Lecture 21: Recurrent actions

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0.1 Several equivalent definitions of amenable actions

In this section we study amenable actions. Assume a discrete group Γ acts on a set X, that is there is a map $(g, x) \mapsto gx$ from $\Gamma \times X$ to X such that $(gh)x = g(h(x))$ for all g, h in Γ. Then the group acts on the space of functions on X by $g.f(x) = f(g^{-1}x)$. A mean on X is a linear functional $m \in l^\infty(X)^*$ which satisfies $m(1_X) = 1$, $m(f) \geq 0$ for all $f \in l^\infty(X)$. This automatically implies $\|m\| \leq 1$. Denote the set of all means on X by $M(X)$. The group Γ acts on $M(X)$, a fixed point of this action (if such exists) is an invariant mean.

There is one-to-one correspondence between means on X and finitely additive probability measures on X, see Appendix **??** for detailed overview.

Definition 0.1.1. An action of a discrete group Γ on a set X is amenable is X admits an invariant mean.

We will identity the set $\{0,1\}^X$ of all sequences indexed by X with values in $\{0,1\}$ with set of all subsets of X.

Theorem 0.1.2. Let a discrete group Γ act on a set X. Then the following are equivalent:

(i) An action of a discrete group Γ on a set X is amenable;

(ii) There exists a map $\mu : \{0,1\}^X \to [0,1]$, which satisfies

- μ is finitely additive, $\mu(X) = 1$,
- $\mu(gE) = \mu(E)$ for all $E \subseteq X$ and $g \in \Gamma$.

(iii) *Følner condition.* For every finite set $E \subset \Gamma$ and for every $\varepsilon > 0$ there exists $F \subseteq X$ such that for every $g \in E$ we have:

$$|gF \Delta F| \leq \varepsilon \cdot |F|$$

(iv) *Reiters condition (or approximate mean condition).* For every finite set $S \subset \Gamma$ and for every $\varepsilon > 0$ there exists a non-negative function $\phi \in l^1(X)$ such that $\|\phi\| = 1$ and $\|g.\phi - \phi\|_1 \leq \varepsilon$.

Proof. The proof of the theorem is exactly the same as in Lemma **??**. $\square$

Remark 0.1.3. Note that if G-invariant finitely additive probability measure μ on X gives a full weight to a subset X', i.e., $\mu(X') = 1$, then we can define a finitely additive probability measure on X by $\overline{\mu}(A) = \mu(A \cap X')$. It is easy to check that it remains G-invariant. Moreover, going through the equivalences in the last theorem, it is immediate that the Følner sets can be chosen as subsets of X' as well as the approximate means can be chosen to be supported on X'. This simple observation will be important for the applications in the next sections.

Let $\Phi : X \to Y$ be a map. Then we have a canonical map $\overline{\Phi} : l^\infty(Y) \to l^\infty(X)$ defined by $\overline{\Phi}(f) = f \circ \Phi$ for all $f \in l^\infty(Y)$. Consider the dual $\overline{\Phi}^* : l^\infty(X)^* \to l^\infty(Y)^*$ of $\overline{\Phi}$. **A push-forward** of a mean $\mu \in l^\infty(X)^*$ with respect to Φ is the mean $\overline{\Phi}^*(\mu)$, we will denote it by $\Phi_*\mu$. It is straightforward that if Φ is a Γ-map then the push-forward of a Γ-invariant mean is Γ-invariant. Let $m \in M(M(X))$ be a mean on the space of means.

The **barycenter** of m is $\overline{m} \in M(X)$ defined by $\overline{m}(f) = m(\mu \mapsto \mu(f))$ for all $f \in l^\infty(X)$. It is easy to check that if m is Γ-invariant then $\overline{m}$ is also Γ-invariant.

Theorem 0.1.4. If Γ acts amenably on a set X and the stabilizer of each point of X is amenable, then Γ is amenable.

Proof. Define a Γ-map $\Phi : X \to M(\Gamma)$ as follows. For each point $x \in X$, the stabilizer of it acts amenably on $M(\Gamma)$, thus it has a fixed point μ_x. Let Y be a set of orbit representatives and $X = \bigcup\limits_{x \in Y} Orb(x, \Gamma)$ be a decomposition of X into the (disjoint) orbits of Γ. Then if $x \in Y$ we define $\Phi(x) = \mu_x$ and if $y = gx$ for $x \in Y$ we define $\Phi(y) = g.\mu_x$. In other words, Φ is the orbital map. It is straightforward to check that Φ is a Γ-map.

