Lecture 7: Minimal subshifts. Finite generation of the commutator subgroup.

by Kate Juschenko

The aim of this section is to prove that every commutator subgroup of a Cantor minimal subshift is finitely generated, Theorem 0.0.3. This result is due to Hiroki Matui, [69]. Through this section we assume that T is a minimal homeomorphism of the Cantor set.

Let U be a clopen set in $\mathbf{C}$, such that U, $T(U)$ and $T^{-1}(U)$ are pairwise disjoint. Define

$$f_U(x) = \begin{cases} T(x), & x \in T^{-1}(U) \cup U \\ T^{-2}(x), & x \in T(U) \\ x, & \text{otherwise} \end{cases}$$

Obviously, f_U is a homeomorphism of $\mathbf{C}$ and $f_U \in [[T]]$. Moreover, we claim that f_U is in the commutator subgroup $[[T]]'$. To verify the claim, define a symmetry in $[[T]]$:

$$g(x) = \begin{cases} T(x), & x \in T^{-1}(U) \\ T^{-1}(x), & x \in U. \end{cases}$$

One verifies that $f_U = [g, f_U]$ by identifying f_U with cycle (123) and g with cycle (12).

Consider the following set of elements of $[[T]]'$

$\mathcal{U} = \{f_U : U \text{ is clopen set and } U, T(U), T^{-1}(U) \text{ are pairwise disjoint}\}$

Lemma 0.0.1. The commutator subgroup of the full topological group $[[T]]'$ is generated by $\mathcal{U}$.

Proof. Let H be a subgroup of $[[T]]$ generated by $\mathcal{U}$. We start by showing that if $g \in [[T]]$ and $g^3 = e$, then g is in H. Since each f_U is of order 3, this will imply that H is normal. Therefore because of simplicity of $[[T]]'$ we would be able to conclude that $H = [[T]]'$.

By Lemma **??** and Lemma **??** we can find a clopen set A such that A, $g(A)$ and $g^2(A)$ are pairwise disjoint and $supp(g) = A \sqcup g(A) \sqcup g^2(A)$. Let now B_i be a clopen partition of $\mathbf{C}$ such that the restriction of g to each B_i coincides with a certain power of T. Consider the following partitions of A:

$$\mathcal{P}_0 = \{B_i \cap A\}_{1 \le i \le n},$$
$$\mathcal{P}_1 = g^{-1}\{B_i \cap g(A)\}_{1 \le i \le n},$$
$$\mathcal{P}_2 = g^{-2}\{B_i \cap g^2(A)\}_{1 \le i \le n}.$$

Denote the common refinement of $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ by $\{A_j\}_{1\leq j\leq m}$. It has the property that for every $1 \leq j \leq m$ there are integers k_j, l_j such that

$$g|_{A_j} = T^{k_j}|_{A_j}, \quad g|_{g(A_j)} = T^{l_j}|_{g(A_j)}, \quad g|_{g^2(A_j)} = T^{-k_j-l_j}|_{g^2(A_j)}.$$

Now we can decompose $g = g_1 \dots g_m$ as a product of commuting elements of $[[T]]$ defined by the restriction of g onto $A_j \cup g(A_j) \cup g^2(A_j)$. This implies that it is sufficient to consider the case when g is in $[[T]]$ of the order 3 and there exists a clopen set $A \subset \mathbf{C}$ such that there are k and l with

$$g|_A = T^k|_A, \quad g|_{g(A)} = T^l|_{g(A)}, \quad g|_{g^2(A)} = T^{-k-l}|_{g^2(A)}.$$

Since for any $x \in A$ there exists a clopen neighborhood $U_x \subseteq A$ such that $\{T^i(U_x)\}_{1\leq i\leq k+l}$ are pairwise disjoint and A is compact, we can select a finite family U_x that covers A. Let $C_1, \dots, C_n$ be the partition of A generated by these finite family. Let $g = g_1 \dots g_n$ be the decomposition of g into a product of commuting elements of $[[T]]$ defined by taking g_i to be the restriction of g to $C_i \cup g(C_i) \cup g^2(C_i)$.

