Sofic groups

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CHAPTER 1

Introduction

The idea of soficity has its origins in the work of Gromov, who aimed to formulate a weak kind of finite approximation property for groups that encompasses both amenability and residual finiteness and is sufficient to imply surjunctivity [**28**]. A discrete group G is said to be *surjunctive* if, for every finite set A, if one considers the left shift action $G \curvearrowright A^G$ then every continuous G-equivariant injective map from A^G to itself is surjective. This can be viewed as a G-equivariant version of Dedekind finiteness for a set X, which asks that every injective map from X to itself be surjective. In [**23**] Gottschalk posed the problem of whether all countable discrete groups are surjunctive. Gromov's result from [**28**] that all sofic groups are surjunctive remains the state of the art in this direction, and in fact it is still unknown whether nonsofic groups exist. The term "sofic" itself was coined by Weiss, who in [**50**] consolidated the basic theory of these groups and gave another proof of their surjunctivity.

Beyond these roots in surjunctivity, sofic groups have generated a remarkable spectrum of applications over the last fifteen years, ranging from a new theory of entropy to the verification of several conjectures in the sofic case which remain open for general groups. Soficity takes the internal finite approximation of amenability in terms of Følner sets and externalizes it to abstract finite sets on which the group approximately acts. One loses the detailed structural picture that one has for amenable groups (as exemplified by quasitilings and the Rokhlin lemma), but the kind of Dedekind-like finiteness expressed by the invariant mean definition of amenability persists, not only in a qualitative sense but in the concrete form of finite approximation. For this reason soficity lends itself both to the formulation of asymptotic numerical invariants like entropy and to problems involving Dedekind-like finiteness such as Kaplansky's direct finiteness conjecture. It can be contrasted with a property like exactness, which is also a generalization of amenability but in an opposite direction in which compressibility phenomena dominate (an exact group is one which admits an amenable action on a compact space, and such an action cannot admit an invariant probability measure if the group is nonamenable).

It is important to stress that soficity, unlike amenability and residual finiteness, is a local property in the strict sense, as a Banach space theorist might use the expression. This means that one can detect soficity by testing a finite approximation property for each finite subset and its multiplication table without knowledge of the rest of the group. For amenability one must search for this finite approximation *inside* the group, while residual finiteness requires the existence of a separating family of globally defined homomorphisms into finite groups. One can obstruct (and in fact characterize the absence of) amenability by means of paradoxical decomposability, as prototypically exhibited by the free group on two generators, while simplicity is enough to preclude the property of residual finiteness for nontrivial groups. The local nature of soficity explains why it has been so hard to come up with possible obstructions, assuming that nonsofic groups do indeed exist. Similarly local in nature is the operator-algebraic analogue of the question of whether nonsofic groups exist, namely Connes's embedding problem, which dates back earlier to the 1970s and also remains open.

These notes aim to provide an introduction to soficity for groups, highlighting its applications to various conjectures as well as its use in the theory of entropy and related invariants. In Chapter 2 we begin with the quasi-action definition of a sofic group and prove its equivalence with Gromov's original graph-theoretic formulation, discuss the behaviour of soficity within the space of marked groups, examine two important subclasses, the amenable and residually finite groups, and present Cornulier's example of a sofic group which is not a limit of amenable groups. Chapter 3 examines soficity from the ultraproduct viewpoint, while Chapter ??? discusses operations on groups which preserve soficity. In Chapter 5 we show how several open problems concerning discrete groups can be resolved in the sofic case, namely Kaplansky's direct finiteness conjecture, Connes's embedding problem, and the determinant conjecture. Chapter 6 is devoted to entropy theory for actions of sofic groups. This includes a proof of surjunctivity using topological entropy, as well as the computation of measure entropy for Bernoulli actions and a discussion of their classification. In Chapter 7 we present sofic dimension for groups and equivalence relations and establish a formula for free products with amagalmation over an amenable subrelation.

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CHAPTER 2

Sofic groups - definition and basic properties

1. Definition of sofic groups

In this section we give the definition and basic properties of sofic groups.

Consider the permutation group of n elements, $S(n)$, with the following distance, called *Hamming distance*:

$$d_{hamm}(\sigma_1, \sigma_2) = \frac{1}{n}|\{i : \sigma_1(i) \neq \sigma_2(i)\}|$$

The following definition is apparently the strongest version among all definitions of sofic groups.

DEFINITION 1.1. A discrete group Γ is **sofic** if for every finite set $F \subseteq \Gamma$ containing e and every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and a map ϕ from F to $S(n)$ such that the following conditions hold:

(i) $\phi(e) = e$,

(ii) $d(\phi(gh), \phi(g)\phi(h)) < \epsilon$ for all g, h, such that $gh \in F$,

(iii) $\phi(g)$ does not have fixed points, i.e. $d(\phi(g), e) = 1$, for every $g \in F \backslash \{e\}$.

We will call such a ϕ an (F, ϵ)**-approximation** of Γ.

It is straightforward from the definition that

(i) A subgroup of a sofic group is sofic.
(ii) A group is sofic if and only if all finitely generated subgroups are sofic.
(iii) A direct product of sofic groups is again sofic.
(iv) A direct limit of sofic groups is sofic. This follows from the property of $G = \lim G_i$: for every finite subset F in G there is an index i and a homomorphism $\pi : G_i \to G$ such that π is a bijection on F_i and $\pi(F_i) = F$.
(v) An inverse limit of sofic groups is sofic. Indeed, by definition, inverse limit of groups is a subgroup of their product.

2. Weak form of the definition of sofic groups. A technical lemma

The following simple and useful lemma will be plugged into many proofs later on. It is designed to weaken the definition of soficity to the case when we have to deal with partially defined maps on a finite set.

LEMMA 2.1. Assume that for every $g \in \Gamma$ there exists a constant $0 \leq C_g < 1$ such that for every $\epsilon > 0$ and a finite subset F of Γ containing identity there is a finite set A and a map ϕ from F into partially defined maps on A which satisfies:

(i) for every $g \in F$ there is a subset A_g of A with $|A \backslash A_g| \leq \epsilon |A|$ where the map $\phi(g)$ is defined and injective

(ii) $|\{a \in A_e : \phi(e)(a) \neq a\}| \leq \epsilon |A|$

(iii) $|\{a \in A_h \cap A_{gh} \cap \{b \in A_h : \phi(h)(b) \in A_g\} : \phi(gh)(a) \neq \phi(g) \circ \phi(h)(a)\}| \leq \epsilon |A|$ for all $g, h \in F$.

(iv) $|\{a \in A_g : \phi(g)(a) \neq a|\} \geq (1 - C_g)|A|$ for every $g \in F \backslash \{e\}$.

then Γ is sofic.

PROOF. Let F be a finite set in Γ, $\epsilon > 0$ and ϕ be as above.

In order to obtain the condition 3 consider the following map: $\phi_n(g) := \phi(g) \otimes \ldots \otimes \phi(g)$ which acts diagonally on $A_g^n = A_g \times \ldots \times A_g$. Verifying conditions of lemma for ϕ_n we obtain:

(i) $\{a \in A_e^n : \phi(e)(a) \neq a\} \leq \epsilon |A^n|$.

(ii) $|\{a \in A_h^n \cap A_{gh}^n \cap \{b \in A_h^n : \phi_n(h)(b) \in A_g^n\} : \phi(gh)(a) \neq \phi_n(g) \circ \phi_n(h)(a)\}| \leq n\epsilon |A^n|$ for all $g, h, gh \in F$.

(iii) $|\{a \in A_g^n : \phi(g)(a) \neq a|\} \geq (1 - C_g^n)|A^n|$ for every $g \in F \backslash \{e\}$.

Now for every $\varepsilon > 0$ we can choose n and ϵ such that $2C_g^n |F| < \varepsilon$ and $2n\epsilon |F| < \varepsilon$. For such chosen n denote the intersection A_g^n with $\bigcap_{g \in F \backslash \{e\}} \{a \in A_g^n : \phi(g)(a) \neq a\} \cap \{a \in A_e^n : \phi(e)(a) = a\}$ again by A_g and denote A^n by A and ϕ^n again by ϕ. We have arrived to the following situation:

(i) $\phi(e) = id_A$.

(ii) $|\{a \in A_h \cap A_{gh} \cap \{b : \phi(h)(b) \in A_g\} : \phi(gh)(a) \neq \phi(g) \circ \phi(h)(a)\}| \leq \varepsilon|A|$ for all $g, h, gh \in F$.

(iii) $\phi(g)$ does not have fixed points on A_g.

One of the possible extensions of the map $\phi(g)$ to a permutation map $\overline{\phi}(g)$ on A is the following. Define $\overline{\phi}(g)(a) = \phi(g)(a)$ for $a \in A_g$. Since $\phi(g)$ is injective on A_g the cardinalities of the sets $\phi(g)(A_g)\backslash A_g$ and $A_g\backslash\phi(g)(A_g)$ are equal and we can extend $\phi(g)$ to $\phi(g)(A_g)\backslash A_g$ as an arbitrary isomorphism between sets $\phi(g)(A_g)\backslash A_g$ and $A_g\backslash\phi(g)(A_g)$. Define $\overline{\phi}(g)$ on the rest of the set A, namely on the set $A\backslash(\phi(g)(A_g) \cup A_g)$, as a permutation which does not have fixed points. It is obvious now that the conditions (1), (2) and (3) from the definition of the soficity are satisfied for $\overline{\phi}$. □

3. Gromov's definition of sofic groups. Initially subamenable graphs

In [**28**] Gromov defined the class of sofic groups using the property of their Cayley graphs. In this section we present the definition of Gromov and prove its equivalence with the definition given in Section 2. This equivalence was first established in [**20**].

Let S be a finite set, we will call it the set of colors. An edge-colored graph (V, E) is a directed graph with the property that to each edge an element from the set of colors S is assigned.

Definition 3.1. An edge colored graph $G = (V, E)$ is *initially subamenable* if for every $r \in \mathbb{N}$, $\varepsilon > 0$ and for every ball $B_r(G)$ of radius r in G there exists an edge-colored finite graph $G' = (E', V')$ and a finite set W in V' such that

(i) G' is r-locally isometric to G. That is all r-balls $B_r(G', w)$ around every point $w \in W$ are isomorphic (as colored graphs) to $B_r(G)$.

(ii) W is $(1 - \varepsilon)$-large with respect to V, i.e. $|W| > (1 - \varepsilon)|V|$.

Let Γ be a finitely generated group with generating set S. The the Cayley graph of Γ is an edge-colored graph (Γ, S) with vertex set Γ

and there exists an edge between g and h with color $g^{-1}h$ if and only if $g^{-1}h$ is in S. Note that all balls of the same radius in the Cayley graph are isomorphic as edge-colored graphs.

THEOREM 3.2. A finitely generated group Γ is sofic if and only if its Cayley graph with respect to any finite set of generators is initially subamenable.

PROOF. Let S be a set of generators. Assume firstly that Γ is sofic. Let $\varepsilon > 0$, $r \in \mathbb{N}$ and $\phi : B_{2r+2}(\Gamma, S) \to S(n)$ be $(B_{2r+2}(\Gamma, S), \varepsilon)$-approximation of the ball of radius r. Define an edge-colored graph G' with vertex set $\{1, \dots n\}$ and such that (i, j) is an edge colored by $s \in S$ is and only if $\phi(s)i = j$. It is easy to check that conditions 1, 2 of the Definition 3.1 are satisfied.

Assume now that the Cayley graph (Γ, S) is initially subamenable. Note that in order to prove that Γ is sofic it is sufficient to find $(B_r(\Gamma), \varepsilon)$-approximation for every $\epsilon > 0$ and every $r > 0$. By assumption for every $\varepsilon > 0$ and every $r \in \mathbb{N}$ there exists an edge-colored (by elements from S) finite graph G' such that G' is r-locally isomorphic to (Γ, S) on a $(1 - \varepsilon)$-large subset W of V. Define a map ϕ from $B_r(\Gamma, S)$ to the set of maps from W to V as follows. For every $s \in S$ and $w \in W$ let $\phi(s)w$ be an element w' in G' such that (w, w') is an edge colored by s. It is easy to check that ϕ satisfies Lemma 2.1. □

4. Topology on the space of marked groups

The following topology on the set of finitely generated groups on a fixed number of generators (marked groups) was introduced by Grigorchuk in [**24**] and perfectly fits into the framework of sofic groups. For the introductory expositions see [**24**], [**10**], [**11**], [**38**].

Let Γ_i for $i \in \mathbb{N}$, and Γ be finitely generated groups on n generators (possibly taken with repetitions) for some $n \in \mathbb{N}$. Let $\phi_i : \mathbb{F}_n \to \Gamma_i$, $i \in \mathbb{N}$ and $\phi : \mathbb{F}_n \to \Gamma$ be canonical surjections, namely ϕ_i and ϕ are bijections on the set of generators. Denote the kernels of ϕ_i and ϕ by N_i and N correspondingly. Then the sequence of groups Γ_i converges to Γ in the space of marked groups (also called the Cayley topology or Grigorchuk's topology) if

$$\sup\{k \in \mathbb{N} : N_i \cap B_k(\mathbb{F}_n) = N \cap B_k(\mathbb{F}_n)\} \to \infty, \text{ when } i \to \infty.$$

In particular, this convergence implies that for every $k \in \mathbb{N}$ there exists i_0 such that the balls (in Cayley graphs) of radius k in Γ_i and Γ coincide as labeled graphs for all $i \geq i_0$. Since every finite set of a group is contained in some ball we have proved the following.

PROPOSITION 4.1. A limit of sofic groups in the space of marked groups is sofic.

5. Isolated points in the space of marked groups

Since a limit of sofic groups is sofic, it is natural to study the question of whether non-amenable finitely generated groups that are isolated in the space of marked groups are sofic. This class of groups potentially has a lot of non-sofic groups. In this section we present examples of isolated groups.

The extensive study of isolated points in Grigorchuk's topology have been done by de Cornulier, Guyot and Pitsch in [**15**]. As a characterization of isolated groups in Grigorchuk's topology they prove the following theorem.

THEOREM 5.1. A finitely generated group Γ is isolated if and only if the following two conditions hold:

(i) Γ is finitely presented

(ii) Γ is finitely discriminable. Namely, there exists a finite subset $F \subseteq \Gamma \backslash \{e\}$ such that for every normal subgroup $N \neq \{e\}$ in Γ we have $N \cap F \neq \emptyset$.

PROOF. Assume firstly that Γ is finitely presented and finitely discriminable. Let Γ_i be a sequence of groups converging to Γ. Since Γ is finitely presented we have that for every finite subset F in Γ there exists $i_0 \in \mathbb{N}$ such that for every $i \geq i_0$ there exists a homomorphism $\psi_i : \Gamma \to \Gamma_i$ which is onto and $\psi_i(g) \neq e$ for every $g \in F$ and $i \geq i_0$. Taking F to be discriminating set we have that $F \cap ker(\psi_i) = \emptyset$, thus $ker(\psi_i) = \{e\}$ and $\Gamma_i \simeq \Gamma$ for every $i \geq i_0$. Therefore Γ is an isolated group.

To prove the converse assume that Γ is isolated. Let $g_1, \ldots, g_n$ be the set of generators of Γ and $\{\omega_i(g_1, \ldots, g_n)\}_{i \in \mathbb{N}}$ be an enumeration of the set of words in $\mathbb{F}_n$ that are equal to the identity element in Γ. Then $\Gamma_i = \langle g_1, \ldots, g_n : w_j(g_1, \ldots, g_n) = e, 1 \leq j \leq i \rangle$ converges to Γ. Since Γ is isolated we have that there exists i_0 such that $\Gamma_{i_0} \simeq \Gamma$, thus Γ is finitely presented. To reach a contradiction assume that Γ is not finitely discriminable. Let $\{F_i\}_{i \in \mathbb{N}}$ be an increasing to $\Gamma \backslash \{e\}$ sequence of finite sets. Thus if there are non-trivial normal subgroups N_i of Γ such that $F_i \cap N_i = \emptyset$, then Γ / N_i converges to Γ, which is a contradiction. Thus Γ is finitely discriminable. $\square$

From the Theorem 5.1 it follows that all finitely presented groups that have finite number of normal subgroups are isolated. In particular, all finitely presented simple groups are isolated. However we don't know any examples of simple non-amenable sofic finitely presented groups. The only finitely generated simple sofic groups that are known are amenable. In fact the existence of finitely generated simple amenable group was not known until recently: Matui, [**35**], showed that the commutator subgroups of the full topological groups of Cantor minimal subshifts are simple and finite generated. Grigorchuk and Medynets in [**25**] conjectured that all this groups groups are amenable. This was proved in affirmative by the first named author and Monod, [**31**]. We will discuss some classes of finitely presented simple groups as well as isolated groups.

Example 5.2. Here we will present three Thompson's groups that are isolated and none of which is known to be sofic, see [**9**] for an introductory survey on Thompson's groups.

(i) Consider the one-dimensional sphere S^1 as the interval $[0, 1]$ with identified ends. Thompson's group T is the group of piecewise linear homeomorphisms of S^1 that map dyadic rationals to dyadic rationals such that they are differentiable except at finitely many images of rational dyadic numbers and on the intervals of differentiability the derivatives are powers of 2. The group T is know to be finitely presented and simple.

