\in E\Delta\{p\}} w(x))d\mu(w)$$

$$= \int_{\{0,1\}^X} f_n\chi_{\{w_p=0\}}(w)exp(i\pi \sum_{x\in E} w(x))d\mu(w)$$

$$= \hat{f}_n(E)$$

Thus $\hat{f}_n$ is $\{p\}$-invariant. Since $\hat{f}_n \in l^2(\mathcal{P}_f(X))$ are almost invariant, we have that $\hat{f}_n^2 \in l_1(\mathcal{P}_f(X))$ are also almost invariant. Indeed, this follows from Cauch-Schwarz inequality:

$$\|gf^2 - f^2\|_1 = \|(gf - f)(gf + f)\|_1$$
$$\leq \|(gf - f)\|_2\|(gf + f)\|_2$$
$$\leq 2\|(gf - f)\|_2,$$

for every unit vector $f \in l^2(\mathcal{P}_f(X))$ and $g \in G$.

Taking a cluster point of the net $\hat{f}_n^2$ in weak$*$-topology we obtain an $\bigoplus_X \mathbb{Z}/2\mathbb{Z}\rtimes G$-invariant mean on $\mathcal{P}_f(X)$, thus the action is extensively amenable.

To show that (ii) implies (iii), assume that m is $\mathcal{P}_f(X) \rtimes G$-invariant mean on $\mathcal{P}_f(X)$. Let $\mathcal{F}_p$ be the set of all finite subsets of X containing p. Then $\{p\}.\mathcal{F}_p = \mathcal{F}_p^c$, therefore we have $m(\mathcal{F}_p) = m(\{p\}.\mathcal{F}_p) = m(\mathcal{F}_p^c) = 1/2$.

(iii) $\Longrightarrow$ (iv). Let now m be a G-invariant mean on $\mathcal{P}_f(X)$ with $m(\mathcal{F}_p) = C$ for some $C > 0$.

Fix $k \in \mathbb{N}$ and define G-equivariant map

$$U_k : \mathcal{P}_f(X)^k \to \mathcal{P}_f(X)$$

by

$$U_k(F_1, \ldots, F_k) = F_1 \cup \ldots \cup F_k$$

5

Let $m^{\times k}$ be a product measure on $\mathcal{P}_f(X)^k$, i.e., for all $F_1, \ldots, F_k \in \mathcal{P}_f(X)$ we have
$$m^{\times k}(F_1, \ldots, F_k) = m(F_1) \cdot \ldots \cdot m(F_k).$$

Note that $m^{\times k}$ is invariant under diagonal action of G.

Define m_k to be the push-forward of $m^{\times k}$ with respect to U_k. Since U_k and $m^{\times k}$ are G-invariant, m_k is also G-invariant. Moreover, for all $E \in \mathcal{X}$ we have
$$U_k^{-1}(\mathcal{F}_p^c) \subseteq \mathcal{F}_p^c \times \ldots \times \mathcal{F}_p^c.$$

Thus $1 - m_k(\mathcal{F}_p) \leq (1 - m(\mathcal{F}_p))^k = (1 - C)^k$. Taking a cluster point in the weak*-topology we obtain a mean that satisfies (v).

(v) $\Longrightarrow$ (iv). It is obvious.

(v) $\Longrightarrow$ (v). For a finite set $E \subset X$ the set of finite subsets that contain E is $\bigcap_{x \in E} \mathcal{F}_x$ Since the action of G on X is transitive we have $\mu(\mathcal{F}_x) = 1$ for every $x \in X$, thus $\mu(E) = 1$ and (v) follows.

(iv) $\Longrightarrow$ (i). Let m be G-invariant mean giving a full weight to $\mathcal{F}_p$. Then there exists a net $m_n \in l_1(\mathcal{P}_f(X))$ such that $\|g.m_n - m_n\|_1 \to 0$ for all $g \in G$ and $m_n(\mathcal{F}_p) = 1$. For a sequence $w \in \{0,1\}^X$, define

$$f_n(w) = \sum_{E \in \mathcal{P}_f(X)} m_n(F) 2^{|F|} \chi_F(w),$$

where $\chi_F(w)$ is 1 is $F \cap w \neq \emptyset$ and 0 otherwise. Since m_n is supported on $\mathcal{F}_p$ we have $f_n \cdot \chi_{\{(w_x) \in \{0,1\}^X : w_p = 0\}} = f_n$. Since $\|\chi_F\|_1 = 2^{-|F|}$ we have that $\|f_n\| = 1$.