Thus this reduces the argument to the case when $g \in [[T]]$ has the following property: $g^3 = id$ and there exists a clopen set $A \subset \mathbf{C}$ such that there are k and l with

$$g|_A = T^k|_A, \quad g|_{g(A)} = T^l|_{g(A)}, \quad g|_{g^2(A)} = T^{-k-l}|_{g^2(A)},$$

and $T^i(A) \cap T^j(A) = \emptyset$ for all $1 \leq i, j \leq k + l$. This element can be considered as a cycle $(k\ l\ k+l)$ of the permutation group S_{k+l+1} and each cycle $(i-1\ i\ i+1)$ is given by $f_{T^i(A)}$. Moreover, g is in the alternating group A_{k+l+1}, which contains all 3-cycles. Thus g is a product of elements of $\mathcal{U}$, which finishes the lemma. □

Note that the proof of lemma shows slightly more. Namely, for every prime number p and an element of order p in $[[T]]$, this element belongs to the commutator subgroup.

Lemma 0.0.2. Let U and V be clopen subsets of $\mathbf{C}$, then the following holds

(i) If $T^2(V)$, $T(V)$, V, $T^{-1}(V)$, $T^{-1}(V)$ are pairwise disjoint and $U \subseteq V$, then for $\tau_U = f_{T^{-1}(U)} f_{T(U)}$ we have

$$\tau_V f_U \tau_V^{-1} = f_{T(U)}$$
$$\tau_V^{-1} f_U \tau_V = f_{T^{-1}(U)}$$

(ii) If V, U, $T^{-1}(U)$, $T(U)\cup T^{-1}(V)$, $T(V)$ are pairwise disjoint then

$$[f_V, f_U^{-1}] = f_{T(U)\cap T^{-1}(V)}.$$

Proof. The proof of the lemma boils down to the identification of the elements involved in the statement with permutations.

(i). The support of $\tau_{V\setminus U}$ is disjoint from supports of other homomorphism, thus

$$\tau_V f_U \tau_V^{-1} = \tau_U f_U \tau_U^{-1} = f_{T(U)},$$

where the last identity is the consequence of the identity in the permutation group $(01234)(123)(04321) = (012)$.

(ii). Let $C = T(U)\cap T^{-1}(V)$. We can decompose $f_U = f_{T^{-1}(C)} f_{U\setminus T^{-1}(C)}$ and $f_V = f_{T(C)} f_{V\setminus T(C)}$. Thus

$$[f_V, f_U^{-1}] = [f_{T(C)}, f_{T^{-1}(C)}^{-1}] = f_{T(C)} f_{T^{-1}(C)}^{-1} f_{T(C)}^{-1} f_{T^{-1}(C)} = f_C,$$

where the last identity is equivalent to the identity in the permutation group:

$$(234)(021)(243)(012) = (123).$$

□

Theorem 0.0.3. Let $T \in Homeo(\mathbf{C})$ is minimal homeomorphism. The commutator subgroup $[[T]]'$ is finitely generated if and only if T is conjugate to a minimal subshift.

Proof. Assume that $T \in Homeo(\mathbf{C})$ is a minimal subshift, i.e., T acts as a shift on the Cantor set $A^{\mathbb{Z}}$ for some finite alphabet A and there exists a clopen T-invariant subset $X \subset A^{\mathbb{Z}}$ such that the action of T on X is minimal. Moreover, enlarging the alphabet and using the characterization of the minimal subshifts in terms of homogeneous sequences, we can assume that $x(i) \neq x(j)$ for every $|i-j| < 4$ and $x \in X$.

For every $n, m \in \mathbb{N}$ and $a_i \in A$, $-m \leq i \leq n$, define the cylinder sets in X by

$$\langle\langle a_{-m}\dots a_{-1}\underline{a_0}a_1\dots a_n\rangle\rangle = \{x \in X : x(i) = a_i, -m \leq i \leq n\},$$

here the underlining of a_0 means that a_0 is in the 0's coordinate of $\mathbb{Z}$-enumeration. Since $x(i) \neq x(j)$ for every $|i-j|<4$ we have that for every cylinder set U the sets $T^{-2}(U), T^{-1}(U), U, T(U), T^2(U)$ are pairwise disjoint. Let H be a subgroup of $[[T]]'$ generated by the finite set of cylinders:

$$\{f_U : U = \langle\langle a\underline{b}c\rangle\rangle, a, b, c \in A\}.$$

We will show that $H = [[T]]'$. By Lemma 0.0.1 it is sufficient to show that for every cylinder sets $U \in X$, we have $f_U \in H$.