(ii) Thompson's group F is the group of all orientation-preserving piecewise linear homeomorphisms of $[0, 1]$ that map dyadic rationals to dyadic rationals such that they are differentiable except at finitely many images of rational dyadic numbers and on the intervals of differentiability the derivatives are powers of 2. The group F is finitely generated and has trivial center. The commutator F' of the group F is simple, thus each normal subgroup contains it. Therefore F is isolated.

(iii) Thompson's group V is the group of right-continuous bijections of S^1 that map images of dyadic rational numbers to images of dyadic rational numbers, that are differentiable except at finitely many images of dyadic rational numbers, such that on each maximal interval where the function is differentiable, the function is linear with derivative a power of 2. The group V is know to be finitely presented and simple.

6. Residually finite groups.

DEFINITION 6.1. A group Γ is called **residually finite** if for every nontrivial element g in Γ there is a homomorphism ϕ from Γ to a finite group such that $\phi(g) \neq e$.

It is easy to show that a group is residually finite if and only if it embeds into a direct product of finite groups. Since soficity is stable under taking a subgroup and direct product we have that residually finite groups are sofic.

From the definition it is immediately follows that simple groups as well as groups which do not admit finite quotients are not residually finite.

Examples of residually finite groups include:

(i) Finite groups,
(ii) Finitely generated abelian groups are residually finite,
(iii) Finitely generated nilpotent groups,
(iv) Polycyclic-by-finite groups,
(v) Free groups,
(vi) Finitely generated linear groups,
(vii) Fundamental groups of 3-manifolds.

THEOREM 6.2 (Malcev). Every finitely generated subgroup G of the linear group $GL_n(K)$ is residually finite.

The following useful notion for groups which are not finitely presented was introduces by Vershik and Gordon in [**48**].

DEFINITION 6.3. A group Γ is **locally embeddable into finite groups** if for every finite set F in Γ there is injective map ϕ from F to a finite group such that if x, y and xy are in F then $\phi(xy) = \phi(x)\phi(y)$.

REMARK 6.4. From the definition it follows that the groups which are finitely generated locally embeddable into finite groups are limits of finite groups in the space of marked groups. If Γ is finitely presented and locally embeddable into finite groups then Γ is residually finite. Indeed, let $\Gamma = \langle g_1, g_2, \dots g_n : \omega_i(g_1, g_2, \dots, g_n) = e, 1 \leq i \leq k\rangle$, where $\omega_i(g_1, g_2, \dots, g_n)$ is a word on generators $g_1, g_2, \dots g_n$. Let ϕ_F satisfy the definition on the set $F \cup \{g : |g| \leq \max_i |\omega_i(g_1, g_2, \dots, g_n)|\}$. Then, obviously, ϕ_F extends to a homomorphism of Γ into a finite group. Since F is arbitrary finite set in Γ we have that $\{\phi_F\}_{F\subset\Gamma}$ separates points of Γ and thus Γ is residually finite.

The following class of groups will be useful to construct examples of non-residually finite groups. A group Γ is **Hopfian** if for every homomorphism $\phi : \Gamma \to \Gamma$ which is onto we have that ϕ is an isomorphism. We will show the classical result of Malcev that finitely generated residually finite groups are Hopfian. We need the following simple lemma.

Lemma 6.5. Let $\phi : G \to H$ be surjective homomorphism of groups G and H. Let $N < H$ be a subgroup. Then $[H : N] = [G : \phi^{-1}(N)]$.

Proof. Let $k = [H : N]$. Since $H = h_1 N \sqcup h_2 N \sqcup \ldots \sqcup h_k N$ for some $h_1, \ldots, h_k \in H$ and ϕ is onto we have that

$$G = \phi^{-1}(H) = \phi^{-1}(h_1 N) \sqcup \ldots \sqcup \phi^{-1}(h_k N).$$

Let $g_i \in G$ be such that $\phi(g_i) = h_i$, then it is straightforward to check that

$$\phi^{-1}(h_i N) = g_i \phi^{-1}(N).$$

So $G = g_1 \phi^{-1}(N) \sqcup \ldots \sqcup g_k \phi^{-1}(N)$, thus ϕ^{-1} is of index k in G. $\square$

Theorem 6.6 (Malcev). Finitely generated residually finite groups are Hopfian.

Proof. Let Γ be a finitely generated residually finite group and let $k \in \mathbb{N}$. Let $\phi : \Gamma \to \Gamma$ be surjective epimorphism and let $S_k = \{N : [\Gamma : N] = k\}$ be the set of subgroups of Γ of index k. Then by Lemma 6.5 we have $\phi^{-1}(N) \in S_k$ for every $N \in S_k$.

Since every finitely generated group has a finite number of subgroups of index k we have that the set S_k is finite. Since an finite intersection of finite index subgroups is of finite index we have that the index of $N_k = \bigcap_{N \in S_k} N$ is finite. In particular, the group N is non-trivial. Then $\phi^{-1}(N_k) = \bigcap_{N \in S_k} \phi^{-1}(N) = N_k$. Since Γ is residually finite, $\bigcap_k N_k = \{e\}$. Let now $g \in \ker \phi$ then $g \in \phi^{-1}(N_k) = N_k$ for all k, thus $g = e$ and ϕ is injective.

$\square$

We will show that the following group is not residually finite. Moreover, it has only 2 finite quotients.

Example 6.7. Let $\Gamma = \langle a, t : t^4 = e, (t^{-1}at)^{-1} a (t^{-1}at) = a^2 \rangle$. Assume that ϕ is a homomorphism from Γ into a finite group and n is the order of $\phi(a)$. By induction one can show that for all $k \in \mathbb{N}$:

$$\phi(t^{-1}at)^{-k} \phi(a) \phi(t^{-1}at)^k = \phi(a)^{2^k}$$

Thus $\phi(a) = \phi(a)^{2^n}$ and n must divide $2^n - 1$. It is easy to see that n never devides $2^n - 1$, therefore $\phi(a) = e$ and all finite quotients of Γ are either $\mathbb{Z}_2$ or $\mathbb{Z}_4$.

EXAMPLE 6.8. The free group $\mathbb{F}_n$ is residually finite for all $n \in \mathbb{N} \cup \{\infty\}$. In fact in order to prove this it is enough to show that $\mathbb{F}_2$ is residually finite. Indeed, let a and b be free generators of $\mathbb{F}_2$, then b, aba^{-1}, a^2ba^{-2}, ..., $a^{n-1}ba^{-(n-1)}$ generate $\mathbb{F}_n$. We will show that $\mathbb{F}_2$ is a subgroup of $SL_2(\mathbb{Z})$. The later group is residually finite, since every element of $SL_2(\mathbb{Z})$ is non-trivial in the reduction mod p for large enough p. Define a homomorphism $\phi : \mathbb{F}_2 \to SL_2(\mathbb{Z})$ by:

$$\phi(a) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \phi(b) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

In order to verify that ϕ gives the desired embedding we apply the following lemma to $\phi(a)$, $\phi(b)$ and the sets $X = \mathbb{R}^2$, $X_1 = \{(x, y) \in X : |x| > |y|\}$ and $X_2 = \{(x, y) \in X : |x| < |y|\}$.

LEMMA 6.9. (Ping-Pong, see [**29**]) Let Γ act on a set X and let a and b be elements of infinite order in Γ. Suppose that there are two disjoint sets X_1 and X_2 in X such that

(i) for every $n \in \mathbb{Z}$, $n \neq 0$ we have $a^n(X_2) \subseteq X_1$,

(ii) for every $n \in \mathbb{Z}$, $n \neq 0$ we have $b^n(X_1) \subseteq X_2$,
then a and b are free, i.e. they generate $\mathbb{F}_2$ inside Γ.

EXAMPLE 6.10. Abels groups.

Let $n \geq 2$, p be a prime number and $GL_n(\mathbb{Z}[\frac{1}{p}])$ the general linear group over the ring $\mathbb{Z}[\frac{1}{p}]$. Consider a subgroup $A_n < GL_n(\mathbb{Z}[\frac{1}{p}])$ which consists of all upper-triangular matrices with positive elements on the diagonal such that the $(1,1)$-th and (n, n)-th entry are equal to 1:

(1)
$$A_n = \{ \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} :$$

$$a_{ii} \in p^{\mathbb{Z}} \text{ for all } 2 \leq i \leq n-1, a_{ij} \in \mathbb{Z}[\frac{1}{p}] \text{ for all } 1 \leq i \leq j \leq n.\} \quad (2)$$

Taking quotients of $\mathbb{Z}[\frac{1}{p}]$ to $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, where k is mutually prime with p, we have that A_n is residually finite for all $n \in \mathbb{N}$. Note that A_2

is isomorphic to $\mathbb{Z}[\frac{1}{p}]$ where $\mathbb{Z}[\frac{1}{p}]$ is considered as a group with respect to addition. It follows that A_2 is not finitely generated. In [**27**] Groves showed that also A_3 is not finitely presented. It was proved by Abels for $n = 4$ and extended to arbitrary $n \geq 4$ by Brown that A_n is finitely presented, see [**1**], [**2**] and [**7**].

We will show that a certain quotient of A_n is not Hopfian, thus it is not residually finite. Historically it was the first example of finitely presentable non-Hopfian solvable group. It is easy to check that the center of A_n is the set of matrices with 1 on the diagonal, the $(1, n)$-th entry is $\mathbb{Z}[\frac{1}{p}]$ and the rest of entries are zeros:

$$Z(A_n) = \{ \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,n} \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} : a_{1,n} \in \mathbb{Z}[\frac{1}{p}]\}.$$

Consider $\mathbb{Z}$ as a subgroup of $Z(A_n) \simeq \mathbb{Z}[\frac{1}{p}]$. Then for $n \geq 4$ the group $A_n/\mathbb{Z}$ is finitely presented since it is a quotient of finitely presented group by finitely generated one. Moreover, $A_n/\mathbb{Z}$ is not Hopfian. Indeed, consider an automorphism ϕ of A_n given by the conjugation by $diag(p, 1, \ldots, 1)$:

$$\phi \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & pa_{1,2} & \cdots & pa_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Thus ϕ acts on $Z(A_n)$ as multiplication by p. Let $\hat{\phi}$ be an endomorphism from $A_n/\mathbb{Z}$ onto $\phi(A_n)/\mathbb{Z}$ induced by ϕ. Then $\hat{\phi}$ is not injective because

$$\hat{\phi} \begin{pmatrix} 1 & 0 & \cdots & \frac{1}{p} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} = 1_n.$$

Therefore $A_n/\mathbb{Z}$ is not Hopfian.

Modifications of Abel's groups are very fruitful. In [**16**], de Cornulier gave an example of an Abel's type property (T) group which is non-Hopfian. In particular we will see in Section 8 that there are Abel's type groups that are sofic and isolated in Grigochuk's topology.

7. Amenable groups

In this section we present another class of sofic groups: amenable groups. We give several characterizations of amenability and show that sofic-by-amenable groups are sofic.

DEFINITION 7.1. A group Γ is amenable if it admits a left invariant mean. Namely, if there exists a positive linear functional μ on $l_\infty(\Gamma)$ with $\mu(1) = 1$ and such that for all $g \in \Gamma$ and $f \in l_\infty(\Gamma)$ we have $\mu(g.f) = \mu(f)$.

DEFINITION 7.2. (Følner condition). A group Γ satisfies the Følner condition if for every finite subset $E \subseteq \Gamma$ and $\epsilon > 0$ there is a finite subset $F \subseteq \Gamma$ such that for each $g \in E$:

$$|gF\Delta F| < \epsilon|F|$$

THEOREM 7.3. Let Γ be a discrete group. Then the following are equivalent:

(i) Γ is amenable.
(ii) Γ satisfies Følner condition.
(iii) Γ satisfies Reiter's condition: for evey finite subset $E \subseteq \Gamma$ and $\epsilon > 0$, there is f in $l_1(\Gamma)$ with $||f||_1 = 1$ such that $||f - g.f||_1 < \epsilon$ for every $g \in E$.

EXAMPLES 7.4. Amenable groups include the following classes of groups:

(i) Finite groups
(ii) Abelian groups
(iii) Groups that have subexponential growth.

Amenability is stable under taking subgroups, factorgroups, extensions, quotients, inductive limits.

Groups that can be obtained from finite and abelian groups using above operations are called *elementary amenable* groups. In particular, this class includes solvable groups.

EXAMPLE 7.5. A group Γ has **a paradoxical decomposition** if there are pairwise disjoint subsets $F_1, \ldots, F_n$, $E_1, \ldots, E_m$ of Γ and elements $g_1, \ldots g_n$, $h_1, \ldots, h_m \in \Gamma$ such that Γ can be expressed as

$$\Gamma = \bigsqcup_i g_i F_i = \bigsqcup_j h_j E_j$$

If Γ admits a paradoxical decomposition then it is not amenable. Indeed, let μ be a left-inveriant mean on $l_\infty(\Gamma)$ then from the paradoxical decomposition of Γ we have

$$\begin{aligned}1 = \mu(1_\Gamma) &\geq \sum_i \mu(1_{F_i}) + \sum_j \mu(1_{E_j})\\ &= \sum_i \mu(1_{g_iF_i}) + \sum_j \mu(1_{h_jE_j})\\ &= 2\mu(1_\Gamma) = 2,\end{aligned}$$

which is a contradiction.

The free group $\mathbb{F}_2$ of rank two is not amenable. Let a and b be the free generators of $\mathbb{F}_2$ and $\omega(x)$ be the set of all reduced words that begin with x. Then $\mathbb{F}_2$ admits the following paradoxical decomposition:

$$\begin{aligned}\mathbb{F}_2 &= \{e\} \sqcup \omega(a) \sqcup \omega(a^{-1}) \sqcup \omega(b) \sqcup \omega(b^{-1})\\ &= \omega(a) \sqcup a\omega(a^{-1}) = \omega(b) \sqcup b\omega(b^{-1}).\end{aligned}$$

Definition 7.6. A group Γ is **initially subamenable** if for every finite set F in Γ there is an injective map ϕ from F to an amenable group such that if x, y and xy are in F then $\phi(xy) = \phi(x)\phi(y)$.

Similarly to the case of local embeddability into finite groups, see Remark 6.4, we have that finitely presented initially subamenable groups are residually amenable. Moreover, from definition it follows that finitely generated initially amenable groups are limits of amenable groups in Grigorchuk's topology.

Theorem 7.7. Amenable groups are sofic. More generally, initially subamenable groups are sofic.

Note that Elek and Szabo constructed an example of a sofic group which is not residually amenable, see [**19**]. However their group is initially subamenable. Answering the question of Gromov, de Cornulier provides an example of sofic but not initially subamenable group, see Section 8 for more on this example.

The following theorem will be useful to construct of an example of a sofic group which is not initially subamenable.

Theorem 7.8. All sofic-by-amenable groups are sofic. More precisely, if a group Γ has a sofic normal subgroup N such that Γ/N is amenable, then Γ is sofic.

Proof. Let F be a finite subset of Γ, $\epsilon > 0$ and $q : \Gamma \to \Gamma/N$ be the canonical quotient map. For every $f \in \Gamma/N$, denote a lifting of f to Γ by $\hat{f}$, i.e. $\hat{f}$ is an element of Γ with the property $q(\hat{f}) = f$.

Since Γ/N is amenable there exists a finite set $E \subseteq \Gamma/N$ such that

$$|gE\Delta E| \leq \epsilon|E|, \text{ for every } g \in q(F).$$

Denote $H = N \cap \hat{E}^{-1} \cdot F \cdot E$. Since N is sofic there exists (H, ϵ)-approximation ϕ from N into $S(n)$ for some $n \in \mathbb{N}$.

Consider the following map Φ from F into the set of partially defined injective maps on $A = \{1, \ldots, n\} \times E$:

$$\Phi(g)(i, h) = (\phi(\widehat{q(g)h}^{-1} g\hat{h})(i), q(g)h), \text{ for } g \in F \text{ and } h, gh \in E.$$

We will show that Φ satisfies all conditions of Lemma 2.1.

Firstly, it is well defined and does not have fixed points on $A_g = \{1, \ldots, n\} \times \{h \in E : gh \in E\}$, indeed, $q(\widehat{q(g)h}^{-1} g\hat{h}) = e$ and thus $\widehat{q(g)h}^{-1} g\hat{h} \in N$. Moreover, $|A \backslash A_g| \leq \epsilon |A|$.

The condition 2 of Lemma 2.1 is satisfied, since $\Phi(1)(i, h) = (\phi(1)(i), h)$ for all $(i, h) \in A$.

