

Lecture 8: Basilica group

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0.1 Presentations of Basilica group

Basilica group is defined by 3-state automaton, see Figure 1.

As a group acting on finite binary tree, Basilica group is defined by

$$\begin{aligned} a(0v) &= 0v, & a(1v) &= 1b(v) \\ b(0v) &= 1v, & b(1v) &= 0a(v), \end{aligned}$$

for $v \in \{0, 1\}^{\mathbb{N}}$.

We will also consider its wreath product presentation constructed by automaton on Figure 1:

$$a = (1, b) \text{ and } b = (1, a)\varepsilon,$$

where ε is a non-trivial element of the group $\mathbb{Z}/2\mathbb{Z}$.

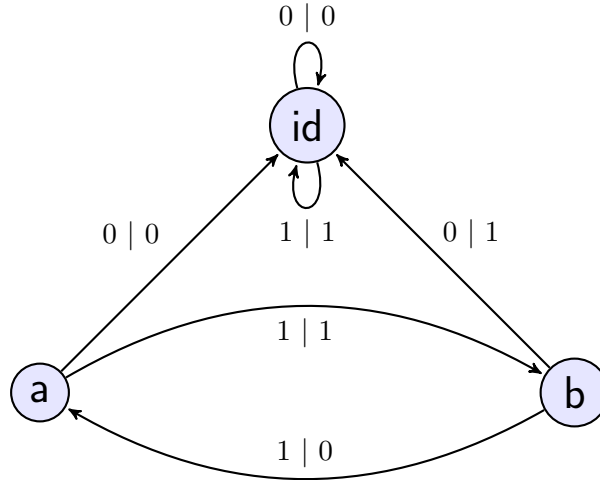


Figure 1: Automaton diagram of the Basilica group

0.2 Growth of Basilica group

Theorem 0.2.1. The semigroup generated by a and b is the free non-abelian group \mathbb{F}_2 .

Proof. To reach a contradiction consider two different words V and W that represents the same element in G and such that $\rho = |V| + |W|$ is minimal. It is easy to check that ρ can not be 0 or 1.

From the wreath product representation of G we see that the parity of occurrences of b in V and in W should be the same. Both words V and W are products of the elements of the form

$$a^n = (1, b^n), \quad n \geq 0$$

or of the form

$$ba^mb = (1, a)\varepsilon(1, b^m)(1, a)\varepsilon = (b^ma, a), \quad m \geq 0.$$

Assume that one of the words does not contain b , say $W = a^n$. Then, since a is of infinite order, V must contain b . Projecting V to the first coordinate we obtain a word which is equal to identity. Moreover the length of this word is strictly less ρ , which gives a contradiction. Thus we can assume that both words contain b .

Suppose that the number of b 's in W and V is one, then by minimality these words should have the form ba^n and a^mb . But $ba^n = (b^n, a)\varepsilon$ and $a^mb = (1, b^ma)\varepsilon$, which represent two different words.

Multiplying by b both words V and W , we can assume that b appears even number of times. This can increase the length of V and W by at most 1. Suppose that in both words b appears twice.

Thus either both words contain two b 's and $|V| + |W| = \rho$ or one of them contains at least four b 's, in which case we have $|U| + |V| \leq \rho + 2$. In the first case, taking the projection of the words onto the second coordinate we see that ρ decreases by 2 for these projections, which contradict to minimality. Similarly, in the second case ρ decreases by 3 which again contradicts minimality. Hence, the statement follows. \square

As a straightforward consequence of the theorem we obtain the following.

Corollary 0.2.2. Basilica group has exponential growth.

0.3 Elementary subexponentially amenable groups

Recall that a group Γ generated by a finite set S is of subexponential growth if $\lim_n |B(n)|^{1/n} = 1$, where $B(n)$ is the ball of radius n in the Cayley graph (Γ, S) .

Let SG_0 be the class of groups whose all finitely generated subgroups have exponential growth. Let $\alpha > 0$ be an ordinal such that we have already defined SG_β for all $\beta < \alpha$. Then if α is a limit ordinal, we set

$$SG_\alpha = \bigcup_{\beta < \alpha} SG_\beta$$

and if α is not a limit ordinal, we set SG_α to be the class of groups which can be obtained from $SG_{\alpha-1}$ by applying either extension or a direct union one time. Define

$$SG = \bigcup_{\alpha} SG_\alpha$$

Theorem 0.3.1. The class SG is the smallest class of groups which contain all groups of sub-exponential growth and is closed under taking subgroups, quotients, extensions, and direct unions.