Moreover f_n is G-invariant. Indeed,

$$\|g.f_n - f_n\| \leq \sum_{E \in \mathcal{P}_f(X)} \|m_n(E) 2^{|F|} g.\chi_F - m_n(E) 2^{|F|} \chi_F\|_1$$

$$= \sum_{E \in \mathcal{P}_f(X)} \|m_n(gE) 2^{|F|} \chi_F - m_n(E) 2^{|F|} \chi_F\|_1$$

$$= \sum_{E \in \mathcal{P}_f(X)} \|m_n(gE) - m_n(E)\|_1$$

$$= \|g.m_n - m_n\|.$$

Since $\|g.f_n^{1/2} - f_n^{1/2}\|_2^2 \leq \|g.f_n - f_n\|_1$ we obtain the desired.

$\square$

The following lemma is an interesting observation of the fact that the group $\mathbb{Z}/2\mathbb{Z}$ is not very essential for extensively amenable actions and can be replaced by any other amenable group.

Lemma 0.0.4. Assume that the action of G on X is transitive and extensively amenable. Then for any amenable group H the action of $\bigoplus_X H \rtimes G$ on $\bigoplus_X H$ is amenable;

Proof. Fix a point p in X. We may assume that G is generated by a finite set S. Let A be a finite subset of H. For every $\varepsilon > 0$ we need to find a finite set E in $\bigoplus_X H$ such that it is ε-invariant for both S and $\{(e, h\delta_p) : h \in A\}$.

Fix $\varepsilon > 0$. By assumptions we can find a finite set $F \subset \mathcal{P}_f(X)$, which is (S, ε)-invariant and such that all elements of F contain the point p. Without loss of generality we may assume that all elements of F are of the same size which is equal to k. Indeed, since the action of G preserves cardinality, we can decompose F as disjoint union $F = F_1 \cup \ldots \cup F_n$ each element of F_i has the same cardinality. Assume that none of these components is $(S, |S| \cdot \varepsilon)$-invariant, i.e., for every F_i there exists $s \in S$ such that

$$|sF_i \backslash F_i| > \varepsilon \cdot |S| \cdot |F_i|.$$

Note that $|sF\backslash F| = \sum_{i=1}^{n} |sF_i\backslash F_i|$. Summing the last equation by s we obtain:

$$\sum_{s \in S} |sF_i\backslash F_i| = \sum_{s \in S}\sum_{i=1}^{n} |sF_i\backslash F_i| > \varepsilon \cdot |S| \cdot \sum_{i=1}^{n} |F_i| = \varepsilon \cdot |S| \cdot |F|,$$

which is impossible.

Let F_A be a (A, ε)-Følner set. Consider the set taken with multiplicities:

$$E = \{\phi \in \bigoplus_X H : \mathrm{supp}(\phi) \in F, \phi(X) \subset F_A\}.$$

Here each ϕ comes with multiplicity $\{|\{C \in F : supp(\phi) \subseteq C\}|\}$. Then $|E| = |F_A|^k \cdot |F|$ and

$$|sE\backslash E| \leq \varepsilon|F_A| \cdot |F| = \varepsilon|E|.$$

Moreover, for all $h \in A$ we have

$$|(e, h\delta_p)E \backslash E| \leq |hF_A \backslash F_A| \cdot |F_A|^{k-1} \cdot |F| = \varepsilon / |F_A| \cdot |E|.$$

Define a function $f : \bigoplus_X H \to \mathbb{R}_+$:

$$f(\nu) = \sum_{\phi \in E} \delta_\phi(\nu) \text{ for all } \nu \in \bigoplus_X H,$$

Here we write sum instead of χ_E in order to specify that the values of f can depend on the multiplicities that appear in the set E. It is immediate that $\|f\|_1 = |E|$ and $\|g.f - f\|_1 = |sE \Delta E|$, therefore we have the statement of the lemma. $\qquad \square$

We will use the following theorem for applications in the later sections.

Theorem 0.0.5. Assume that the action of G on X is extensively amenable and $H < G$ is a subgroup. Assume, in addition, that a set $Y \subset X$ is invariant under the action of H. Then the action of H on Y is extensively amenable, in particular, it is amenable.

Proof. Define a $\mathcal{P}_f(Y) \rtimes H$-equivariant map on $\mathcal{P}_f(X)$ into $\mathcal{P}_f(Y)$ by intersecting a finite subset of X with Y. The push-forward of $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$-invariant mean on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ along our H-map is $\bigoplus_Y \mathbb{Z}/2\mathbb{Z} \rtimes H$-invariant mean on $\bigoplus_Y \mathbb{Z}/2\mathbb{Z}$. $\qquad \square$

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14

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