Lecture 5: Examples of non-elementary amenable groups. The full topological group of Cantor minimal system.

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We begin with basic definitions. The Cantor space is denoted by $\mathbf{C}$, it is characterized up to a homeomorphism as a compact, metrizable, perfect and totally disconnected topological space. The group of all homeomorphisms of the Cantor space is denoted by $Homeo(\mathbf{C})$. *A Cantor dynamical system* $(T, \mathbf{C})$ is the Cantor space together with its homeomorphism T.

Let A be a finite set, we will call it an alphabet. A basic example of Cantor space is the set of all sequences in A indexed by integers, $A^{\mathbb{Z}}$, and considered with product topology. A sequence $\{\alpha_i\}$ converges to α in this space if and only if for all n there exists i_0 such that for all $i \geq i_0$, we have that α_i coincides with α on the interval $[-n, n]$.

The basic example of a Cantor dynamical system is *the shift* on $A^{\mathbb{Z}}$, i.e., the map $s : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by

$$s(x)(i) = x(i+1)$$

for all $x \in A^{\mathbb{Z}}$.

The system $(T, \mathbf{C})$ is *minimal* if there is no non-trivial closed T-invariant subset in $\mathbf{C}$. Equivalently, the closure of the orbit of T of any point p in $\mathbf{C}$ coincides with $\mathbf{C}$:

$$\overline{\{T^i p : i \in \mathbb{Z}\}} = \mathbf{C}$$

One of the basic examples of the Cantor minimal system is *the odometer*, defined by the map $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$:

$$\sigma(x)(i) = \begin{cases} 0, & \text{if } i < n, \\ 1, & \text{if } i = n, \\ x(i), & \text{if } i > n \end{cases}$$

where n is the smallest integer such that $x(n) = 0$, and $\sigma(1) = 0$. One can verify that the odometer is minimal homeomorphism.

While shift is not minimal, one can construct many Cantor subspaces of $A^{\mathbb{Z}}$ on which the action of the shift is minimal. Closed and shift-invariant subsets of $A^{\mathbb{Z}}$ are called *subshifts*.

A sequence $\alpha \in A^{\mathbb{Z}}$ is *homogeneous*, if for every finite interval $J \subset \mathbb{Z}$, there exists a constant $k(J)$, such that the restriction of α to any interval of the size $k(J)$ contains the restriction of α to J as a subsequence. In other words, for any interval J' of the size $k(J)$, there exist $t \in \mathbb{Z}$ such that $J + t \subset J'$ and $\alpha(s + t) = \alpha(s)$ for every $s \in J$.

Theorem 0.0.1. Let A be a finite set, T be the shift on $A^{\mathbb{Z}}$, $\alpha \in A^{\mathbb{Z}}$ and

$$X = \overline{Orb_T(\alpha)}.$$

Then the system (T, X) is minimal if and only if α is homogeneous.

Proof. Assume that the sequence $\alpha \in A^{\mathbb{Z}}$ is homogeneous. Let $\beta \in \overline{Orb_T(\alpha)}$. It is suffice to show that $\alpha \in \overline{Orb_T(\beta)}$. Fix $n > 0$, then there exist $k(n, \alpha)$ such that the restriction of the sequence α to any interval of the length $k(n, \alpha)$ contains a copy of the restriction of α to the interval $[-n, n]$. Thus, since $\beta \in \overline{Orb_T(\alpha)}$ then the restriction of β to the interval $[-k(n, \alpha), k(n, \alpha)]$ contains a copy of α restricted to $[-n, n]$. This implies that there exists a power i of T such that $T^i(\beta)(j) = \alpha(j)$ for all $j \in [-n, n]$. Since n is arbitrary large, we can find a sequence i_n of powers of T, such that $T^{i_n}(\beta)$ converges to α, therefore (T, X) is minimal.

Assume (T, X) is a minimal system. To reach a contradiction assume that α is not homogeneous. Then there exists an interval $[-n, n]$, such that for any k there exists a subinterval of length k in α, which does not contain the restriction of α to $[-n, n]$. Thus there exists a sequence m_k such that the interval $[-k, k]$ of $T^{m_k}(\alpha)$ does not contain the restriction of α to $[-n, n]$. Since the space is compact we can find a convergent subsequence in $T^{m_k}(\alpha)$. Let β be a limit point. Then $\alpha \notin \overline{Orb_T(\beta)}$, which gives a contradiction. Hence α is homogeneous. □

The full topological groups. The central object of this Chapter is the full topological group of a Cantor minimal system.