In order to prove the condition 3 take $g_1, g_2 \in F$ and $h \in E$ such that $g_1 h, g_2 h \in E$. Note that $|E \backslash \{h' \in E : g_1 h', g_2 h' \in E\}| \leq \epsilon |E|$. By definition of Φ:

$$\begin{aligned}\Phi(g_1)\Phi(g_2)(i, h) &= \Phi(g_1)(\phi(\widehat{q(g_2)h}^{-1} g_2\hat{h})(i), q(g_2)h) \\ &= (\phi(\widehat{q(g_1g_2)h}^{-1} g_1 \widehat{q(g_2)h}) \cdot \phi(\widehat{q(g_2)h}^{-1} g_2\hat{h})(i), q(g_1g_2)h).\end{aligned}$$

One the other hand:

$$\Phi(g_1g_2)(i, h) = (\phi(\widehat{q(g_1g_2)h}^{-1} g_1g_2\hat{h})(i), q(g_1g_2)h).$$

Since $d(\phi(gf), \phi(g)\phi(f)) \leq \epsilon n$ for every $g, f \in H$ we have that the condition 3 of Lemma 2.1 is satisfied with 2ϵ.

To prove the last condition of Lemma 2.1 consider first $e \neq g \in N \cap F$, then

$$\Phi(g)(i, h) = (\phi(\hat{h}^{-1} g\hat{h})(i), h).$$

Since ϕ does not have fixed points we have that neither does Φ.

Let now $g \in F$ and $g \notin N$, then by assumptions on E we have

$$|\{(i, h) \in A : \Phi(g)(i, h) = (\phi(\widehat{q(g)h}^{-1} g\hat{h})(i), q(g)h) = (i, h)\}| \leq \epsilon n |E|.$$

Thus the Condition 4 is satisfied and Γ is sofic. $\square$

It would be interesting to decide if all sofic-by-(residually amenable) groups are sofic. In a relation to the Proposition 7.8 we have the following open question:

Question 7.9. Are sofic-by-sofic or amenable-by-sofic groups sofic?

Let us discuss some variations of this question. Given two groups Γ and H there are two variations of the wreath product: **the unrestricted wreath product** $\Gamma \wr H$ and **the restricted wreath product**

$\Gamma \wr_r H$. Namely, let $\Gamma_\infty := \prod_{h\in H}\Gamma$ and $\Gamma_0 := \bigoplus_{g\in H}\Gamma$ be the direct product and the direct sum of copies of Γ. Then H acts on Γ_∞ and Γ_0 by shifting: $h((\gamma_g)_{g\in H}) = (\gamma_{h^{-1}g})_{g\in H}$. The unrestricted and the restricted wreath products are defined as semidirect products $\Gamma_\infty \rtimes H$ and $\Gamma_0 \rtimes H$ correspondingly. It is well known that Γ embeds into the unrestricted wreath product of Γ/H and H, refereed as universal embedding theorem. Indeed, let $c : \Gamma\times H \to \Gamma/H$ be the canonical cocycle associated to the quotient map, i.e.

$$c(g,h) = \widehat{q(q)t}^{-1} g\hat{t} \text{ for } t \in H, g \in \Gamma.$$

Then the following map is an isomorphism of Γ with a subgroup of $H \wr \Gamma/H$:

$$g \mapsto ((c(g,t))_{t\in H}, q(g)).$$

Thus the positive answer to the following question will imply the positive answer to the Question 7.9:

Question 7.10. Wether the unrestricted wreath product of an amenable (or sofic) group and a sofic group is necessarily sofic?

In Section 3 we will show that the restricted wreath product of amenable and sofic group is sofic. However the case of unrestricted wreath product should be of different nature and most likely the answer to the Question 7.10 it is negative.

8. Example of sofic group which is not in a limit of amenable groups

In Section 7 we have shown that amenable groups are sofic and a limit of sofic groups are sofic. The natural question posted by Gromov in [**28**] is whether there is a sofic group which is not a limit of amenable groups. In this section we will discuss an example of a modification of Abels' group due to de Cornulier which answers this question positively. More precisely, we will show that there exists a finitely presented non-amenable, sofic-by-amenable group which is an isolated point in Grigorchuk's topology. By Proposition 7.8 this group is sofic. Since it is isolated and not amenable it is not in a limit of amenable groups.

As in example 6.10, let p be a prime number and $GL_n(\mathbb{Z}[\frac{1}{p}])$ the generalized linear group over the ring $\mathbb{Z}[\frac{1}{p}]$. Consider the subgroup $\Gamma < GL_n(\mathbb{Z}[\frac{1}{p}])$ given by matrices:

$$\{\begin{pmatrix} a & b & u_{13} & u_{14} & u_{15} \\ c & d & u_{23} & u_{24} & u_{25} \\ 0 & 0 & p^n & u_{34} & u_{35} \\ 0 & 0 & 0 & p^k & u_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), u_{ij} \in \mathbb{Z}[\frac{1}{p}], n, k \in \mathbb{Z}\}.$$

Consider the following normal subgroups $M_{\mathbb{Z}[\frac{1}{p}]}, M_{\mathbb{Z}}$ of Γ formed by the following set of matrices:

$$\{\begin{pmatrix} 1 & 0 & 0 & 0 & m_1 \\ 0 & 1 & 0 & 0 & m_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : m_1, m_2 \in \mathbb{Z}[\frac{1}{p}] \text{ or } m_1, m_2 \in \mathbb{Z} \text{ correspondingly}\}.$$

We will show that $G := \Gamma/M_{\mathbb{Z}}$ is sofic and not a limit of amenable groups. In fact, much more than soficity can be proved: the group G is (locally residually finite)-by-abelian. Namely there exists a locally residually finite (each finitely generated subgroup of it is residually finite), normal subgroup Λ of G such that G/Λ is abelian.

Using homological properties of Γ and criteria of finitely presentability of Abels, [**2**], de Cornulier showed that $\Gamma/M_{\mathbb{Z}}$ is finitely presented. The reader can also adapt the proof of Abels, [**1**], in order to see the direct proof of finitely presentability of $\Gamma/M_{\mathbb{Z}}$. Thus, by Proposition 5.1, in order to prove that $\Gamma/_{M_{\mathbb{Z}}}$ is isolated in Grigorchuk's topology it is enough to show the following:

THEOREM 8.1. The group $\Gamma/M_{\mathbb{Z}}$ is finitely discriminable.

The proof will consist of several additional lemmas, which are based on ideas from [**15**]. Let Λ be a normal subgroup of Γ with the property:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and let U be a normal subgroup of Λ defined by

$$U = \{[a_{ij}] \in \Lambda : a_{ii} = e \text{ for every } 1 \leq i \leq 5\}.$$

It is easy to check that U is nilpotent.

LEMMA 8.2. Assume N is a normal subgroup of Γ containing $M_{\mathbb{Z}}$ and such that $N \cap U = M_{\mathbb{Z}}$, then $N = M_{\mathbb{Z}}$.

PROOF. Since $N \cap U = M_{\mathbb{Z}}$ we have that $N/M_{\mathbb{Z}} \cap U/M_{\mathbb{Z}} = e$. Thus $[N/M_{\mathbb{Z}}, U/M_{\mathbb{Z}}] = e$. In other words, $[\gamma, u] \in M_{\mathbb{Z}}$ for every $\gamma \in N$ and

$u \in U$. We will show that under these conditions $\gamma \in U$. For fixed $\gamma \in N$ and $u \in U$ let $t \in M_{\mathbb{Z}}$ be such that

$$\gamma u = t u \gamma \tag{3}$$

Then in order to find the required conditions on γ it is more efficient to write (3) in its matrix form and consider submatrices of this equation. Define $E_{s,t}$ to be the matrix with e on the (s,t)-th entry and 0 on the rest of the entries.

Writing $\gamma = [\gamma_{ij}]$ in its matrix form and using $[\gamma, 1 + E_{i,i+1}] \in M_{\mathbb{Z}}$ for every $2 \leq i \leq 4$ it is easy to check that $\gamma_{22} = \ldots = \gamma_{55} = 1$. Note that $1 + E_{1,2} \notin U$. The condition

(4)
$[\gamma, 1 + E_{1,3}] \in M_{\mathbb{Z}}$ implies $\gamma_{11} = 1, \gamma_{23} = \gamma_{21} + 1$

(5)
$[\gamma, 1 + E_{2,3}] \in M_{\mathbb{Z}}$ implies $\gamma_{12} = 0$

(6)
$[\gamma, 1 + E_{1,3} + E_{2,3}] \in M_{\mathbb{Z}}$ together with (4) and (5) implies $\gamma_{21} = 0$.

Thus $\gamma \in U$ and therefore $N \subseteq U$, but then $N = N \cap U = M_{\mathbb{Z}}$. □

From the lemma if follows that every normal subgroup N of $\Gamma/M_{\mathbb{Z}}$ has non-trivial intersection with $U/M_{\mathbb{Z}}$. But $U/M_{\mathbb{Z}}$ is nilpotent, as a factor of nilpotent group. So by the next lemma N has non-trivial intersection with the center of $U/M_{\mathbb{Z}}$.

Lemma 8.3. If G is nilpotent group then any non-trivial normal subgroup of G has non-trivial intersection with $Z(G)$.

Proof. Let $G_0 < G_1 < \ldots < G_n = G$ be the upper central series of G, i.e. $G_0 = Z(G)$ and $G_i = q_i^{-1}(Z(G/G_{i-1}))$, where $q_i : G \to G/G_i$ is the canonical quotient map. Let N be a normal subgroup of G. To reach a contradiction assume that N does not intersect $Z(G)$. Let i be such that $H := N \cap G_{i+1} \neq e$ and $N \cap G_i = e$. Note that $[G_{i+1}, G] \subseteq G_i$, since G_{i+1}/G_i is in the center of G/G_i. Since H is normal in G we have that $[H, G] \subseteq [G_{i+1}, G] \cap H \subseteq G_i \cap H = e$ thus H is a subgroup of $G_0 = Z(G)$, which contradicts to $N \cap G_0 = 1$. □

Now, to finish the proof of the theorem it is sufficient to show that $Z(U/M_{\mathbb{Z}}) = M/M_{\mathbb{Z}} \simeq \mathbb{Z}[p^{-1}]/\mathbb{Z} \times \mathbb{Z}[p^{-1}]/\mathbb{Z}$ is finitely discriminable. Since $M/M_{\mathbb{Z}}$ is abelian we need to provide a finite set $F \subset M/M_{\mathbb{Z}}$ such that any one-generated subgroup of $M/M_{\mathbb{Z}}$ has non-trivial intersection with F. Let $E = \{\frac{n}{p} + \mathbb{Z} : n = 0, 1, \ldots p-1\} \subset \mathbb{Z}[p^{-1}]/\mathbb{Z}$ and $F = E \times E$. Then the set F is discriminable. Indeed, let $z = (\frac{a}{p^k} + \mathbb{Z}, \frac{b}{p^m} + \mathbb{Z}) \in$

$M/M_{\mathbb{Z}}$ with a and b are not divisible by p. Without loss of generality we may assume that $k \geq m$. If $k > m$ then $p^{k-1}z \in F$ and if $k = m$ then $p^k z \in F$. Thus $\Gamma/M_{\mathbb{Z}}$ is finitely discriminable which proves Theorem 8.1.

THEOREM 8.4. The group $\Gamma/M_{\mathbb{Z}}$ is sofic.

PROOF. Let Γ_0 be a normal subgroup of Γ satisfying $n = k = 0$. Since $\Gamma/\Gamma_0 \simeq \mathbb{Z}^2$ by Proposition 7.8 it enough to show that $\Gamma_0/M_{\mathbb{Z}}$ is sofic. We will show that $\Gamma_0/M_{\mathbb{Z}}$ is locally residually finite, i.e every finitely generated subgroup of $\Gamma_0/M_{\mathbb{Z}}$ is residually finite, and hence it is sofic. Define an increasing family of subsets $\{\Gamma_n\}_{n\in\mathbb{N}}$ by the following restrictions:

$$u_{13}, u_{23}, u_{34}, u_{45} \in p^{-n}\mathbb{Z},$$
$$u_{14}, u_{24}, u_{35} \in p^{-2n}\mathbb{Z},$$
$$u_{15}, u_{25} \in p^{-3n}\mathbb{Z}.$$

It is easy to check that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a family of groups increasing to Γ. To prove the statement it is enough to show that $\Gamma_n/M_{\mathbb{Z}}$ is residually finite. Trivially $\Gamma_n/M_{\mathbb{Z}} \simeq (\Gamma_n \cap \Lambda)/M_{\mathbb{Z}} \rtimes SL_2(\mathbb{Z})$. Thus it is left to show that a semidirect product of finitely generated residually finite groups is residually finite. Indeed, let N be a residually finite finitely generated group and G be a residually finite group of automorphisms of N. Fix $(h, g) \in N \rtimes G$ with $(n, g) \neq e$. If $g \neq e$ then there exists a homorphism $\phi : G \to G_0$ with $\phi(g) \neq e$ and G_0 finite group, taking composition of homomorphisms ϕ and the canonical quotient $q : N \rtimes G \to G$ we have $\phi \circ q : N \rtimes G \to G_0$ is such that $\phi \circ q(n, g) \neq e$. Now assume $g = e$ and $n \neq e$. Let N_0 be a normal subgroup of N, $[N : N_0] = k < \infty$ and $n \notin N_0$. Then the set of subgroups $\{N' : N' \subseteq N, [N : N'] = k\}$ is finite. Also the index of $g(N_0)$ is k, thus the set $\{g(N_0) : g \in G\}$ is finite. Hence $N_1 = \bigcap_{g\in G} gN_0$ is of finite index in N and N_1 is normal and G-invariant. Then $N_1 \times e$ is normal in $N \rtimes G$. Let $\varphi : N \rtimes G \to N/N_1 \rtimes G$. Note that $\varphi(n, e) \neq e$. Let $G_1 = \{g \in G : g(h) = h \text{ for every } h \in N/N_1\}$, i.e. G_1 is the kernel of the action of G on N/N_1. Hence we have that G_1 is normal and G/G_1 is finite, since it is a subgroup of automorphisms of a finite group. Thus $\pi : N/N_1 \rtimes G \to N/N_1 \rtimes G/G_1$ is a map into a finite group with $\pi(\phi(n, e)) \neq e$, which finishes the proof. $\square$

CHAPTER 3

Ultraproduct constructions

1. Ultraproducts of groups with an invariant metric.

In this Section we give examples of groups with metric and consider their relation to sofic groups.

Let $d : \Gamma \times \Gamma \to \mathbb{R}_+$ be a metric on a discrete group Γ. A function d is called **an invariant metric** if for every x, y and g in Γ we have $d(x, y) = d(gx, gy) = d(xg, yg)$.

Let (Γ_i, d_i) be a family of discrete groups with invariant metric on them. Denote by $\prod_i \Gamma_i$ the direct product of Γ_i and choose a non-principal ultrafilter $\mathcal{U}$ on natural numbers. Then it is easy to check that

$$N = \{x \in \prod_i \Gamma_i : \lim_{\mathcal{U}} d(x_i, e) = 0\}$$

is normal subgroup of Γ. Denote by $\prod_{\mathcal{U}} \Gamma_i$ the quotient group $\prod_i \Gamma_i/N$. We will call $\prod_{\mathcal{U}} \Gamma_i$ the **the metric ultraproduct** of (Γ_i, d_i). Note that the group $\prod_{\mathcal{U}} \Gamma_i$ is complete topological group, which follows from the assumption that $\mathcal{U}$ is non-principal. It has invariant metric defined by

$$d(xN, yN) = \lim_{\mathcal{U}} d_i(x_i, y_i),$$

where $(x_i), (y_i)$ are representatives of the class of x and y.

Consider several important examples of invariant metrics on groups:

Example 1.1. The Hamming distance defined on the symmetric group $S(n)$ in the Section 2 is an invariant metric. Recall that for $\sigma, \tau \in S(n)$:

$$d_{hamm}(\sigma, \tau) = \frac{1}{n}|\{i : \sigma(i) \neq \tau(i)\}|.$$

Example 1.2. The uniform norm on the group of unitary operators acting on a Hilbert space H:

$$d_{norm}(u, v) = \frac{1}{2}||u - v|| \text{ for every } u, v \in U(H),$$

where $U(H)$ is the group of unitary operators on H. Since every discrete group can be represented by left regular representation as a group of unitary operators on $l_2(\Gamma)$ we have that there always exists an invariant metric on a discrete group.

EXAMPLE 1.3. If Γ is a subgroup of unitary n by n matrices over the field of complex numbers, denote by $U(n)$, then it poses the normalized Hilbert-Schmidt metric defined by

$$d_{tr}(u,v) = ||u-v||_2 = tr_n((u-v)^*(u-v)),$$

where tr_n is the normalized trace on $M_n(\mathbb{C})$.

2. Universal sofic groups. One more definition of soficity.

Consider a non-principal ultrafilter $\mathcal{U}$ on the set of natural numbers and let $f:\mathbb{N}\to\mathbb{N}$ be a function such that $\lim\limits_{\omega} f(i) = +\infty$. Consider the Hamming distance d_{hamm} on the symmetric group $S(n)$. The metric ultraproduct $\prod_{\mathcal{U}}(S(f(i)), d_{hamm})$ is called **universal sofic group**, denote by S_f. Then we have the following equivalent definition of soficity.

THEOREM 2.1. A group Γ is sofic if and only if Γ is a subgroup of a universal sofic group.