Let $\Phi_*\mu \in M(M(\Gamma))$ be the push-forward of μ. Then its barycenter is an invariant mean on Γ. $\square$

Lemma 0.1.5. If X has subexponential growth then an action of any finitely generated group on it is amenable.

0.2 Recurrent actions: definition and basic properties

Let G be a finitely generated group with finite symmetric generating set S.

If G acts transitively on X the **Schreier graph** $\Gamma(X, G, S)$ is the graph with the set of vertices identified with X, the set of edges is $S \times X$, where an edge (s, x) connects x to $s(x)$.

Choose a measure μ on G such that support of μ is a finite generating set of G and $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Consider the Markov chain on X with transition probability from x to y equal to $p(x, y) = \sum_{g \in G, g(x)=y} \mu(g)$.

Definition 0.2.1. The action is called **recurrent** if the probability of returning to x_0 after starting at x_0 is equal to 1 for some (hence for any) $x_0 \in X$. An action is transient if it is not recurrent.

It is well known (see [101, Theorems 3.1, 3.2]) that recurrence of the described Markov chain does not depend on the choice of the measure μ, if the measure is symmetric, and has finite support generating the group.

Definition 0.2.2. An action of G on X is **recurrent** if a Markov chain that corresponds to a measure that supported on a symmetric finite generating set of G is recurrent.

Note that, the action of G on itself is recurrent if and only if G is virtually $\{0\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$. Moreover, all recurrent actions are amenable.

The following theorem is a part of more general Nash-Williams criteria for recurrency. It will be very useful in the applications.

Theorem 0.2.3. Let Γ be a connected graph of uniformly bounded degree with set of vertices V. Suppose that there exists an increasing sequence of finite subsets $F_n \subset V$ such that $\bigcup_{n \geq 1} F_n = V$, the subsets ∂F_n are pairwise disjoint, and

$$\sum_{n \geq 1} \frac{1}{|\partial F_n|} = \infty,$$

where ∂F_n is the set of vertices of F_n adjacent to the vertices of $V \setminus F_n$. Then the simple random walk on Γ is recurrent.

We will also use a characterization of transience of a random walk on a locally finite connected graph (V, E) in terms of electrical network. The **capacity** of a point $x_0 \in V$ is the quantity defined by

$$cap(x_0) = \inf \left\{ \left(\sum_{(x,x') \in E} |a(x) - a(x')|^2 \right)^{1/2} \right\}$$

where the infimum is taken over all finitely supported functions $a : V \to \mathbb{C}$ with $a(x_0) = 1$. We will use the following

Theorem 0.2.4 ([101], Theorem 2.12)**.** The simple random walk on a locally finite connected graph (V, E) is transient if and only if $cap(x_0) > 0$ for some (and hence for all) $x_0 \in V$.

0.3 Recurrent actions are extensively amenable

In this section we discuss the relation of recurrent actions to extensive amenability. The connecting point is the definition of extensive amenability given in Theorem **??** (**??**).

Let $\mathcal{H}_i$ be a collection of Hilbert spaces indexed by a set I. Fix a sequence of normal vectors $\xi_i \in \mathcal{H}_i$. Then the algebraic (incomplete) tensor product of $\mathcal{H}_i$ is the set of all linear combinations of $\bigotimes_{i \in I} \phi_i$, where all but finitely many ϕ_i are equal to ξ_i. It carries an inner product, which is defined by

$$\langle \bigotimes_{i \in I} \phi_i, \bigotimes_{i \in I} \nu_i \rangle = \prod_{i \in I} \langle \phi_i, \nu_i \rangle_{\mathcal{H}_i}$$

An infinite tensor product of Hilbert spaces is the Hilbert space is defined to be the completion of the algebraic tensor product by the norm defined by the above inner product.

Consider a Hilbert space of square integrable functions $L_2(\{0,1\}^X, \mu)$ with respect to the measure μ given by the product of measure m on $\{0, 1\}$, where $m(0) = m(1) = \frac{1}{2}$.

It is natural to consider the Hilbert space $L_2(\{0,1\}^X, \mu)$ as an infinite tensor power of the Hilbert space $L_2(\{0,1\}^X, m)$.

A function $f \in L_2(\{0,1\}^X, \mu)$ is called *a product of independent random variables* if there are functions $f_x : \{0, 1\} \to \mathbb{C}$ such that $f(w) = \prod_{x \in X} f_x(w_x)$.