The full topological group of $(T, \mathbf{C})$, denoted by $[[T]]$, is the group of all $\phi \in Homeo(\mathbf{C})$ for which there exists a continuous function $n : \mathbf{C} \to \mathbb{Z}$ such that

$$\phi(x) = T^{n(x)}x \text{ for all } x \in \mathbf{C}.$$

Since C is compact, the function $n(\cdot)$ takes only finitely many values. Moreover, for every its value k, the set $n^{-1}(k)$ is clopen. Thus, there exists a finite partition of C into clopen subsets such that $n(\cdot)$ is constant on each piece of the partition.

Kakutani-Rokhlin partitions. Let T be a minimal homeomorphism of the Cantor space $\mathbf{C}$, we can associate a partition of $\mathbf{C}$ as follows.

Let D be a non-empty clopen subset of $\mathbf{C}$. It is easy to check that for every point $p \in \mathbf{C}$ the minimality of T implies that the forward orbit $\{T^k p : k \in \mathbb{N}\}$ is dense in $\mathbf{C}$. Define the first return function $t_D : D \to \mathbb{N}$:

$$t_D(x) = \min(n \in \mathbb{N} : T^n(x) \in D).$$

Since $t_D^{-1}[0, n] = T^{-n}(D)$, it follows that t_D is continuous. Thus we can find natural numbers $k_1, \ldots, k_N$ and a partition

$$D = D_1 \sqcup D_2 \sqcup \ldots \sqcup D_N$$

into clopen subsets, such that t_D restricted to D_i is equal to k_i for all $1 \leq i \leq N$.

This gives a decomposition of $\mathbf{C}$, called *Kakutani-Rokhlin partition*:

$$\begin{aligned}\mathbf{C} =& (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \\ & \sqcup (D_2 \sqcup T(D_2) \sqcup \ldots \sqcup T^{k_2}(D_2)) \sqcup \ldots \\ & \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N))\end{aligned}$$

The family $D_i \sqcup T(D_i) \sqcup \ldots \sqcup T^{k_i}(D_i)$ is called *a tower* over D_i. The base of the tower is defined to be D_i and the top of the tower is $T^{k_i}(D_i)$.

Refining of the Kakutani-Rokhlin partitions. Let $\mathcal{P}$ be a finite clopen partition of $\mathbf{C}$ and let

$$\begin{aligned}\mathbf{C} =& (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \\ & \sqcup (D_2 \sqcup T(D_2) \sqcup \ldots \sqcup T^{k_2}(D_2)) \sqcup \ldots \\ & \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N))\end{aligned}$$

be the Kakutani-Rokhlin partition over a clopen set D in $\mathbf{C}$. There exist a refinement of the partition of $D_i = \bigsqcup_{j=1}^{j_i} D_{i,j}$ such that the partition

$$(D_{1,1} \sqcup T(D_{1,1}) \sqcup .. \sqcup T^{k_1}(D_{1,1})) \sqcup .. \sqcup (D_{1,j_1} \sqcup T(D_{1,j_1}) \sqcup .. \sqcup T^{k_1}(D_{1,j_1})) \ldots$$
$$(D_{N,1} \sqcup T(D_{N,1}) \sqcup .. \sqcup T^{k_N}(D_{N,1})) \sqcup .. \sqcup (D_{N,j_N} \sqcup T(D_{N,j_N}) \sqcup .. \sqcup T^{k_N}(D_{N,j_N}))$$

of $\mathbf{C}$ is a refinement of $\mathcal{P}$. Indeed, this can be obtained as follows. Assume there exists a clopen set $A \in \mathcal{P}$ such that $A \cap T^i(D_j) \neq \emptyset$ and $A \Delta T^i(D_j) \neq \emptyset$ for some i, j. Then we refine the partition $\mathcal{P}$ by the sets $T^s(T^{-i}(A) \cap D_j)$, $0 \leq s \leq k_j$. Since $\mathcal{P}$ is finite partition this operation is exhaustive.

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