PROOF. Assume Γ is sofic. Let $\phi_i : \Gamma \to S(n_i)$ be an $(F_i, \frac{1}{i})$-approximation, where $\{F_i\}_{i\in\mathbb{N}}$ is an increasing to Γ sequence of finite subsets. Then from the condition 2 on approximation ϕ_i it follows that the map $\Phi:\Gamma\to\prod\limits_{\mathcal{U}}(S_{n_i}, d_{hamm})$ defined by

$$\Phi(g) = (\phi_i(g))_{\mathcal{U}}$$

is a homomorphism into a universal sofic group. Since ϕ_i satisfy also condition 3 we have that Φ is injective. Thus Γ is a subgroup of a universal sofic group.

To prove the converse assume that $\Gamma < S_f$ for some function $f:\mathbb{N}\to\mathbb{N}$. Let $F\subset\Gamma$ be a finite subset and fix $\varepsilon>0$. Considering coordinate projections we obtain a family of maps

$$\phi_i:\Gamma\to S(f(i))$$

with the property that there exists $i_0\in\mathbb{N}$ such that for all $i>i_0$ we have

(i) $|\{k : \phi_i(e)(k)\neq k\}| \leq \varepsilon f(i)$,

(ii) $|\{k : \phi_i(g)\phi_i(h)(k) \neq \phi_i(gh)\}| \leq \varepsilon f(i)$ for every g, h such that $gh \in F$,

(iii) For every $g \in F$ there exists a constant $C_g > 0$ such that $|\{k : \phi_i(g)(k) \neq k\}| \geq (1 - C_g)f(i)$.

Thus all conditions of Lemma 2.1 are satisfied and hence Γ is sofic group. □

Since $\mathcal{U}$ is a non-principal ultrafilter we have that universal sofic groups are complete topological groups. It was firstly noticed by Elek and Szabo [**20**] that all universal sofic groups are simple. Thus if Γ is sofic then it is a subgroup of a countable simple sofic group. Note however, that there are no examples of finitely presented (or even finitely generated) simple sofic groups. The main idea of the proof of the following theorem is based on the fact that permutations that are equal on large subsets represent the same element in a universal sofic group. In particular, changing 2 values of a permutation we may assume that it belongs to an alternating group, hence $\prod_{\mathcal{U}}(S(f(i)), d_{hamm}) = \prod_{\mathcal{U}}(A(f(i)), d_{hamm})$, where $A(n)$ is the alternating group.

In order to prove that all universal groups are sofic we will need the following simple lemma from [**5**].

LEMMA 2.2. Let n be odd and let g be an element of the alternating group $A(n)$ which does not have fixed points. Denote by $C_g := \{tgt^{-1} : t \in A(n)\}$ the conjugacy class of g. Then $C_g \cdot C_g$ contains all n-cycles.

PROOF. Consider a decomposition of g into the product of disjoint cycles, $g = \sigma_1 \cdot \sigma_2 \cdot \ldots \cdot \sigma_r$, where

$$\begin{aligned}
\sigma_1 &= (1, \ldots, k(1)),\\
\sigma_2 &= (k(1)+1, \ldots, k(2)),\\
&\cdot \quad \cdot \quad \cdot\\
\sigma_r &= (k(r-1)+1, \ldots, k(r)).
\end{aligned}$$

Let $t = (k(1), k(1)+1)(k(2), k(2)+1)\ldots(k(r-1), k(r-1)+1)$. Then it is straight forward to check that $gt^{-1}gt$ is a full cycle. Taking the conjugacy of $gt^{-1}gt$ we have that $C_g \cdot C_g$ contains all n-cycles. □

The next lemma was noticed by A. M. Gleason in 1962 and explicitly stated in [**30**].

LEMMA 2.3. Every element of $A(n)$ is a product of two n-cycles.

Proof. For $n = 1$ the statement is trivial. Assume that the statement is true for $A(k)$ for every $k < n$. Let $g \in A(n)$, then we may assume that $g(1) = 2$. Let $g' = (321)g$, then $g'(1) = 1$. Hence $g' \in A(n-1)$ and $g' = g_1g_2$ is a product of two $n-1$-cycles g_1, g_2 that fix point 1. Now g can be written as a product of two n-cycles, namely $g = (31) \cdot (21)g_1 \cdot (31)^{-1} \cdot (31)g_2$. □

Theorem 2.4. All universal sofic groups are simple.

Proof. Let S_f be a universal sofic group defined by a function $f : \mathbb{N} \to \mathbb{N}$ and an ultrafilter $\mathcal{U}$. We will show that the normal closure of any element of S_f coincides with the whole group.

Choose a lifting $(g_i)_{i\in\mathbb{N}}$ in the direct product $\prod\limits_{i\in\mathbb{N}} S(f(i))$ with the property that there are a constant $C > 0$ and $i_0 \in \mathbb{N}$ such that

$$|\{k : g_i(k) \neq k\}| > Cf(i) \text{ for all } i \geq i_0.$$

As it was remarked above we may assume that $g_i \in A(f(i))$. We may assume as well that $f(i)$ is odd. Thus combining Lemma 2.2 and Lemma 2.3 it is sufficient to show that the normal closure of g has a representative $(s_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} S(f(i))$ such that $s_i \in S(f(i))$ does not have fixed points for every $i \in \mathbb{N}$. Let V_i be the maximal subset of $\{1, \dots, f(i)\}$ where g_i does not have fixed points, we may assume that $|V_i|$ is odd. Then by Lemma 2.2 there exists $g' \in C_g \cdot C_g$, where C_g is the conjugacy class of g in S_f such that it has a lifting $(g'_i)_{i\in\mathbb{N}}$ to the direct with the property that g'_i is a standard $|V_i|$-cycle on the set V_i. Without loss a generality we may assume that $V_i = \{1, \dots, n_i\}$, where $n_i = |V_i|$. It is easy to see that if $\sigma \in S(f(i))$ is of the form $\sigma = (1, k)(2, k+1)\dots(n_i, k+n_i+1)$ then $g'_i\sigma g'_i\sigma^{-1}$ does not have fixed points in the set $\{1, \dots, n_i\} \cup \{k, \dots, k+n_i+1\}$. Denote by $\sigma_k = (1, kn_i+1) \cdot \dots \cdot (n_i, (k+1)n_i)$ the permutation conjugating by which we obtain shift of the support of g'_i from the set $\{1, \dots, n_i\}$ to the set $\{kn_i+1, \dots, (k+1)n_i\}$. Now if g'_i has more than n_i fixed points then denote

$$g''_i = \prod_{1\leq k\leq[\frac{f(i)}{n_i}]} g'_i\sigma_k g'_i\sigma_k^{-1}.$$

Since $\frac{f_i}{n_i} \leq \frac{1}{C}$ and C does not depend on i we have that the new sequence of $(g''_i)_{\mathbb{N}}$ is again a lifting of an element from the normal closure of g. Moreover, if $\{k_i, \dots, f(i)\}$ is the maximal set of fixed points of g''_i then $f(i) - k_i < n_i$. To finish the proof consider $t_i = g''_i\sigma g'_i\sigma^{-1}$, where $\sigma = (1, f(i)-n_i) \cdot \dots \cdot (n_i, f(i))$. Then $(t_i)_{i\in\mathbb{N}}$ is again a lifting of an element in the normal closure of g and t_i do not have fixed points, hence Lemma 2.2 and Lemma 2.3 applied to $(t_i)_{i\in\mathbb{N}}$ gives the statement. $\square$

CHAPTER 4

Sofic equivalence ralations

1. Free products of sofic groups with amalgamation over amenable group

2. HNN extension of sofic groups

Let H and K be two isomorphic subgroups of G with a given isomorphism $\pi : H \to K$. An HNN (Higman-Neumann-Neumann) extension of G with respect to H, denoted by $G*_\pi$, is the group generated by G and an extra generator t with relation $t^{-1}ht = \pi(h)$ for all $h \in H$. The following theorem appeared in [**18**].

THEOREM 2.1. The HNN extension of a sofic group with respect to amenable group is sofic.

PROOF. The proof is nothing but construction of HNN extension from a free product of sofic groups amalgamated over an amenable group using operations that preserve soficity. Let H and K be two isomorphic subgroups of G with a given isomorphism $\pi : H \to K$. For the HNN extension $G*_\pi$ it is well known that it is isomorphic to a semidirect product $K \rtimes \mathbb{Z}$ where K is the subgroup of G generated by $\cup_{k\in\mathbb{Z}} t^{-k}Gt^k$ with action of $\mathbb{Z}$ on K given by conjugation by t. Moreover, K is a direct limit of groups $K_{n,m} := \cup_{k\in\mathbb{Z}\cap[n,m]} t^{-k}Gt^k$. Each $K_{n,m}$ can be obtained as a free product of G amalgamated by H, namely $K_{n,m+1} = K_{n,m} *_H G$ and $K_{n-1,m} = G *_H K_{n,m}$ via canonical isomorphisms. Thus G is sofic. □

3. Soficity of a wreath product of amenable and sofic groups.

CHAPTER 5

Some conjectures that are valid for sofic groups

1. Kaplansky's direct finiteness conjecture

In early forties Kaplanski have formulated several important conjectures for the group ring $K[\Gamma]$, where K is field and Γ is a discrete group. One of the following conjectures of Kaplanski is still open and became even more exciting in the context of sofic groups:

Conjecture 1.1. Let Γ be a discrete group. Is it true that for every field K and $a, b \in K[\Gamma]$ the equation $ab = 1$ implies $ba = 1$?

Ara, O'Meara and Perera verified this conjecture for residually amenable groups, see [**3**]. Recently Elek and Szabo, [**21**] extended their result to the case of sofic groups.

Note that if a and b are elements of the group ring $\mathbb{C}[\Gamma]$ then $ab = 1$ implies $ba = 1$, see [**32**], [**26**] and [**37**]. We will give an elementary proof of this fact. The group ring $\mathbb{C}[\Gamma]$ is a subalgebra of von Neumann algebra $vN(\Gamma) = \lambda(\Gamma)''$, where λ is the left regular representation of Γ on $l_2(\Gamma)$. Define a functional on $\mathbb{C}$ by:

$$\tau(\lambda_0 e + \lambda_1 g_1 + \ldots + \lambda_n g_n) = \lambda_0.$$

Clearly it is extendable to $vN(\Gamma)$ by $\tau(a) = \langle a\delta_e, \delta_e \rangle$ for every $a \in vN(\Gamma)$. The properties of τ that are needed for our purposes are:

(i) τ is unital and tracial: $\tau(1) = 1$ and $\tau(ab) = \tau(ba)$ for every $a, b \in \Gamma$,

(ii) positive: $\tau(a^*a) \geq 0$ for every $a \in vN(\Gamma)$,

(iii) faithful: $\tau(a^*a) = 0$ then $a = 0$ for every $a \in vN(\Gamma)$.

From the equation $ab = 1$ follows that ba is an idempotent with $\tau(ba) = 1$. Now, in order to show that $ba = 1$ it is enough to show that if p is an idempotent in $vN(\Gamma)$ with $\tau(p) = 1$ then $p = 1$. To prove the last statement we will give a simple trick of Burger and Valette, [**8**]. The trick was used in order to give a simple proof of Zallesskii's theorem, [**51**], which states that τ can take only rational values on the idempotents of $\mathbb{C}[\Gamma]$. Let $p \in vN(\Gamma)$ be an idempotent. Then $z = 1+(p^*-p)^*(p^*-p) = 1-p^*-p+p^*p+pp^*$ is invertible in $vN(\Gamma)$. Let $q = pp^*z^{-1}$, then $pp^*z = (pp^*)^2$ and hence $q^2 = q$. Since z commutes with p we have $q^* = q$. Moreover $pq = q$ and $qp = p$. Therefore $\tau(q) = \tau(pq) = \tau(qp) = \tau(p) = 1$. Now $\tau((1-q)^*(1-q)) = 1-\tau(q) = 0$. Since τ is faithful we have that $q = 1$ and thus $p = 1$.

In fact in the prove above we work in the closure of $\mathbb{C}[\Gamma]$ in the norm topology of $B(l_2(\Gamma))$. In [**37**] showed that $\mathbb{C}[\Gamma]$ satisfies Kaplanski working only inside the group ring $\mathbb{C}[\Gamma]$.

Since every separable field of characteristic 0 is a subfield of $\mathbb{C}$ we have that $K[\Gamma]$ is directly finite for every Γ and every field K of characteristic 0.

The idea behind the proof of Kaplanski conjecture for sofic groups is quite similar to the case of the group rings over the complex field. For a fixed field K and a discrete group Γ, we will embed the group ring $K[\Gamma]$ into a ring $\mathcal{R}$ where the Kaplanski condition: $ab = 1$ implies $ba = 1$ is satisfied for all elements $a, b \in \Gamma$. Let $\mathcal{R}_\infty = \prod_\alpha M_{n_\alpha}(K)$ be a direct product of matrix algebras over the field K. Define a *pseudo-rank function* of E as follows:

$$N(\{r_\alpha\}) = \lim_\omega \frac{\dim_K rank(r_\alpha)}{n_\alpha}.$$

It satisfies the following obvious properties:

(i) $N(0) = 0$, $N(1) = 1$, $N(x) \in [0, 1]$ for every $x \in \mathcal{R}_\infty$.

(ii) $N(x + y) \leq N(x) + N(y)$.

(iii) $N(xy) \leq \min\{N(x), N(y)\}$ for every $x, y \in \mathcal{R}_\infty$.

(iv) $N(e + f) = N(e) + N(f)$ if $e^2 = e$, $f^2 = f$ and $ef = fe = 0$.

Using the above property 3 it follows that the kernel of N is an ideal in $\mathcal{R}_\infty$. Then the ring that we are looking for is $\mathcal{R} := \mathcal{R}_\infty / \ker(N)$. In Theorem 1.3 we will show that Kaplanski's condition is satisfied in $\mathcal{R}$.

The key property of $\mathcal{R}$ is that it admits a natural pseudo-rank function $\overline{N} : \mathcal{R} \to [0,1]$ defined by

$$\overline{N}(x + \ker(N)) = N(x), x \in \mathcal{R}.$$

Obviously, $\overline{N}$ is faithful, i.e., if $\overline{N}(x) = 0$ for some $x \in \mathcal{R}$ then $x = 0$.

Using the following lemma we can lift every idempotent from $\mathcal{R}$ to $\mathcal{R}_\infty$.

Lemma 1.2. Let $e \in \mathcal{R}$ be such that $e^2 = e$. Then there exist $\hat{e} = \{e_\alpha\} \in \mathcal{R}_\infty$ such that $\hat{e} \in e + \ker(N)$ and $e_\alpha^2 = e_\alpha$.

Proof. Consider arbitrary representative of e, $e = e' + \ker(N)$ with $e' \in \mathcal{R}$. Let $e' = \{r_\alpha\}$ where $r_\alpha \in M_{n_\alpha}(D)$. Then we have

$$\lim_\omega \frac{\dim_K rank(r_\alpha^2 - r_\alpha)}{n_\alpha} = 0 \iff \lim_\omega \frac{\dim_K \ker(r_\alpha^2 - r_\alpha)}{n_\alpha} = 1.$$

Let $V_\alpha = \ker(r_\alpha^2 - r_\alpha) \subseteq D^{n_\alpha}$. Then for every $v \in V_\alpha$ we have $r_\alpha^2 v = r_\alpha v$ and $r_\alpha v \in V_\alpha$. Thus $r_\alpha(V_\alpha) \subseteq V_\alpha$ and r_α is an idempotent on V_α. Let $e_\alpha \in M_{n_\alpha}(D)$ be an idempotent that coincides with r_α on V_α, for instance one can take e_α to be 0 on a complement to V_α. Then $\{e_\alpha\} = \{r_\alpha\}$ in $\mathcal{R}$. Indeed, $\dim_D rank(r_\alpha - e_\alpha) = n_\alpha - \dim\ker(r_\alpha - p_\alpha)$, but $V_\alpha \subseteq \ker(r_\alpha - e_\alpha)$, so $\lim_\omega \frac{\dim V_\alpha}{n_\alpha} = 1$, therefore $\overline{N}(\{r_\alpha\} - \{e_\alpha\}) = 0$. □

Theorem 1.3. Let $a, b \in \mathcal{R}$ and $ab = 1$ then $ba = 1$.

Proof. Let $e = ba$ then $e^2 = baba = ba = e$. So by Lemma 1.2 there exists $\hat{e} = \{e_\alpha\}$ in $\mathcal{R}_\infty = \prod M_{n_\alpha}(K)$ such that $e = x + \ker(N)$ and e_α are idempotents in M_{n_α}. We have

$$1 = \overline{N}(1) = \overline{N}(ab) \leq \overline{N}(b) \leq 1 \implies \bar{b} = 1$$

$$\overline{N}(e) = \overline{N}(ba) \geq \overline{N}(bab) \geq \overline{N}(b) = 1 \implies \overline{N}(e) = 1.$$

It follows that $N(\hat{e}) = 0$. Since $\hat{e}$ and $1 - \hat{e}$ are orthogonal idempotents we have

$$1 = N(1) = N(\hat{e} + (1 - \hat{e})) = N(\hat{e}) + N(1 - \hat{e}) \implies N(1 - \hat{e}) = 0.$$

Thus $\hat{e} \in 1 + \ker(N)$ and therefore $e = ba = 1$ in $\mathcal{R}$.