Equivalently, if we consider $L_2(\{0,1\}^X, \mu)$ as the infinite tensor power, then the condition that f is a product of random independent variables means that f is an elementary tensor in $L_2(\{0,1\}^X, \mu)$.

Theorem 0.3.1. Let G be a finitely generated group acting transitively on a set X and fix a point p in X. There exists a sequence of functions $\{f_n\}$ in $L_2(\{0,1\}^X, \mu)$ with $\|f_n\|_2 = 1$ given by a product of random independent variables that satisfy

(i) $\|gf_n - f_n\|_2 \to 0$ for all $g \in G$,

(ii) $\|f_n \cdot \chi_{\{(w_x)\in\{0,1\}^X : w_p = 0\}}\|_2 \to 1$,

if and only if the action of G on X is recurrent.

Proof 1. Denote by (X, E) the Schreier graph of the action of G on X with respect to S. Suppose that the simple random walk on (X, E) is recurrent. By Theorem 0.2.4, there exists $a_n = (a_{x,n})_x$ a sequence of finitely supported functions such that $a_{p,n} = 1$ and

$$\sum_{x\sim x'} |a_{x,n} - a_{x',n}|^2 \to 0.$$

Without loss of generality we may assume that $0 \le a_{x,n} \le 1$. Indeed, we can replace all values $a_{x,n}$ that are greater than 1 by 1 and those that are smaller than 0 by 0, this would not increase the differences $|a_{x,n} - a_{x',n}|$.

For $0 \le t \le 1$ consider the unit vector $\xi_t \in L_2(\{0,1\}, m)$ defined by

$$(\xi_t(0), \xi_t(1)) = (\sqrt{2}\cos(t\pi/4), \sqrt{2}\sin(t\pi/4)).$$

Define $f_{x,n} = \xi_{1-a_{x,n}}$ and $f_n = \bigotimes_{x\in X} f_{x,n}$.

To show that $\|gf_n - f_n\|_2 \to 0$ for all $g \in G$, it the same as to show that $\langle gf_n, f_n\rangle \to 1$ for all $g \in \Gamma$. It is sufficient to show this for $g \in S$. Since $\cos(x) \ge e^{-x^2}$, whenever $|x| \le \pi/4$, we have

$$
\begin{aligned}
\langle g f_n, f_n\rangle &= \prod_x \langle f_{x,n}, f_{gx,n}\rangle \\
&= \prod_x \cos\frac{\pi}{4}(a_{x,n} - a_{gx,n}) \\
&\geq \prod_x \exp\left(-\frac{\pi^2}{16}(a_{x,n} - a_{gx,n})^2\right) \\
&\geq \exp\left(-\frac{\pi^2}{16}\sum_{x\sim x'} |a_{x,n} - a_{x',n}|^2\right)
\end{aligned}
$$

By the selection of $a_{x,n}$, the last value converges to 1.

Since $f_{p,n} = \xi_0 = (1,0)$ we have

$$f_n \chi_{\{(w_x)\in\{0,1\}^X : w_p = 0\}} = f_n.$$

Let us prove the other direction of the theorem. Define the following pseudometric on the unit sphere of $L_2(\{0,1\}, m)$ by

$$d(\xi,\eta) = \inf_{w\in\mathbb{C}, |w|=1} \|w\xi - \eta\| = \sqrt{2 - 2|\langle\xi,\eta\rangle|}.$$

Assume that there is a sequence of products of random independent variables $\{f_n\}$ in $L_2(\{0,1\}^X, \mu)$ that satisfy the conditions of the theorem, i.e.,

$$f_n(w) = \prod_{x\in X} f_{n,x}(w_x).$$

We can assume that the product is finite. Replacing $f_{n,x}$ by $f_{n,x}/\|f_{n,x}\|$ we can assume that $\|f_{n,x}\|_{l_2(\{0,1\},m)} = 1$. Define $a_{x,n} = d(f_{x,n}, 1)$. It is straightforward that $(a_{x,n})_{x\in X}$ has finite support and

$$\lim_n a_{p,n} = \sqrt{2 - \sqrt{2}} > 0.$$

Moreover for every $g \in G$

$$
\begin{aligned}
|\langle g f_n, f_n\rangle| &= \prod_x |\langle f_{n,x}, f_{n,gx}\rangle| \\
&= \prod_x (1 - d(f_{n,x}, f_{n,gx})^2/2) \\
&\leq \exp\left(-\sum_x d(f_{n,x}, f_{n,gx})^2/2\right).
\end{aligned}
$$

