Lecture 3: Elementary amenable groups

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Mahlon Day in [27] defined the following class of groups.

Definition 0.0.1. The smallest class of groups that contain finite and abelian groups and is closed under taking subgroups, quotients, extensions and directed unions is called the class of *elementary amenable groups.* We denote this class by EG.

As we proved in the previous section all elementary amenable groups are amenable. This was already known to von Neumann and remained the only source of examples for several decades.

Here we list several examples of elementary amenable groups.

Theorem 0.0.2. Every solvable group is in EG. In particular, every nilpotent group is in EG.

Proof. Let Γ be a solvable group. By definition it admits a composition series

$$\{1\} = \Gamma_0 \trianglelefteq \Gamma_1 \trianglelefteq \ldots \trianglelefteq \Gamma_k = \Gamma,$$

such that Γ_i/Γ_{i-1} is an abelian group, for $i = 1, \ldots, k$. Since Γ can be constructed inductively using extensions and abelian groups, we have that Γ is in EG.

One of the definitions of nilpotent group Γ is the existence of upper central series that terminates at it:

$$\{1\} = Z_0 \trianglelefteq Z_1 \trianglelefteq \ldots \trianglelefteq Z_k = \Gamma,$$

where $Z_{i+1} = \{x \in \Gamma : \forall y \in \Gamma : [x, y] \in Z_i\}$. In particular, Z_{i+1}/Z_i is the center of Γ/Z_i, which implies that Γ is solvable. □

One of the numerous examples of nilpotent groups is *the Heisenberg group.*

In [24], Chou describes several important properties of the class of elementary amenable groups. Milnor, [72], and Wolf, [102], showed that if a finitely generated solvable group has subexponential growth then it contains a nilpotent subgroup of finite index. Chou managed to extend this result to elementary amenable groups. We refer to Osin's paper, [86], for more results in this direction. In particular, Chou's result implies that every elementary amenable group has either exponential or polynomial growth. Moreover, he proved that the operations of taking subgroups and quotient are redundant in the definition of elementary amenable groups. This gave a start in finding examples of amenable groups which are not elementary amenable. We discuss the constructions of Chou in the next two sections.

0.1 Simplification of the class of elementary amenable groups

The main goal of this section is to simplify the set of operations needed for construction of elementary amenable groups.

Let EG_0 be the class of all finite and abelian groups. Assume that α is an ordinal such that for all ordinals $\beta < \alpha$ we have already constructed the class EG_β. If α is a limit ordinal we set

$$EG_\alpha = \bigcup_{\beta<\alpha} EG_\beta.$$

Otherwise we set EG_α as the class of groups which can be obtained from $EG_{\alpha-1}$ by taking one time either extension or direct union.

Lemma 0.1.1. For every ordinal α the class EG_α is closed under taking subgroups and quotients.

Proof. The statement of the lemma is trivial for EG_0. Assume that $\alpha > 0$ and for all $\beta < \alpha$ the class EG_β is closed under taking subgroups and quotients. Let $G \in EG_\alpha$, K be a subgroup of G and H be a normal subgroup of G with canonical quotient map $q : G \to G/H$. We have to show that both G/H and K are in EG_α.

If α is a limit ordinal then by the definition $G \in EG_\beta$ for some $\beta < \alpha$, therefore G/H and K are in EG_α.

If $\alpha - 1$ exists then we have two possible cases: either G is an extension of two groups in $EG_{\alpha-1}$ or G is a direct union of groups from $EG_{\alpha-1}$.

In the first case we have a short exact sequence $e \to F \to G \to E \to e$ for some groups F and E in the class $EG_{\alpha-1}$. Then K is the extension of

$F \cap K$ by a subgroup of E. Thus K is in EG_α. In its turn G/H is in EG_α, since it is the extension of $q(F)$ and $q(E)$.

Now assume that G is the direct union of $\{G_i\}_{i\in I}$ for some index set I. Then K is the direct union of $\{G_i \cap K\}_{i\in I}$ and G/H is the direct union of $\{q(G_i)\}_{i\in I}$. By assumption, $G_i \cap K$ and $q(G_i)$ are in $EG_{\alpha-1}$, therefore K and G/H are in EG_α. By transfinite induction, we obtain the statement of the lemma. □

Now we are ready to simplify the class of elementary amenable groups.

Theorem 0.1.2. The class of elementary amenable groups is the smallest class, which contains all finite and all abelian groups and is closed under taking extensions and direct limits.

Proof. We will show that

$$EG = \bigcup \{EG_\alpha : \alpha \text{ is ordinal}\},$$

which concludes the statement.

Obviously, $\bigcup_\alpha EG_\alpha$ is closed under taking extensions and, by the previous lemma, it is also closed under taking subgroups and quotients. To see that it is also closed under taking direct limits, let G be a direct union of $\{G_i\}$ for $G_i \in \bigcup_\alpha EG_\alpha$. Thus for each G_i there exists an ordinal α_i such that $G_i \in EG_{\alpha_i}$. Then for $\alpha = \sup_i \alpha_i$ we have $G \in EG_{\alpha+1}$. Therefore, $\bigcup_\alpha EG_\alpha$ coincides with the class of elementary amenable groups. □

As a consequence of the previous theorem we have.

Corollary 0.1.3. Every finitely generated simple elementary amenable is finite.

Proof. Let G is a finitely generated simple group in EG. Since G is finitely generated we have that if α is the smallest ordinal such that $G \in EG_\alpha$, then α is not a limit ordinal. If $\alpha > 0$ then since G is simple it must be a direct union of groups $\{G_i\}$ in $G_{\alpha-1}$. Since G is finitely generated it belongs to one of the G_i, which contradicts minimality of α. Thus $\alpha = 0$ which implies that G is finite. □

0.2 Growth of elementary amenable groups

Let Γ be a group generated by a finite set S. We say that a non-decreasing function f dominates a non-decreasing function h, $f \preceq h$, if there exists a constant $\alpha, C > 0$ such that for for all $n > 1$ we have

$$h(n) \leq Cf(\alpha n)$$

Functions f and h are equivalent, $f \sim h$, if we have both $h \preceq f$ and $f \preceq h$.

The growth function of Γ is defined to be the size of the n-th ball along the generating set S:

$$\gamma_G^S = |B_n(S)|.$$

It is easy to check that if S' is another generating set then $\gamma_G^S \sim \gamma_G^{S'}$. Thus we can omit the subscript that corresponds to the generating set and write γ_G instead.

The growth function γ is *polynomial* is $\gamma(n) \preceq n^\beta$ for some $\beta > 0$. It is *exponential* if $\gamma \succeq e^n$. If a growth function γ is neither polynomial or exponential, we say that γ is of *intermediate growth*. The growth function is *subexponential* if $\lim_{n\to\infty} \gamma^{1/n}(n) = 1$.

Milnor, [73], was the first to notice that the $\lim_{n\to\infty} \gamma^{1/n}(n)$ exists. Thus each finitely generated group has either exponential or subexponential growth.

The combination of the results of Milnor, [72], and Wolf, [102], give the following theorem.

Theorem 0.2.1 (Milnor-Wolf)**.** Every solvable group has either exponential or polynomial growth.

The proof of following theorem of Chou relies heavily on the previous theorem.

Theorem 0.2.2 (Chou)**.** Every elementary amenable group has either polynomial or exponential growth.

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