□

Now in order to verify Kaplanski conjecture for sofic groups is is sufficient to embed the group ring $K[\Gamma]$ into $\mathcal{R}$.

Theorem 1.4. Let Γ be sofic group and K be a field. Then $K[\Gamma]$ is a subring of $\mathcal{R}$.

PROOF. Let $\{F_i\}_{i\in\mathbb{N}}$ be an increasing to Γ sequence of finite sets. Let $\phi_i : \Gamma \to S(n_i)$ be an $(F_i, \frac{1}{i})$-approximation of Γ. By canonically identifying $S(n_i)$ with n_i by n_i permutation matrices we define a map $\Phi : K[\Gamma] \to \mathcal{R}_\infty$ by

$$\Phi(g) = (\phi_i(g))_{i\in\mathbb{N}}, g \in \Gamma.$$

Denote by $\overline{\Phi}$ the composition of Φ with the quotient map $\mathcal{R} \to \mathcal{R}_\infty / \ker(N) = \mathcal{R}$. We will show that $\overline{\Phi}$ is an injective homomorphism.

From the definition of $(F_i, \frac{1}{i})$-approximation we have:

$$\begin{aligned} N(\Phi(g)\Phi(h) - \Phi(gh)) &= \lim_\omega \frac{\dim_K rank(\phi_i(g)\phi_i(h) - \phi_i(gh))}{n_i} \\ &\leq \lim_\omega \frac{|\{j : \phi_i(g)\phi_i(h)j \neq \phi_i(gh)(j)\}|}{n_i} \\ &= \lim_\omega \frac{1}{i} = 0. \end{aligned}$$

To show the injectivity of $\overline{\Phi}$ consider a finite sum $\sum_{s\in S} k_s s \in K[\Gamma]$ with non-zero coefficients $k_s \in K$. We will show that $N(\sum_{s\in S} k_s\phi_i(s))) \neq 0$.

Let $X_i \subseteq \{1 \dots n\}$ be a maximal set with the property:

(7) $\quad \phi_i(s_1)(k) \neq \phi_i(s_2)(l)$ for every $s_1, s_2 \in S$ and $k, l \in X_i, k \neq l$.

Denote $Y_i = \{j \in \{i, \dots, n\} : \phi_i(s_1)(j) \neq \phi_i(s_2)(j)$ for all $s_1, s_2 \in S\}$. We may assume $S \subseteq F_i$ for i large enough. Then from the definition of $(S, \frac{1}{i})$-approximation we have $|Y_i| \geq (1 - \frac{|S|^2}{i})n_i$. It is easy to check that $\sum_{s\in S} k_s\phi_i(s))$ is injective on $span\{e_k : k \in X_i \cap Y_i\}$ thus

$$\dim_K rank(\sum_{s\in S} k_s\phi_i(s))) \geq |X_i \cap Y_i|.$$

In order to estimate $|X_i \cap Y_i|$ consider $j \in \{1, \dots, n\}\backslash X_i$. Since X_i is maximal with property 7 there exist s_1, s_2 in S and $k \in X_i$ such that

$$\phi_i(s_1)(j) = \phi_i(s_2)(k).$$

Since $\phi_i(s_2)(k)$ can take at most $|S||X_i|$ different values when s_2 and k vary and $\phi_i(s)$ is injective on $\{1, \dots, n\}$ for every $s \in S$ we have that the number of all possible values of j from 7 is at most $|S| \cdot |X_i$. Thus $n_i - |X_i| \leq |S|^2|X_i|$ and

$$
\begin{aligned}
N(\sum_{s\in S} k_s\Phi(s)) &= \lim_{\omega} \frac{\dim_K rank(\sum_{s\in S} k_s\phi_i(s))}{n_i} \\
&\geq \lim_{\omega} \frac{|X_i\cap Y_i|}{n_i} \\
&\geq \lim_{\omega} \frac{1}{1+|S|^2} - \frac{1}{i} \\
&= \frac{1}{1+|S|^2}.
\end{aligned}
$$

Thus Φ is an injective homomorphism. □

2. Connes' embedding conjecture for sofic groups

Let $\omega \in \beta(\mathbb{N})\setminus\mathbb{N}$ be a free ultrafilter on $\mathbb{N}$ and R be the hyperfinite II_1-factor with faithful tracial normal state τ. Then the subset I_ω in $l^\infty(\mathbb{N}, R)$ consisting of $(x_1, x_2, \ldots)$ with $\lim_{n\to\omega} \tau(x_n^* x_n) = 0$ is a closed ideal in $l^\infty(\mathbb{N}, R)$ and a quotient algebra $R^\omega = l^\infty(\mathbb{N}, R)/I_\omega$ is a von Neumann II_1-factor called *ultrapower* of R. It is naturally endowed with a faithful tracial normal state

$$\tau_\omega((x_n) + I_\omega) = \lim_{n\to\omega} \tau(x_n).$$

The following have been conjectured by Alain Connes in [**14**]:

The Connes Embedding Conjecture: Any separable II_1-factor embeds into the ultrapower $\mathcal{R}^\omega$ of the hyperfinite factor $\mathcal{R}$.

Connes' embedding problem is known to be equivalent to a number of different problems, in large part due to a remarkable paper [**34**] of Kirchberg. We refer also to the survey [**36**], the Chapter on QWEP in Pisier's book [**41**] and the papers [**42**], [**43**], [**44**], [**6**], [**45**], [**49**], [**12**], [**46**], [**17**] for results with bearing on Connes' embedding problem.

In particular the conjecture is open in the case of the group von Neumann algebras. A group Γ is called *hyperlinear* if its group von Neumann algebra $vN(\Gamma)$ embeds into $\mathcal{R}^\omega$. It is well known that Γ is hyperlinear if and only if it is a subgroup of the unitary group of $\mathcal{R}^\omega$. We will show that all sofic groups are hyperlinear.

3. Approximation of L_2-invariants: the Determinant conjecture

Let N be a von Neumann algebra with a faithful normal trace τ. **The spectral density function** associated to a positive operator $\Delta \in N$ is a function $F_\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$F_\Delta(\lambda) = \tau(\chi_{[0,\lambda]}(\Delta)).$$

Using spectral densities we can define a regularized determinant, called **Fuglede-Kadison determinant**, by

$$\ln \det_N(\Delta) = \begin{cases} \int\limits_{0+}^{+\infty} \ln(\lambda) dF_\Delta(\lambda), & \text{if the integral converges,} \\ -\infty, & \text{otherwise.} \end{cases}$$

Let $L\Gamma$ be the group von Neumann algebra of a discrete group Γ acting on $l_2(\Gamma)$ with faithful normal trace defined by

$$\tau_\Gamma(a) = \langle a\delta_e, \delta_e \rangle, \text{ for } a \in L\Gamma.$$

Denote by τ the linear functional on $M_d(L\Gamma)$ defined by

$$\tau((a_{ij})) = (1/d) \sum_{i=1}^{n} \tau(a_{ii}) \text{ for } (a_{ij}) \in M_d(L\Gamma).$$

The Determinant Conjecture: $\ln \det(\Lambda) \geq 0$ for every positive $\Lambda \in M_d(\mathbb{Z}\Gamma) \subset M_d(L\Gamma)$.

The conjecture is known to be true for the large class of groups mainly due to the work of Schick [**47**]. Recently Elek and Szabo in [**20**] verified the conjecture for sofic groups. Here is the proof from [**20**].

To get the positivity of the determinant for sofic groups we will use the following theorem due to Schick, [**47**].

Theorem 3.1. Let N be a von Neumann algebra with a faithful normal trace τ and let $\Delta \in N$ with $\Delta \geq 0$. Assume that there is a sequence of von Neumann algebras N_n with traces τ_n and positive elements $\Delta_n \in N_n$, with $\|\Delta_n\|$ uniformly bounded above, and such that

(i) $\lim\limits_{n\to\infty} \tau_n(\Delta_n^k) = \tau(\Delta^k)$ for every $k \geq 1$,

(ii) $\ln \det_{N_n}(\Delta_n)$ is uniformly bounded below.

Then condition 1 implies that

$$\ln \det_N(\Delta) \geq \limsup_{n\to\infty} \ln \det_{N_n}(\Delta_n), \tag{8}$$

and 1 and 2 together imply that

(9) $$\lim_{n\to\infty} F_{\Delta_n}(0) = F_\Delta(0), \text{ where } F_{\Delta_n}(\lambda) = \tau_n(\chi_{[0,\lambda]}(\Delta_n)).$$

Proof. Condition 1 implies by linearity that $\lim_{n\to\infty} \tau_n(p(\Delta_n)) = \tau(p(\Delta))$ for any polynomial p. Since $\|\Delta_n\|$ is uniformly bounded above, there is $K > 0$ such that $\|\Delta_n\|, \|\Delta\| \le K$. Weierstrass theorem gives that

$$\lim_{n\to\infty} \tau_n(p(\Delta_n)) = \tau(p(\Delta))$$

for any continuous function p on $[0, K]$.

Define

$$\ln^\varepsilon(x) = \begin{cases} \ln(x), \text{ if } x \ge \varepsilon \\ 0, \text{ if } x < \varepsilon. \end{cases} = \int_\varepsilon^\infty \ln(\lambda) d\chi_{[0,\lambda]}(x).$$

We have $\ln\det_N(\Delta) = \lim_{\varepsilon\to 0+} \tau(\ln^\varepsilon(\Delta))$. Indeed,

$$\ln\det_N(\Delta) = \int_{0+}^\infty \ln(\lambda) dF_\Delta(\lambda) = \lim_{\varepsilon\to 0+} \int_\varepsilon^\infty \ln(\lambda) d\tau(\chi_{[0,\lambda]}(\Delta)) =$$

$$= \lim_{\varepsilon\to 0+} \tau\left(\int_\varepsilon^\infty \ln(\lambda) d\chi_{[0,\lambda]}(\Delta)\right) = \lim_{\varepsilon\to 0+} \tau(\ln^\varepsilon(\Delta)).$$

To prove the first conclusion of the theorem, note that if $1 > \varepsilon > \varepsilon'$ then $\ln^{\varepsilon'} \le \ln^\varepsilon$. Thus for $\varepsilon < 1$ the function $\varepsilon \mapsto \tau(\ln^\varepsilon(\Delta))$ is non-increasing. Let $1 > \varepsilon > \varepsilon'$ and $g_{\varepsilon,\varepsilon'}$ be a continuous function on $[0, K]$ such that $\ln^\varepsilon(x) \ge g_{\varepsilon,\varepsilon'}(x) \ge \ln^{\varepsilon'}(x)$. Thus we have

$$\tau(\ln^\varepsilon(\Delta)) \ge \tau(g_{\varepsilon,\varepsilon'}(\Delta)) = \lim_{n\to\infty} \tau_n(g_{\varepsilon,\varepsilon'}(\Delta_n)) \ge \limsup_{n\to\infty} \tau_n(\ln^{\varepsilon'}(\Delta_n)) \ge \limsup_{n\to\infty} \ln\det_{N_n}(\Delta_n).$$

We now prove the second part of the theorem. Consider continuous function $\phi_m : [0, K] \to \mathbb{R}$ given by

$$\phi_m(\lambda) = \begin{cases} (1 - m\lambda)\ln(\frac{1}{m}), \text{ if } 0 \le \lambda \le 1/m \\ 0, \text{ if } 1/m < \lambda \le K. \end{cases}$$

By the remark at the beginning of the proof we have

$$\tau(\phi_m(\Delta)) = \lim_{n\to\infty} \tau_n(\phi_m(\Delta_n)).$$

On the other hand, since for every $\varepsilon > 0$ we have $\ln^{\varepsilon}(x) \leq \chi_{[\varepsilon,1]}\phi_m(x) + \ln(K)$ on $x \in [0, K]$, we obtain

$$\tau_n(\ln^{\varepsilon}(\Delta_n)) \leq \tau_n(\chi_{[\varepsilon,1]}\phi_m(\Delta_n)) + \ln(K)$$
$$\lim_{\varepsilon\to 0+} \tau_n(\ln^{\varepsilon}(\Delta_n)) \leq \tau_n(\chi_{(0,1]}\phi_m(\Delta_n)) + \ln(K).$$

Note that for $m \geq 1$ we have $\chi_{(0,1]}\phi_m \leq 0$ thus $\tau_n(\chi_{(0,1]}\phi_m(\Delta_n)) \leq 0$. By the condition 2 of the theorem there exists D such that $\lim_{\varepsilon\to 0+} \tau_n(\ln^{\varepsilon}(\Delta_n)) = \ln \det_{N_n}(\Delta_n) \geq D$, thus

$$D - \ln(K) \leq \tau_n(\chi_{(0,1]}\phi_m(\Delta_n)) \leq 0.$$

Dividing by $-\ln(m)$ we obtain:

$$0 \leq \tau_n\left(\left(\chi_{(0,1]}\frac{\phi_m}{-\ln(m)}\right)(\Delta_n)\right) \leq \frac{\ln(K) - D}{\ln(m)}$$

Let $\psi_m = -\phi_m/\ln(m)$, then

$$\tau_n((\chi_{(0,1]}\psi_m)(\Delta_n)) = \tau_n(\chi_{[0,1]}\psi_m(\Delta_n)) - \tau_n((\chi_{\{0\}}\psi_m)(\Delta_n)),$$

$\chi_{[0,1]}\psi_m = \psi_m$ on $[0, K]$ and $\chi_{\{0\}}\psi_m = \chi_{\{0\}}$. Thus we have

$$0 \leq \tau_n(\psi_m(\Delta_n)) - F_{\Delta_n}(0) \leq \frac{\ln(K) - D}{\ln(m)}. \tag{10}$$

Since $\psi_m \to \chi_{\{0\}}$ when $m \to \infty$ pointwise and $\{\psi_m\}$ is a uniformly bounded sequence we have $\tau(\psi_m(\Delta)) \to \tau(\chi_{\{0\}}(\Delta)) = F_{\Delta}(0)$. Taking first $n \to \infty$ and then $m \to \infty$ in (10) we obtain that the limit $\lim_{n\to\infty} F_{\Delta_n}(0)$ exists and is equal to $F_{\Delta}(0)$.

□

Now the determinant conjecture for sofic groups follows from the following constructions and three lemmas after it (see [**47**]).

Note first that we can write $\Lambda \in M_d(\mathbb{Z}\Gamma)$ as a finite sum $\Lambda = \sum_{\gamma\in\Gamma} \Lambda_\gamma \gamma$, with $\Lambda_\gamma \in M_d(\mathbb{Z})$. Let $S = \{\gamma | \Lambda_\gamma \neq 0\}$. Then $\tau(\Lambda) = tr_d(\Lambda_e)$ and more generally

$$\tau(\Lambda^k) = \sum_{\gamma_1,\dots,\gamma_k\in S:\gamma_1\cdots\gamma_k=e} tr_d(\Lambda_{\gamma_k}\cdots\Lambda_{\gamma_1}).$$

Note that since Λ is positive, we have $\Lambda^* = \Lambda$, and hence $\Lambda_\gamma = \Lambda_{\gamma^{-1}}$. In particular, S is a symmetric subset of Γ. Let Γ_0 be the subgroup generated by S. Since Γ is sofic, Γ_0 is sofic.

Let $V = V_{r,\delta}$ be a directed $S-$labeled graph from the definition of soficity of Γ_0. Denote by V_0 the subset of vertices v with unit ball $B_V(v, r) \simeq B_\Gamma(e, 1)$, and by V_1 the set of those v that $B_V(v, r) \simeq B_\Gamma(e, r)$. Note that $V_1 \subset V_0$.

Define M as the von Neumann algebra of all linear maps of the space $l^2(V, \mathbb{C}^d)$. Any element of M is given by a map $V^2 \to M_d(\mathbb{C})$.

Define $A \in M$ by the rule

$$A(w,v) = \begin{cases} \Lambda_\gamma \text{ if } v \in V_0 \text{ and } w = \gamma v \\ 0 \text{ if } v \notin V_0 \end{cases} \tag{11}$$

In particular, if $A(w,v) \neq 0$ then there is an edge from v to w.

Also, if both $v, w \in V_0$ then $A(w,v) = A(v,w)$, since $\Lambda_\gamma = \Lambda_{\gamma^{-1}}$ for all $\gamma \in S$.

LEMMA 3.2. If $2k < r$ then

$$\left|\tau((A^*A)^k) - \tau(\Lambda^{2k})\right| \leq \delta \left(\tau(\Lambda^{2k}) + |S|^{2k} (\max_\gamma \|\Lambda_\gamma\|)^{2k} \right)$$

PROOF. Since $2k < r$ we have for $v \in V_1$

$$(A^*A)^k(w,v) = \sum A(w, v_{2k-1}) A(v_{2k-1}, v_{2k-2}) \cdots A(v_2, v_1) A(v_1, v),$$

where the sum is over all paths of length $2k$ from v to w. Indeed, it suffices to note that $v \in V_1$ implies that $v_i \in V_0$ for any $i \geq 2k-1$, and hence $A^*(v_{i+1}, v_i) = A(v_{i+1}, v_i)$.