The class SG will be called as *the class of elementary subexponentially amenable groups*. Obviously, the class SG contains the class of elementary amenable groups EG .

In this section we will show that Basilica group is not elementary subexponentially amenable, which was proved in [48].

Theorem 0.3.2. The Basilica group is not elementary subexponentially amenable. In particular, it is not elementary amenable.

The proof will be subdivided into two lemmas. We will use the following notations for the lemmas. Since G acts on the rooted a binary tree, we denote the stabilizer of the n -th level of the tree by $St_G(n)$.

Then there are naturally defined 2^n projections p_k , $1 \leq k \leq 2^n$, defined on $St_G(n)$ as a restriction to the k -th tree on the n -th level.

Lemma 0.3.3. For the Basilica group G we have $G' \geq G' \times G'$.

Proof. It is sufficient to show that the projections onto both coordinates of $St_G(1)$ coincide with G . Direct computation shows

$$\begin{aligned} a^b &= (a^{-1}, 1)\varepsilon(1, b)(1, a)\varepsilon = (a^{-1}ba, 1) = (b^a, 1) \in St_G(1) \\ b^2 &= (a, a) \in St_G(1), \end{aligned}$$

and moreover $a = (1, b) \in St_G(1)$. Thus, $p_1(St_G(1)) = p_2(St_G(1)) = G$.

Note that

$$[a, b^2] = (1, b^{-1})(a^{-1}, a^{-1})(1, b)(a, a) = (1, b^{-1}a^{-1}ba) = (1, [b, a]).$$

Summarizing above, we obtain

$$G' \geq \langle [a, b^2]^G \rangle \geq \{e\} \times \langle [b, a] \rangle^G = \{e\} \times G'.$$

Moreover, $(\{e\} \times G')^b = G' \times \{e\}$, therefore we have $G' \geq G' \times G'$. \square

Lemma 0.3.4. For every normal subgroup H of the Basilica group G there exists a subgroup $K < H$ such that K admits a quotient group containing G .

Proof. Let g be a non-trivial element. Then let n be a level of the tree which is fixed by g and the next level is not fixed by g . Then for some $1 \leq k \leq 2^n$ g has the form

$$g = (g_1, \dots, g_{k-1}, \bar{g}, g_{k+1}, \dots, g_{2^n})$$

where \bar{g} is an automorphism of the tree which does not fix the first level. In other words, $\bar{g} = (g_1, g_2)\varepsilon$, for some $g_1, g_2 \in G$.

For every $\bar{h} \in G'$ put $h = (1, \dots, 1, \bar{h}, 1, \dots, 1)$, where \bar{h} is based on the k -th place. By Lemma 0.3.3, we obtain $h \in G'$ and thus

$$[g, h] = (1, \dots, 1, [\bar{g}, \bar{h}], 1, \dots, 1) \in H.$$

The same lemma implies that \bar{h} can be chosen as $(w, 1)$, where w is any element of G' . Hence

$$[\bar{g}, \bar{h}] = (w, g_1^{-1}w^{-1}g_1).$$

This implies that there exists a subgroup K in H such that $p_1(p_k(K))$ contains G' . Since we have the following expressions

$$[b^{-1}, a] = (b^{-1}, b) \text{ and } [a, b^2] = (1, b^{-1}a^{-1}ba)$$

we obtain that $b, b^a \in p_2(G')$. But

$$b^2 = (a, a) \text{ and } b^a = (b, b^{-1}a).$$

Thus $p_2(p_2(G'))$ contains G , which implies the desired statement. \square

Proof of the Theorem 0.3.2. Since, by the Corollary 0.2.2, the Basilica group has exponential growth we have that it is not in the class SG_0 . To reach a contradiction, assume $G \in SG$ and let α be a minimal ordinal such that $G \in SG_\alpha$. Since α is minimal it is not limit ordinal. Thus G is obtained from the class $SG_{\alpha-1}$ by taking either extension or a direct union. It can not be obtained as a direct union, since otherwise $G \in SG_{\alpha-1}$ which contradicts to minimality of α . Hence G is obtained as an extension of groups from the class $SG_{\alpha-1}$. Therefore we can find a normal subgroup H of G , which is in the class $SG_{\alpha-1}$. Since all these classes are closed by taking subgroups and quotients, by Lemma 0.3.4 we obtain $G \in SG_{\alpha-1}$. □

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