Since by assumption $|\langle gf_n, f_n\rangle| \to 1$ and $\sum_x d(f_{n,x}, f_{n,gx})^2 \geq 0$ we have

$$\sum_x d(f_{n,x}, f_{n,gx})^2 \to 0.$$

By definition of the Schreier graph and the triangle inequality for d,

$$\begin{aligned}\sum_{(x,x')\in E} |a_{x,n} - a_{x',n}|^2 &= \sum_{g\in S}\sum_x |a_{x,n} - a_{gx,n}|^2 \\ &\leq \sum_{g\in S}\sum_x d(f_{n,x}, f_{n,gx})^2 \to 0.\end{aligned}$$

This proves that $cap(p) = 0$ in (X, E), and hence by Theorem 0.2.4 that the simple random walk on (X, E) is recurrent. $\square$

A more direct proof of the amenability of lamps from recurrency of the action is the following.

Direct proof of recurrency implies extensive amenability. We again use the characterization of recurrency in terms of capacity, which implies that there exists $a_n : X \to \mathbb{R}_+$ be a sequence of finitely supported functions such that for a fixed point $p \in X$ we have $a_n(p) = 1$ for all n and

$$\|ga_n - a_n\|_2 \to 0 \text{ for all } g \in G.$$

Moreover, we can assume $0 \leq a_n(x) \leq 1$ for all $x \in X$ and n.

Define $\xi_n : \mathcal{P}_f(X) \to \mathbb{R}_+$ by $\xi_n(\emptyset) = 1$ and

$$\xi_n(F) = \prod_{x\in F} a_n(x).$$

We claim that $\nu_n := \xi_n/\|\xi_n\|_2 \in l_2(\mathcal{P}_f(X)$ is almost invariant under the action of $\mathcal{P}_f(X) \rtimes G$. Thus taking a cluster point in the weak*-topology of $\nu_n^2 \in l_1(\mathcal{P}_f(X))$ we obtain a $\mathcal{P}_f(X) \rtimes G$-invariant mean on $\mathcal{P}_f(X)$.

To prove the claim, note that since $a_n(p) = 1$ for all n the functions ν_n are automatically invariant under the action of $\{p\} \in \mathcal{P}_f(X)$. From the transitivity of action of G on X we have that it is sufficient to show that ν_n are almost invariant under the action of G. Since $\|g\nu_n - \nu_n\| = 2 - 2\langle g\nu_n, \nu_n\rangle$, it is sufficient to show that $\langle g\nu_n, \nu_n\rangle \to 1$. The direct verification shows that

$$
\begin{aligned}
\|\xi_n\|^2 &= \langle \xi_n, \xi_n \rangle \\
&= \prod_{x \in X} \left(1 + a_n(x)^2\right) \\
&= \prod_{x \in X} \left(1 + a_n(g^{-1}x)^2\right)
\end{aligned}
$$

and

$$
\langle g\xi_n, \xi_n \rangle = \prod_{x \in X} \left(1 + a_n(g^{-1}x) a_n(x)\right).
$$

Thus we have

$$
\left(\frac{\langle \xi_n, \xi_n \rangle}{\langle g\xi_n, \xi_n \rangle} \right)^2 = \prod_{x \in X} \frac{(1 + a_n(x)^2)\,(1 + a_n(g^{-1}x)^2)}{\left(1 + a_n(g^{-1}x) a_n(x)\right)^2}
$$

Since $\log(t) \leq t - 1$ for all $t > 0$ and $0 \leq a_n(x) \leq 1$ we have

$$
\begin{aligned}
0 &\leq 2 \log \frac{\langle \xi_n, \xi_n \rangle}{\langle g\xi_n, \xi_n \rangle} \\
&= \sum_{x \in X} \log \frac{(1 + a_n(x)^2)\,(1 + a_n(g^{-1}x)^2)}{\left(1 + a_n(g^{-1}x) a_n(x)\right)^2} \\
&= \sum_{x \in X} \frac{\left(a_n(x) - a_n(g^{-1}x)\right)^2}{\left(1 + a_n(g^{-1}x) a_n(x)\right)^2} \\
&= \|g a_n - a_n\| \to 0.
\end{aligned}
$$

□

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