Since

$$f_{T(\langle\langle\underline{a}\rangle\rangle)} = \prod_{b\in A} f_{\langle\langle a\underline{b}\rangle\rangle}, \quad f_{T^{-1}(\langle\langle\underline{a}\rangle\rangle)} = \prod_{b\in A} f(\underline{b}a),$$

we immediately have $f_{T(\langle\langle\underline{a}\rangle\rangle)}, f_{T^{-1}(\langle\langle\underline{a}\rangle\rangle)}$, thus $\tau_{\langle\langle\underline{a}\rangle\rangle}$ is in H. Applying Lemma 0.0.2 to the sets

$$U = \langle\langle a_{-m}\dots a_{-1}\underline{a}a_1\dots a_n\rangle\rangle \subseteq \langle\langle\underline{a_0}\rangle\rangle = V$$

we obtain:

$$\tau_{\langle\langle\underline{a_0}\rangle\rangle} f_U \tau^{-1}_{\langle\langle\underline{a_0}\rangle\rangle} = f_{T(U)}, \quad \tau^{-1}_{\langle\langle\underline{a_0}\rangle\rangle} f_U \tau_{\langle\langle\underline{a_0}\rangle\rangle} = f_{T^{-1}(U)}.$$

Hence we conclude the statement of the lemma by induction on $m+n$ and applying Lemma 0.0.2 (ii) to the sets $V = \langle\langle a_{-m}\dots a_{-1}\underline{a_0}a_1\rangle\rangle$ and $U = \langle\langle a_1\underline{a_2}\rangle\rangle$.

To prove the converse, assume that $T \in Homeo(\mathbf{C})$ is minimal and $[[T]]'$ is finitely generated. Let $g_1, \dots, g_n$ be the generating set of $[[T]]'$ and $n_i : \mathbf{C} \to \mathbb{Z}$ be continuous maps that satisfy:

$$g_i(x) = T^{n_i(x)}x, \quad x \in \mathbf{C}.$$

Let $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_n\}$ be the common refinement of the partition $\{n_i^{-1}(k)\}_{k\in\mathbb{Z}}$. We will consider $\mathcal{P}$ as a finite alphabet together with shift map $s : \mathcal{P}^{\mathbb{Z}} \to \mathcal{P}^{\mathbb{Z}}$. Define a continuous map $S : X \to \mathcal{P}^{\mathbb{Z}}$ by the property that $S(x)(k) = \mathcal{P}_s$, if $T^k(x) \in \mathcal{P}_s$. It is easy to verify that S is a factor map. Define a homeomorphism $f_i \in Homeo(\mathbf{C})$ by $f_i(z) = S^k(z)$ when $z(0) \subseteq n_i^{-1}(k)$. It is easy to see that $f_i \in [[s]]$ and $Sg_i = f_iS$. It remains to show that S is injective.

Suppose $x, y \in \mathbf{C}$ are distinct and $S(x) = S(y)$. Let $g \in [[T]]'$ such that $g(x) \neq x$ and $g(y) = y$. By assumptions $[[T]]'$ is finitely generated, thus we can write g as a word on the generators $w(g_1, \dots, g_n)$.

$$
\begin{aligned}
Sg(x) &= Sw(g_1, \ldots, g_n)(x) \\
&= w(f_1, \ldots, f_n)S(x) \\
&= w(f_1, \ldots, f_n)S(y) \\
&= Sw(g_1, \ldots, g_n)(y) \\
&= Sg(y) = S(x).
\end{aligned}
$$

Hence, for some k we have $s^k S(x) = S(T(x)) = S(x)$, which contradicts to the minimality of s and thus of T. □

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