Moreover, since $v \in V_1$, any path of length not greater then r which starts at v is determined by it labeling. So finally we have that

$$(A^*A)^k(w,v) = \sum \Lambda_{\gamma_{2k}} \Lambda_{\gamma_{2k-1}} \cdots \Lambda_{\gamma_2} \Lambda_{\gamma_1},$$

where the sum runs over all sequences $\gamma_1, \ldots, \gamma_{2k} \in S$ such that

$$w = (\gamma_{2k}(\ldots(\gamma_1 v)\ldots)) = (\gamma_{2k} \cdots \gamma_1) v.$$

We can now compute the trace of $(A^*A)^k$. To start, note that if $v \in V_1$ and $2k < r$ then $v = (\gamma_{2k} \cdots \gamma_1) v$ if and only if $\gamma_{2k} \cdots \gamma_1 = e$.

$$(A^*A)^k(v,v) = \sum_{\substack{\gamma_{2k}, \ldots, \gamma_1 \in S \\ \gamma_{2k} \cdots \gamma_1 = e}} \Lambda_{\gamma_{2k}} \Lambda_{\gamma_{2k-1}} \cdots \Lambda_{\gamma_2} \Lambda_{\gamma_1} = (\Lambda^{2k})_e$$

On the other hand, if $v \in V \setminus V_1$ then, since each vertex has at most $|S|$ neighbours,

$$\|(A^*A)^k(v,v)\| \leq |S|^{2k} (\max_\gamma \|\Lambda_\gamma\|)^{2k}.$$

Hence

$$\left|\tau((A^*A)^k) - \tau(\Lambda^{2k})\right| \leq \frac{|V| - |V_1|}{|V|} \left(\tau(\Lambda^{2k}) + |S|^{2k} (\max_\gamma \|\Lambda_\gamma\|)^{2k} \right) \leq$$
$$\delta \left(\tau(\Lambda^{2k}) + |S|^{2k} (\max_\gamma \|\Lambda_\gamma\|)^{2k} \right).$$

□

LEMMA 3.3.

$$\|A\| \leq \|\Lambda\| |S|^{1/2}.$$

PROOF. if $x \in l^2(V, \mathbb{C}^d)$ then if $v \in V_0$

$$(xA)(v) = \sum_{w \in V} x(w) A(w, v) = \sum_{\gamma \in S} x(\gamma v) \Lambda_\gamma = (y_v \Lambda)(e),$$

where $y_v \in l^2(\Gamma, \mathbb{C}^d)$ is $y_v = \sum_{\gamma \in S} x(\gamma v) \cdot \gamma^{-1}$. If $v \notin V_0$ then $(xA)(v) = 0$. It follows that if $v \in V_0$ we obtain

$$\|xA\|^2 \leq \|\Lambda\|^2 \sum_{v \in V_0} \|y_v\|^2 \leq |S| \|\Lambda\|^2 \|x\|^2,$$

since there are no more then $|S|$ vertices adjacent to any given vertex.

□

LEMMA 3.4. Since M is finite dimensional, for positive element $B \in M$ we have that $\ln \det_M(B)$ equals to the logarithm of the product of nonzeroeigenvalues of B, divided by $d|V|$. In particular, $\ln \det_M(A^*A) \geq 0$.

PROOF. We have that $\ln \det_M(B) = \lim_{\varepsilon \to 0+} \tau(\ln^\varepsilon(B))$, and it suffices to notice that if $\alpha_1, \ldots, \alpha_s$ are eigenvalues of B greater then ε then $\tau(\ln^\varepsilon(B)) = 1/(d|V|) \sum_i \ln(\alpha_i) = \ln(\prod_i \alpha_i)/(d|V|)$.

Since each Λ_γ, and hence A has integer coefficients, we obtain that the characteristic polynomial of A has integer coefficient. It is left to note that the product of nonzero eigenvalues of A is equal to the coefficient of the nonzero monomial with smallest possible degree in this polynomial. □

CHAPTER 6

Entropy

1. Measure entropy and the classification of Bernoulli actions

By a *standard probability space* (X, μ) we mean a standard Borel space X equipped with a probability measure on its Borel σ-algebra. By a *p.m.p. (probability-measure-preserving) action* of a group G we mean action of G by measure-preserving automorphisms of a standard probability space (X, μ), and we write $G \curvearrowright (X, \mu)$. We similarly also speak of a *p.m.p. transformation*, which can be viewed as a generator for a $\mathbb{Z}$-action. A central goal of ergodic theory is to classify p.m.p. actions up to conjugacy. Two p.m.p. actions $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ of a given group are *conjugate* if there exists an invertible bimeasurable map $\varphi : X \to Y$ such that $\nu(\varphi(A)) = \mu(A)$ for all measurable $A \subseteq X$ and $s\varphi(x) = \varphi(sx)$ for all $s \in G$ and a.e. $x \in X$.

1.1. Kolmogorov-Sinai entropy. In the late 1950s Kolmogorov introduced the idea of entropy into ergodic theory as a conjugacy invariant for p.m.p. transformations. Kolmogorov's main motivation was to resolve the problem of whether there exist nonconjugate Bernoulli shifts. For a Bernoulli shift $T : (Y, \nu)^{\mathbb{Z}} \to (Y, \nu)^{\mathbb{Z}}$, where (Y, ν) is a standard probability space and the action is by coordinate translation, the associated unitary representation $n \mapsto U^n$ of $\mathbb{Z}$ on $L^2(Y, \nu)$ given by $U^n f(x) = f(T^{-n}x)$ (called the *Koopman representation*) is always the direct sum of infinitely many copies of the regular representation along with a copy of the trivial representation, and so Bernoulli shifts cannot be distinguished by spectral means. In contrast, for discrete spetcrum transformations like rotation on a circle, the set of eigenvalues of the corresponding unitary operator counted with multiplicity is a complete invariant, as shown by Halmos and von Neumann. While the measure of an intersection of two sets naturally registers in the Koopman representation via the inner product, which can thereby be used to express notions like ergodicity and weak mixing, entropy reflects the higher-order statistics of set intersections under interation and hence is an more of an algebraic phenomenon. Indeed the product structure of a Bernoulli shift $T : (Y, \nu)^{\mathbb{Z}} \to (Y, \nu)^{\mathbb{Z}}$ is the prototype for entropy, which in this case is equal to the Shannon entropy of the base (Y, ν). A celebrated theorem of Ornstein asserts that entropy complete invariant for Bernoulli shifts, and this was extended to Bernoulli actions of countable amenable groups by Ornstein and Weiss (Theorem 1.1).

Write $H(\mathcal{P})$ for the Shannon entropy $\sum_{A\in\mathcal{P}} -\mu(A)\log\mu(A)$ of a partition $\mathcal{P}$. This is the integral of the *information function* given by $x \mapsto \log\mu(A)$ where $x \in A$, which is a measure of how much information is gained in learning that the otherwise unknown point x lies in a particular partition element A. The smaller the measure of A, the more likely we are able to distinguish x from a random point which we are similarly only able to locate up to membership in some element of $\mathcal{P}$. Kolmogorov showed that, for a p.m.p. transformation T of a standard probability space (X, μ), the limit

$$\lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}),$$

which exists by subadditivity, is always the same among partitions $\mathcal{P}$ which are *generating* in the sense that the partitions $T^n\mathcal{P}$ for $n \in \mathbb{Z}$ generate the σ-algebra up to sets of measure zero. Kolmogorov's definition of entropy as this common value was extended by Sinai to general transformations T by taking the supremum

$$h_\mu(T) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{n} H(\mathcal{P} \vee T^{-1}\mathcal{P} \vee \cdots \vee T^{-n+1}\mathcal{P}) \tag{12}$$

over all finite partitions $\mathcal{P}$. The Kolmogorov-Sinai theorem asserts that this supremum is achieved on any finite generating partition. Note that the Kolmogorov-Sinai entropy (12) is invariant under *conjugacy*, which for p.m.p. actions $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ of a general group G means the existence of an invertible bimeasurable map $\varphi : X \to Y$ such that $\nu(\varphi(A)) = \mu(A)$ for all measurable $A \subseteq X$ and $s\varphi(x) = \varphi(sx)$ for all $s \in G$ and a.e. $x \in X$.

The general theory of entropy for p.m.p. actions of amenable groups was largely developed by Ornstein and Weiss in [ref???]. Given such an action $G \curvearrowright (X, \mu)$, we fix a Følner sequence $\{F_n\}$ for G and define the entropy as

$$h_\mu(X, G) = \sup_{\mathcal{P}} \lim_{n\to\infty} \frac{1}{|F_n|} H\Bigg(\bigvee_{s\in F_n} s^{-1}\mathcal{P}\Bigg) \tag{13}$$

where $\mathcal{P}$ ranges over all finite measurable partitions of X. The limit exists by subadditivity, which can also be used to show that it is independent of the choice of Følner sequence. As for single transformations, the supremum is achieved on generating partitions. This can be used to see that for a Bernoulli action $G \curvearrowright (Y,\nu)^G$ the entropy is equal to the Shannon entropy of the base (Y,μ), defined as

$$H(\nu) = \sup_{\mathcal{P}} -\sum_{A\in\mathcal{P}} \nu(A)\log\nu(A) \tag{14}$$

where $\mathcal{P}$ here ranges over all finite measurable partitions of Y. Note in particular that if Y is a finite set then we obtain a generating partition by considering the $|Y|$ cylinder sets in which membership is determined by the coordinate of a point at the identity elment of G. Extending a celebrated theorem of Ornstein in the case $G = \mathbb{Z}$, Ornstein and Weiss showed that two such Bernoulli actions with the same base entropy are conjugate, showing that entropy is a complete invariant for Bernoulli actions of a countably infinite amenable group:

Theorem 1.1. Let G be a countably infinite amenable group. Then two Bernoulli actions $G \curvearrowright (Y_1,\nu_1)^G$ and $G \curvearrowright (Y_2,\nu_2)^G$ over standard probability spaces are conjugate if and only if $H(\nu_1) = H(\nu_2)$.

In the case $G = \mathbb{Z}$ Bernoulli structure is surprisingly pervasive, occurring for instance in geodesic flows of compact surfaces with negative curvature. On the other hand, in the case of $G = \mathbb{Z}^d$ for $d \geq 2$, for example, smooth G-actions on compact manifolds always have zero entropy.

1.2. Sofic measure entropy. Consider now a p.m.p. action $G \curvearrowright (X,\mu)$ of a countable sofic group. Fix a sofic approximation sequence $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$ for G, as in Section **??**. We write e for the identity element of G.

If Ω is a collection of subsets of a given set, we write $\mathcal{A}(\Omega)$ for the algebra generated by Ω. For $d \in \mathbb{N}$ we write $\mathcal{A}(d)$ for the algebra of all subsets of $\{1,\dots,d\}$. For a measurable partition $\mathcal{P}$ and a finite set $F \subseteq G$ we denote by $\mathcal{P}_F$ the partition $\{\bigcap_{s\in F} sA_s : A \in \mathcal{P}^F\}$ where A_s is the value of A at s.

Definition 1.2. Let $\mathcal{P}$ be a finite measurable partition of X, F a finite subset of G, and $\delta > 0$. Let $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. Define $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)$ to be the set of all homomorphisms $\varphi : \mathcal{A}(\mathcal{P}_F) \to \mathcal{A}(d)$ such that

(i) $\sum_{A\in\mathcal{P}} |\sigma_s\varphi(A)\Delta\varphi(sA)|/d < \delta$ for all $s \in F$, and
(ii) $\sum_{A\in\mathcal{P}_F} \big||\varphi(A)|/d - \mu(A)\big| < \delta$.

For a partition $\mathcal{Q} \leq \mathcal{P}$ we let $|\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)|_{\mathcal{Q}}$ denote the cardinality of the set of restrictions of elements of $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)$ to $\mathcal{Q}$. Note that $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma) \supseteq \mathrm{Hom}_\mu(\mathcal{P}', F', \delta', \sigma)$ and thus $|\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)|_{\mathcal{Q}} \geq |\mathrm{Hom}_\mu(\mathcal{P}', F', \delta', \sigma)|_{\mathcal{Q}'}$ whenever $\mathcal{P} \leq \mathcal{P}'$, $F \subseteq F'$, $\delta \geq \delta'$, and $\mathcal{Q} \geq \mathcal{Q}'$.

Definition 1.3. Let $\mathcal{S}$ be a subalgebra of the Borel σ-algebra of X, and let $\mathcal{Q}$ and $\mathcal{P}$ be finite measurable partitions of X with $\mathcal{P} \geq \mathcal{Q}$. Let F be a nonempty finite subset of G and $\delta > 0$. Set

$$\begin{aligned} h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}, F, \delta) &= \limsup_{i\to\infty} \frac{1}{d_i} \log |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}}, \\ h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}) &= \inf_F \inf_{\delta>0} h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}, F, \delta), \\ h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{S}) &= \inf_{\mathcal{P}} h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}), \\ h_{\Sigma,\mu}(\mathcal{S}) &= \sup_{\mathcal{Q}} h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{S}) \end{aligned}$$

where F in the second line ranges over all nonempty finite subsets of G, $\mathcal{P}$ in the fourth line ranges over all finite partitions $\mathcal{P} \subseteq \mathcal{S}$ which refine $\mathcal{Q}$, and $\mathcal{Q}$ in the last line ranges over all finite partitions in $\mathcal{S}$. When $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)$ is empty for all sufficiently large i we declare that $h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}, F, \delta) = -\infty$.

Definition 1.4. We define the measure entropy $h_{\Sigma,\mu}(X, G)$ of the action $G \curvearrowright (X, \mu)$ with respect to Σ as $h_{\Sigma,\mu}(\mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra of X.

We now aim to establish a version of Kolmogorov-Sinai theorem, Theorem 1.7, which will enable us to determine the measure entropy for Bernoulli actions by reducing the window of computation to a generating subalgebra.

Definition 1.5. A subalgebra $\mathcal{S}$ of the Borel σ-algebra of X is said to be *generating* if every Borel set is contained, modulo a set of measure zero, in the σ-algebra generated by the translates $t\mathcal{S}$ for $t \in G$.

Lemma 1.6. Let $\mathcal{P}$ be a finite measurable partition of X and let $\varepsilon > 0$. Then there is a $\delta > 0$ such that, for every subalgebra $\mathcal{S}$ of the Borel σ-algebra of X such that $\max_{A\in\mathcal{P}} \inf_{B\in\mathcal{S}} \mu(A\Delta B) < \delta$, there is a homomorphism $\theta : \mathcal{A}(\mathcal{P}) \to \mathcal{S}$ such that $\mu(\theta(A)\Delta A) < \varepsilon$ for all $A \in \mathcal{A}(\mathcal{P})$.

Proof. Let $\delta > 0$, to be determined. Fix an enumeration $A_1, \dots, A_n$ of the members of $\mathcal{P}$. We construct a homomorphism $\theta : \mathcal{A}(\mathcal{P}) \to \mathcal{S}$ by recursively defining $\theta(A_i)$ for $i = 1, \dots, n-1$ to be an element of $\mathcal{S}$ contained in the complement of $\theta(A_1) \cup \cdots \cup \theta(A_{i-1})$ such that $\mu(\theta(A_i)\Delta A_i)$ is within δ of the infimum of its possible values, and then declaring $\theta(A_n)$ to be the complement of $\theta(A_1) \cup \cdots \cup \theta(A_{n-1})$. It is then readily seen that if δ is small enough as a function of ε and $\mathcal{P}$ then the assumption $\max_{A\in\mathcal{P}} \inf_{B\in\mathcal{S}} \mu(A\Delta B) < \delta$ will imply that the homomorphism θ has the desired property. $\square$

THEOREM 1.7. Let $\mathcal{S}$ be a generating subalgebra of the Borel σ-algebra of X. Then $h_{\Sigma,\mu}(X,G) = h_{\Sigma,\mu}(\mathcal{S})$.

PROOF. By symmetry it is enough to show that $h_{\Sigma,\mu}(\mathcal{T}) \leq h_{\Sigma,\mu}(\mathcal{S})$ for any other generating subalgebra $\mathcal{T}$ of the Borel σ-algebra of X. Given such a $\mathcal{T}$, let $\mathcal{N}$ be a finite partition in $\mathcal{T}$ and let $\kappa > 0$. As $\mathcal{S}$ is generating, we can find a finite partition $\mathcal{Q} \subseteq \mathcal{S}$ and a nonempty finite set $K \subseteq G$ such that for every $B \in \mathcal{N}$ there is a $\Upsilon_B \subseteq \mathcal{Q}^K$ such that the set $B' = \bigcup_{Y \in \Upsilon_B} \bigcap_{s \in K} sY_s$ satisfies $\mu(B \Delta B') < \varepsilon/16$.

Choose a finite partition $\mathcal{P} \subseteq \mathcal{S}$ with $\mathcal{P} \geq \mathcal{Q}$, a finite set $F \subseteq G$ containing $K \cup \{e\}$, and a $\delta > 0$ less than $\varepsilon/(8|\mathcal{Q}^K||K|)$ such that

$$\limsup_{i\to\infty} \frac{1}{d_i} \log |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}} \leq h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{S}) + \kappa.$$

Since $\mathcal{T}$ is generating we can find a finite partition $\mathcal{M} \subseteq \mathcal{T}$ refining $\mathcal{N}$ and a finite set $E \subseteq G$ containing e such that for each $A \in \mathcal{N}_F$ there exists a subset Λ_A of $\mathcal{M}^E$ for which the set $A' = \bigcup_{Y \in \Lambda_A} \bigcap_{s \in E} sY_s$ satisfies $\mu(A \Delta A') < \delta/(12|\mathcal{N}^F|)$. As $e \in F$ the partition $\mathcal{M}_{FE}$ is a refinement of $\mathcal{M}_E$, and so by Lemma 1.6 we may furthermore assume that $\mu(A\Delta A')$ is small enough for each $A \in \mathcal{P}_F$ to ensure the existence of a homomorphism $\theta : \mathcal{A}(\mathcal{P}_F) \to \mathcal{A}(\mathcal{M}_{FE})$ satisfying

$$\mu(\theta(A)\Delta A) < \min\left(\frac{\delta}{12|\mathcal{P}^F|}, \frac{\varepsilon}{16|\mathcal{Q}^K|}\right)$$

for all $A \in \mathcal{P}_F$. Now take a small enough $\delta' > 0$ less than $\delta/(9|\mathcal{P}^F||\mathcal{M}^E||E|)$ such that for every map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$ and every $\varphi \in \mathrm{Hom}_\mu(\mathcal{M}, FE, \delta'\sigma)$ we are guaranteed that $|\varphi(B)|/d \leq 2\mu(B)$ for every $B \in \mathcal{A}(\mathcal{M}_{FE})$.

Now let $\sigma : G \to \mathrm{Sym}(d)$ be a sofic approximation which is good enough for a purpose to be described shortly. We will show, given a $\varphi \in \mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma)$, that the composition $\varphi^\natural := \varphi \circ \theta$ is an element of $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)$. For every $t \in F$ and $A \in \mathcal{P}$ we have, granted that σ is a good enough sofic approximation,

$$\begin{aligned}\frac{1}{d}|\varphi(tA')\Delta\sigma_t\varphi(A')| &\leq \sum_{Y\in\Lambda_A}\sum_{s\in E}\frac{1}{d}\big(|\varphi(tsY_s)\Delta\sigma_{ts}\varphi(Y_s)| + |\sigma_{ts}\varphi(Y_s)\Delta\sigma_t\sigma_s\varphi(Y_s)| \\ &\qquad + |\sigma_t(\sigma_s\varphi(Y_s)\Delta\varphi(sY_s))|\big) \\ &< 3|\mathcal{M}^E||E|\delta' < \frac{\delta}{3|\mathcal{P}|},\end{aligned}$$

in which case

$$\sum_{A\in\mathcal{P}}\frac{1}{d}|\varphi^\natural(tA)\Delta\sigma_t\varphi^\natural(A)| \leq \sum_{A\in\mathcal{P}}\frac{1}{d}\big(|\varphi(\theta(tA)\Delta tA'))| + |\varphi(tA')\Delta\sigma_t\varphi(A')|$$

$$
\begin{aligned}
&\qquad\qquad + |\sigma_t\varphi(A'\Delta\theta(A))|\big) \\
&< 2|\mathcal{P}|\big(\mu(\theta(tA)\Delta tA) + \mu(tA\Delta tA')\big) + \frac{\delta}{3} + \\
&\qquad\qquad + 2|\mathcal{P}|\big(\mu(A'\Delta A) + \mu(A\Delta\theta(A))\big) < \delta.
\end{aligned}
$$

Moreover, for every $A \in \mathcal{P}_F$ the estimates $\big||\varphi(A')|/d - \mu(A')\big| < \delta' < \delta/(3|\mathcal{P}^F|)$ and

$$
\mu(\theta(A)\Delta A') \leq \mu(\theta(A)\Delta A) + \mu(A\Delta A') < \frac{\delta}{6|\mathcal{P}^F|}
$$

yield

$$
\begin{aligned}
\sum_{A\in\mathcal{P}_F}\left|\frac{|\varphi^\natural(A)|}{d} - \mu(A)\right| &\leq \sum_{A\in\mathcal{P}_F}\left(\frac{|\varphi(\theta(A)\Delta A')|}{d} + \left|\frac{|\varphi(A')|}{d} - \mu(A')\right| + \mu(A'\Delta A)\right) \\
&\leq \sum_{A\in\mathcal{P}_F}\left(2\mu(\theta(A)\Delta A') + 2\cdot\frac{\delta}{3|\mathcal{P}^F|}\right) < \delta.
\end{aligned}
$$

Hence $\varphi^\natural \in \mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)$, as desired.

Let $\Gamma : \mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma) \to \mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)$ be the map $\varphi \mapsto \varphi^\natural$. If we could show that maps in $\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma)$ which differ on $\mathcal{N}$ get sent under Γ to maps which differ on $\mathcal{Q}$, then we would have $|\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma)|_{\mathcal{N}} \leq |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)|_{\mathcal{Q}}$, which would effectively finish the proof. We cannot prove this injectivity exactly, but only in an approximate form which is nevertheless sufficient for our purposes. We express this approximation using the pseudometrics

$$
\rho_{\mathcal{N}}(\varphi,\psi) = \max_{A\in\mathcal{N}}\frac{1}{d}|\varphi(A)\Delta\psi(A)| \quad \text{and} \quad \rho_{\mathcal{Q}}(\varphi,\psi) = \max_{A\in\mathcal{Q}}\frac{1}{d}|\varphi(A)\Delta\psi(A)|
$$

on $\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma)$ and $\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)$, respectively. Set $\varepsilon' = \varepsilon/(8|\mathcal{Q}^K||K|)$. Suppose we are given $\varphi, \psi \in \mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma)$ such that $\rho_\xi(\varphi^\natural, \psi^\natural) < 2\varepsilon'$. Then, for every $B \in \mathcal{N}$,

$$
\begin{aligned}
\frac{1}{d}|\varphi^\natural(B')\Delta\psi^\natural(B')| &\leq \sum_{Y\in\Upsilon_B}\sum_{s\in K}\frac{1}{d}\big(|\varphi^\natural(sY_s)\Delta\sigma_s\varphi^\natural(Y_s)| + |\sigma_s(\varphi^\natural(Y_s)\Delta\psi^\natural(Y_s))| \\
&\qquad\qquad + |\sigma_s\psi^\natural(Y_s)\Delta\psi^\natural(sY_s)|\big) \\
&< |\mathcal{Q}^K||K|(\delta + 2\varepsilon' + \delta) < \frac{\varepsilon}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mu(B\Delta\theta(B')) &\leq \mu(B\Delta B') + \mu(B'\Delta\theta(B')) \\
&< \frac{\varepsilon}{16} + \sum_{Y\in\Upsilon_B}\mu\big(\big(\textstyle\bigcap_{s\in K} sY_s\big)\Delta\theta\big(\bigcap_{s\in K} sY_s\big)\big)
\end{aligned}
$$

$$< \frac{\varepsilon}{16} + |\Upsilon_B| \cdot \frac{\varepsilon}{16|\mathcal{Q}^K|} \leq \frac{\varepsilon}{8}$$

so that

$$\rho_{\mathcal{N}}(\varphi, \psi) \leq \max_{B \in \mathcal{N}} \frac{1}{d}\big(|\varphi(B \Delta \theta(B'))| + |\varphi^\natural(B')\Delta\psi^\natural(B')| + |\psi(\theta(B')\Delta B)|\big)$$
$$\leq 4 \max_{B \in \mathcal{N}} \mu(B \Delta \theta(B')) + \frac{\varepsilon}{2} < \varepsilon.$$

This shows that for every set $W \subseteq \mathrm{Hom}_\mu(\beta, FE, \delta', \sigma)$ which is ε-separated with respect to $\rho_{\mathcal{N}}$, the image $\Gamma(W)$ is ε'-separated with respect to $\rho_{\mathcal{Q}}$. Therefore, writing $N_\varepsilon(\cdot, \rho_{\mathcal{N}})$ for the maximum cardinality of an ε-separated set with respect to $\rho_{\mathcal{N}}$, and similarly for $\rho_{\mathcal{Q}}$, we have

(15)
$$N_\varepsilon(\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma), \rho_{\mathcal{N}}) \leq N_{\varepsilon'}(\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma), \rho_{\mathcal{Q}}) \leq |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma)|_{\mathcal{Q}}.$$

Observe next that, for every $\eta > 0$, $d \in \mathbb{N}$, and $A \subseteq \{1, \dots, d\}$, the set of all $B \subseteq \{1, \dots, d\}$ satisfying $|B \Delta A|/d < \eta$ has cardinality at most $\binom{d}{\lfloor \eta d \rfloor}$, which by Stirling's approximation is less than $e^{\beta d}$ for some $\beta > 0$ not depending on d with $\beta \to 0$ as $\eta \to 0$. As a consequence we see that

$$|\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma)|_{\mathcal{N}} \leq e^{\kappa d} N_\varepsilon(\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma), \rho_{\mathcal{N}})$$

assuming ε is small enough independently of d. Combining this with (15) yields

$$h^{\mathcal{N}}_{\Sigma,\mu}(\mathcal{T}) \leq h^{\mathcal{N}}_{\Sigma,\mu}(\mathcal{M}) \leq \limsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Hom}_\mu(\mathcal{M}, FE, \delta', \sigma_i), \rho_{\mathcal{N}}) + \kappa$$
$$\leq \limsup_{i\to\infty} \frac{1}{d_i} \log |\mathrm{Hom}_\mu(\mathcal{P}, F, \delta, \sigma_i)|_{\mathcal{Q}} + \kappa$$
$$\leq h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{S}) + 2\kappa \leq h_{\Sigma,\mu}(\mathcal{S}) + 2\kappa.$$

Since κ was an arbitrary number greater than zero, we conclude that $h_{\Sigma,\mu}(\mathcal{T}) \leq h_{\Sigma,\mu}(\mathcal{S})$. □

1.3. Bernoulli actions of sofic groups. To compute the entorpy of a Bernoulli action of a sofic group, we will need the following lemma to handle the case when the base is not a finite set.

LEMMA 1.8. Let $G \curvearrowright (X, \mu)$ be a p.m.p. action of a countable sofic group. Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ be finite measurable partitions of X such that $\mathcal{Q}, \mathcal{R} \leq \mathcal{P}$. Then

(i) $h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}) \leq H_\mu(\mathcal{Q})$,
(ii) $h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}) \geq h^{\mathcal{R}}_{\Sigma,\mu}(\mathcal{P}) - H_\mu(\mathcal{R}|\mathcal{Q})$.

Proof. (i). Order the elements of $\mathcal{Q}$ as $A_1, \dots, A_n$. Let $\varepsilon > 0$. Writing ζ for the uniform probability measure on $\{1, \dots, d\}$, we observe by the continuity properties of Shannon entropy that $|H_\mu(\mathcal{Q}) - H_\zeta(\mathcal{K})| < \varepsilon$ for every ordered partition $\mathcal{K} = \{B_1, \dots, B_n\}$ of a finite set $\{1, \dots, d\}$ such that $\sum_{i=1}^n |\mu(A_i) - \zeta(B_i)| < \delta$. Fix d and consider the set T of all $(c_1, \dots, c_n) \in \{1/d, 2/d, \dots, 1\}^n$ satisfying $c_1 + \cdots + c_n = 1$ and $\sum_{i=1}^n |\mu(A_i) - c_i| < \delta$, which has cardinality at most $(2\delta d)^n$. For each $c \in T$, the set of all ordered partitions $\{C_1, \dots, C_n\}$ of $\{1, \dots, d\}$ such that $|C_i|/d = c_i$ for each i is equal to $d!/((c_1 d)! \cdots (c_n d)!)$, which by Stirling'approximation is bounded above by $e^{d(1+\varepsilon)(H_\mu(\mathcal{Q})+\varepsilon)}$ if d is sufficiently large. Therefore the number of homomorphisms $\varphi : \mathcal{A}(\mathcal{Q}) \to \mathcal{A}(d)$ satisfying $\sum_{i=1}^n |\zeta(\varphi(A_i)) - \mu(A_i)| < \delta$ is at most $(2\delta d)^n e^{d(1+\varepsilon)(H_\mu(\mathcal{Q})+\varepsilon)}$. It follows that

$$h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}) \leq h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}, \{e\}, \delta) \leq h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{Q}, \{e\}, \delta) \leq (1+\varepsilon)(H_\mu(\mathcal{Q}) + \varepsilon),$$

establishing (i).

(ii). Let $\varepsilon > 0$. As before order the elements of $\mathcal{Q}$ as $A_1, \dots, A_n$. For each $i = 1, \dots, n$ let $\mathcal{R}_i = \{C_{i,1}, \dots, C_{i,n_i}\}$ be the partition of A_i consisting of the intersections of the members of $\mathcal{R}$ with B_i. With ζ continuing to denote the uniform probability measure on $\{1, \dots, d\}$, we observe that the continuity properties of Shannon entropy imply the existence of a $\delta > 0$ such that $\max_{i=1,\dots,n} |H_{\mu_i}(\mathcal{R}_i) - H_{\zeta_i}(\psi(\mathcal{R}_i))| < \varepsilon$ for every $d \in \mathbb{N}$ and homomorphism $\psi : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(d)$ satisfying $\sum_{A \in \mathcal{P}} |\zeta(\psi(A)) - \mu(A)| < \delta$, where μ_i is $\mu(A_i)^{-1}$ times the restriction of μ to A_i and $H_{\zeta_i}(\psi(\mathcal{R}_i))$ is understood to mean zero if $\zeta(\psi(B_i)) = 0$.

Take a $\delta' > 0$ such that $\delta'(\sum_{i=1}^m H_{\mu_i}(\mathcal{R}_i)) \leq \delta$. Let $\psi : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(d)$ be a homomorphism such that $\sum_{A \in \mathcal{P}} |\zeta(\psi(A)) - \mu(A)| < \delta'$, Using Stirling's approximation as above, the set W_i of all ordered partitions $\{C_1, \dots, C_{n_i}\}$ of $\psi(B_i)$ satisfying $\sum_{j=1}^{n_i} |\mu(B_{i,j}) - \zeta(C_j)| < \delta'$ has cardinality at most $(2\delta' d)^{n_i} e^{d\zeta(\psi(A_i))(1+\varepsilon)(H_{\mu_i}(\mathcal{R}_i)+\varepsilon)}$. Since

$$\begin{aligned}\sum_{i=1}^m \zeta(\psi(A_i)) H_{\mu_i}(\mathcal{R}_i) &< \sum_{i=1}^m (\mu(A_i) + \delta') H_{\mu_i}(\mathcal{R}_i) \\ &\leq \sum_{i=1}^m \mu(A_i) H_{\mu_i}(\mathcal{R}_i) + \delta = H_\mu(\mathcal{R}|\mathcal{Q}) + \delta\end{aligned}$$

and the set of all restrictions to $\mathcal{R}$ of homomorphisms $\varphi : \mathcal{A}(\mathcal{P}) \to \mathcal{A}(d)$ satisfying $\sum_{A \in \mathcal{P}} |\zeta(\varphi(A)) - \mu(A)| < \delta'$ and restrict on $\mathcal{R}$ to ψ has cardinality at most $\prod_{i=1}^m |W_i|$, it follows that

$$\prod_{i=1}^m |W_i| \leq (2\delta' d)^{|\mathcal{Q}| \cdot |\mathcal{R}|} e^{d(1+\varepsilon)(H_\mu(\mathcal{R}|\mathcal{Q})+\delta+\varepsilon)}.$$

Thus for every nonempty finite set $F \subseteq G$ we consequently get

$$h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}, F, \delta') \geq h^{\mathcal{R}}_{\Sigma,\mu}(\mathcal{P}, F, \delta') - (1+\varepsilon)(H_\mu(\mathcal{R}|\mathcal{Q}) + \delta + \varepsilon)$$

and thus, since we can choose δ' to be less than δ,

$$h^{\mathcal{Q}}_{\Sigma,\mu}(\mathcal{P}) \geq h^{\mathcal{R}}_{\Sigma,\mu}(\mathcal{P}) - (1+\varepsilon)(H_\mu(\mathcal{R}|\mathcal{Q}) + \varepsilon).$$

We thus obtain (ii). □

THEOREM 1.9. Let $G \curvearrowright (Y,\nu)^G$ be a Bernoulli action of a countable sofic group. Then $H_{\Sigma,\nu^G}(Y^G, G) = H(\nu)$.

PROOF. Let $\mathcal{S}$ be the algebra of measurable cylinder sets over e, i.e., sets $A \subseteq Y^G$ for which there is a measurable $B \subseteq Y$ such that membership in A depends on whether the coordinate of the given element Y^G at e lies in B. This algebra is generating and so we need only show that $h_{\Sigma,\nu^G}(\mathcal{S}) = H(\nu)$ by Theorem 1.7. By Lemma 1.8(i) it is furthermore enough to prove that $h_{\Sigma,\nu^G}(\mathcal{S}) \geq H(\nu)$. By Lemma 1.8(ii) this will follow once we show that $H^{\mathcal{P}}_{\Sigma,\nu^G}(\mathcal{P}) \geq H_{\nu^G}(\mathcal{P})$ for a given finite partition $\mathcal{P} \subseteq \mathcal{S}$.

Let $\delta > 0$, and let $\eta > 0$ be such that $2\eta < |\mathcal{P}|^{-|F|}\delta$. Let $F \subseteq G$ be a finite set containing e. Let $\sigma : G \to \mathrm{Sym}(d)$ be a sofic approximation which is sufficiently good for purposes to be described. Denote by V the set of all $v \in \{1,\dots,d\}$ such that $\sigma_s^{-1}(v) \neq \sigma_t^{-1}(v)$ for all distinct $s,t \in F$. Write $\mathcal{P} = \{A_1,\dots,A_n\}$ and denote by κ the probability measure on $\{1,\dots,n\}$ such that $\kappa(\{i\}) = \nu^G(A_i)$ for all $i = 1,\dots,n$. Equip $\{1,\dots,n\}^d$ with the product measure κ^d. We view elements of this product as partitions of $\{1,\dots,d\}$, and we aim to show that, outside a set of small measure, every such partition (i) is a good model for $\mathcal{P}$ via the homomorphism $\mathcal{A}(\mathcal{P}_F) \to \mathcal{A}(d)$ it naturally defines, and (ii) occurs with probability at least $e^{-d(H(\kappa)-\delta}$ when d is large enough.

Let $f \in \{1,\dots,\}^F$ and set $C_f = \bigcap_{s\in F} sA_{f(s)}$. For each $\gamma \in \{1,\dots,n\}^d$ we set we set $D_{\gamma,f} = \bigcap_{s\in F} \sigma_s\gamma^{-1}(f(s))$ and write φ_γ for the homomorphism $\mathcal{A}(\mathcal{P}_F) \to \mathcal{A}(d)$ determined by $\varphi_\gamma(C_f) = D_{\gamma,f}$. Observe that, for $s \in F$ and $i = 1,\dots,n$, if we denote by $\Lambda_{s,i}$ for the set of all $g \in \{1,\dots,n\}^F$ such that $g(s) = i$ then we can write $sA_i = \bigsqcup_{g\in\Lambda_{s,i}} C_f$ and $\sigma_s\gamma^{-1}(i) = \bigsqcup_{g\in\Lambda_{s,i}} D_{\gamma,f}$, from which we see that

$$\begin{aligned}|\varphi_\gamma(sA_i)\Delta\sigma_s\varphi_\gamma(A_i)| &\leq |\sigma_s(\gamma^{-1}(i)\Delta\sigma_e\gamma^{-1}(i))| \\ &\leq |\{v \in \{1,\dots,d\} : \sigma_e(v) \neq v\}| < \delta d\end{aligned}$$

assuming σ is a good enough sofic appproximation. This shows that for every γ the homomorphism φ_γ satisfies condition (i) in the definition of $\mathrm{Hom}_{\nu^G}(\mathcal{P}, F, \delta, \sigma)$. Our next goal to show that, with high probability, φ_γ satisfies condition (ii) in the definition and therefore lies in $\mathrm{Hom}_{\nu^G}(\mathcal{P}, F, \delta, \sigma)$.

With f continuing to be fixed, we next derive a bound on the variance $\mathrm{Var}(Z)$ of the random variable on $\{1, \dots, n\}^d$ defined by $Z = \sum_{v=1}^d Z_v$ where Z_v at a point γ takes the value 1 if $v \in V \cap D_{\gamma,f}$ and 0 otherwise. Writing $\mathbb{E}(\cdot)$ for expected value, we note that if $v \notin V$ then $\mathbb{E}(Z_v) = 0$, while if $v \in V$ then

$$\begin{aligned}\mathbb{E}(Z_v) &= \kappa^d\big(\{\gamma \in \{1, \dots, n\}^d : \sigma_s^{-1}(v) \in \gamma^{-1}(f(s)) \text{ for every } s \in F\}\big) \\ &= \prod_{s\in F} \kappa(\{f(s)\}) = \prod_{s\in F} \nu(A_{f(s)}) = \nu^G(C_f).\end{aligned}$$

If $v, w \in \{1, \dots, d\}$ satisfy $\sigma_s^{-1}(v) \neq \sigma_t^{-1}(w)$ for all $s, t \in F$ then Z_v and Z_w are independent, and so the number of such pairs $(v, w) \in V \times V$ such that Z_v and Z_w are not independent is at most $d|F|^2$. Therefore

$$\mathbb{E}(Z^2) = \sum_{v,w=1}^{d} \mathbb{E}(Z_v Z_w) \leq \sum_{v,w=1}^{d} \mathbb{E}(Z_v)\mathbb{E}(Z_w) + d|F|^2 = \mathbb{E}(Z)^2 + d|F|^2,$$

yielding the bound $\mathrm{Var}(Z) = \mathbb{E}(Z^2) - \mathbb{E}(Z)^2 \leq d|F|^2$. Now we apply Chebyshev's inequality to obtain, for all $t > 0$,

$$\mathbb{P}\big(|Z/d - \mathbb{E}(Z)/d| > t\big) \leq \frac{\mathrm{Var}(Z)}{d^2t^2} \leq \frac{|F|^2}{dt^2}. \tag{16}$$

Now if σ is a good enough sofic approximation so that $\zeta(V) \geq 1 - \eta$ we will have, for every γ,

$$\begin{aligned}\big|\zeta(D_{\gamma,f}) - \nu^G(C_f)\big| &\leq \big|\zeta(D_{\gamma,f}) - \zeta(V \cap D_{\gamma,f})\big| + |(Z/d)(\gamma) - \mathbb{E}(Z)/d| \\ &\quad + \big|\zeta(V)\nu^G(C_f) - \nu^G(C_f)\big| \\ &\leq |(Z/d)(\gamma) - \mathbb{E}(Z)/d| + 2\eta\end{aligned}$$

and so when $t > 2\eta$ the estimate (16) yields

$$\mathbb{P}\big(|\zeta(D_{\gamma,f}) - \nu^G(C_f)| > t\big) \leq \frac{|F|^2}{d(t - 2\eta)^2}.$$

In particular, setting $t = n^{-|F|}\delta$ we get, assuming d is large enough,

$$\mathbb{P}\left(\big|\zeta(D_{\gamma,f}) - \nu^G(C_f)\big| > \frac{\delta}{n^{|F|}}\right) \leq \frac{\delta}{n^{|F|}}.$$

Having shown this for an arbitrary $f \in \{1, \dots, n\}^F$ we now obtain

$$\mathbb{P}\left(\big|\zeta(D_{\gamma,f}) - \nu^G(C_f)\big| > \frac{\delta}{n^{|F|}} \text{ for some } f \in \{1, \dots, n\}^F\right) \leq \delta.$$

If the above event does not occur for a given γ then

$$\sum_{f\in\{1,\dots,n\}^F} \bigl|\zeta(\varphi_\gamma(C_f)) - \nu^G(C_f)\bigr| = \sum_{f\in\{1,\dots,n\}^F} \bigl|\zeta(D_{\gamma,f}) - \nu^G(C_f)\bigr| < \delta,$$

so that

$$\mathbb{P}\bigl(\varphi_\gamma \in \mathrm{Hom}_{\nu^G}(\mathcal{P}, F, \delta, \sigma)\bigr) \geq 1 - \delta \tag{17}$$

Finally, we use this to estimate the actual number of γ satisfying $\varphi_\gamma \in \mathrm{Hom}_{\nu^G}(\mathcal{P}, F, \delta, \sigma)$ by observing that the law of large numbers and the independence of the coordinates yield

$$\lim_{d\to\infty} \mathbb{P}\Bigl(\Bigl| -\frac{1}{d}\log \kappa^d(\{\gamma\}) - H(\kappa)\Bigr| > \delta\Bigr) = 0,$$

so that for d large enough we will have $\kappa^d(\{\gamma\}) \leq e^{-d(H(\kappa)-\delta)}$ for all γ in a subset of measure at least $1-\delta$. It follows by (17) that

$$|\mathrm{Hom}_{\nu^G}(\mathcal{P}, F, \delta, \sigma)|_{\mathcal{R}} \geq (1-2\delta)e^{d(H(\kappa)-\delta)},$$

from which we conclude that $h^{\mathcal{P}}_{\Sigma,\nu^G}(\mathcal{P}) \geq H_{\nu^G}(\mathcal{P})$, as desired. $\square$

As in the amenable case, once one has entropy as an invariant for p.m.p. actions of countable sofic groups and computes its value for a Bernoulli action to be the Shannon entropy of the base as we have done above, the classification of Bernoulli actions reduces to the problem of whether a given countable sofic group is Ornstein in the following sense.

Definition 1.10. A group G is *Ornstein* if, for all standard probability spaces (Y_1, ν_1) and (Y_2, ν_2), the equality $H(\nu_1) = H(\nu_2)$ of the Shannon entropies (as defined by (14)) implies that the Bernoulli actions $G \curvearrowright (Y_1, \nu_1)^G$ and $G \curvearrowright (Y_2, \nu_2)^G$ are conjugate.

Whether all countable sofic groups are Ornstein is still an open problem in general but is known in many cases, most notably when the group contains an element of infinite order, as we will record in Theorem 1.13. The key point is Lemma 1.12, which we now aim to prove. For this we require the following notion of coinduced action.

Let G be a countable group and H a subgroup of G. Let $\gamma : G/H \to G$ be a section, meaning that $\gamma(sH) \in sH$ for all $s \in G$. We assume that $\gamma(H) = e$ for convenience. The map $\alpha : G \times G/H \to H$ defined by

$$\alpha(s, a) = \gamma(a)^{-1} s\gamma(s^{-1}a)$$

for all $s \in G$ and $a \in G/H$ is a cocycle, as it satisfies the identity

$$\alpha(st, a) = \alpha(s, a)\alpha(t, s^{-1}a)$$

for all $s, t \in G$ and $a \in G/H$. Given a p.m.p. action $H \curvearrowright (Y, \nu)$, the prescription

$$(sx)(a) = \alpha(s, a)x(s^{-1}a)$$

for $s \in G$, $x \in Y^{G/H}$, and $a \in G/H$ is readily checked to define a p.m.p. action $G \curvearrowright (Y, \nu)^{G/H}$, which we call the action *coinduced* from $H \curvearrowright (Y, \nu)$.

LEMMA 1.11. Let (Y, ν) be a standard probability space, G a countable group, and H a subgroup of G. Then the action $G \curvearrowright ((Y^H)^{G/H}, (\nu^H)^{G/H})$ coinduced from the Bernoulli action $H \curvearrowright (Y^H, \nu^H)$ via a cocycle α as above is conjugate to the Bernoulli action $G \curvearrowright (Y^G, \nu^G)$.

PROOF. We will show that the map $\psi : Y^G \to (Y^H)^{G/H}$ given by

$$\psi(x)(a)(t) = x(\gamma(a)t)$$

for all $x \in Y^G$, $a \in G/H$, and $t \in H$ is a conjugacy for the actions $G \curvearrowright (Y^G, \nu^G)$ and $G \curvearrowright ((Y^H)^{G/H}, (\nu^H)^{G/H})$. One can check that ψ is invertible with inverse given by

$$\psi(y)(s) = y(sH)(\alpha(s, sH))$$

for all $y \in (Y^H)^{G/H}$ and $s \in G$. Moreover, since the map $(a, t) \to \gamma(a)t$ from $G/H \times H$ to G is bijective, the map ψ is effectively a recoordinatization over the base Y and hence pushes ν^G forward to $(\nu^H)^{G/H}$ Finally, we observe that, for all $s \in G$, a.e. $x \in Y^G$, $a \in G/H$, and $t \in H$,

$$(s\psi(x))(a)(t) = x(\gamma(s^{-1}a)\alpha(s, a)^{-1}t) = x(s^{-1}\gamma(a)t) = (sx)(\gamma(a)t) = \psi(sx)(a)(t),$$

so that ψ is a.e. G-equivariant. □

Let $G \curvearrowright (Y_1, \nu_1)^{G/H}$ and $G \curvearrowright (Y_2, \nu_2)^{G/H}$ be actions coinduced from p.m.p. actions $H \curvearrowright (Y_1, \nu_1)$ and $H \curvearrowright (Y_2, \nu_2)$. Let $\varphi : Y_1 \to Y_2$ be a factor map for the H-actions, i.e., φ is measurable, a.e. H-equivariant, and pushes ν_1 forward to ν_2, and its image has full measure. Then the product map $\varphi^{\times G/H} : Y_1^{G/H} \to Y_2^{G/H}$ given by

$$\Phi(x)(a) = \varphi(x(a))$$

for all $x \in Y_1^{G/H}$ and $a \in G/H$ is a factor map for the coinduced actions, as it clearly pushes $\nu_1^{G/H}$ forward to $\nu_2^{G/H}$ and, for $s \in G$, $a \in G/H$, and a.e. $x \in Y_1^{G/H}$,

$$\begin{aligned}\varphi^{\times G/H}(sx)(a) = \varphi(sx(a)) &= \varphi(\alpha(s, a)x(s^{-1}a)) = \alpha(s, a)\varphi(x(s^{-1}a)) \\ &= \alpha(s, a)\varphi^{\times G/H}(x)(s^{-1}a) = (s\varphi^{\times G/H}(x))(a).\end{aligned}$$

In the proof of the following lemma we will use this map in conjunction with Lemma 1.11 to produce a conjugacy between Bernoulli G-actions given a conjugacy between Bernoulli H-actions.

LEMMA 1.12. Let G be a countable group that contains an Ornstein subgroup. Then G itself is Ornstein.

PROOF. Let $G \curvearrowright (Y_1, \nu_1)^G$ and $G \curvearrowright (Y_2, \nu_2)^G$ be Bernoulli actions over standard probability spaces such that $H(\nu_1) = H(\nu_2)$. Take a subgroup H of G which is Ornstein. Then there exists a conjugacy $\varphi : Y_1^H \to Y_2^H$ for the Bernoulli actions $H \curvearrowright (Y_1^H, \nu_1^H)$ and $H \curvearrowright (Y_2^H, \nu_2^H)$. The product map $\varphi^{G/H} : (Y_1^H)^{G/H} \to (Y_2^H)^{G/H}$ as defined above is invertible since φ is, and thus defines a conjugacy for the coinduced actions associated to any given section $G/H \to G$. By Lemma 1.11 this implies that the actions $G \curvearrowright (Y_1, \nu_1)^G$ and $G \curvearrowright (Y_2, \nu_2)^G$ are conjugate. We conclude that G is Ornstein. □

By a *nontorsion* group we mean one which contains an element of infinite order.

THEOREM 1.13. Let G be a nontorsion countable group. Then two Bernoulli actions $G \curvearrowright (Y_1, \nu_1)^G$ and $G \curvearrowright (Y_2, \nu_2)^G$ over standard probability spaces are conjugate if and only if $H(\nu_1) = H(\nu_2)$.

PROOF. By assumption G contains a copy of $\mathbb{Z}$, which is Ornstein by the work of Ornstein [ref???]. Lemma 1.12 then yields the result. □

In [**?**] Bowen showed that every countable group is *almost Ornstein* in the sense that for all standard probability spaces (Y_1, ν_1) and (Y_2, ν_2) which do not consist of precisely two atoms, the equality of $H(\nu_1)$ and $H(\nu_2)$ implies that the Bernoulli actions $G \curvearrowright (Y_1, \nu_1)^G$ and $G \curvearrowright (Y_2, \nu_2)^G$ are conjugate.

2. Topological entropy and Gottschalk's surjunctivity problem

2.1. Gottschalk's surjunctivity problem. A set X is said to be *Dedekind finite* if every injective map from X to is itself is surjective, and *Dedekind infinite* otherwise. Under the axiom of (countable) choice, Dedekind finiteness is equivalent to the usual definition of finiteness, which asks the existence of a bijection from X to $\{1, \dots, n\}$ for some $n \in \mathbb{N}$. Dedekind infiniteness expresses the idea of compressibility in its most basic form, and it has fruitful analogues in various other settings where X has extra structure and the map or operation that compresses X into itself is compatible with this structure. An example in operator algebras is the notion of an infinite projection, which by definition is Murray-von Neumann equivalent to a proper subprojection.

One may go further and strengthen Dedekind compressibility to *paradoxicality*, which means that X can into itself in two disjoint ways. For example, one says that a projection is properly infinite if it is Murray-von Neumann equivalent to each of two orthogonal subprojections. For a discrete group G we can naturally interpret paradoxicality via the relation of equidecomposability, so that it becomes the existence of pairwise disjoint sets $A_1, \dots, A_n, B_1, \dots, B_m \subseteq G$ and $s_1, \dots, s_n, t_1, \dots, t_m \in G$ such that $G = A_1 \sqcup \dots \sqcup A_n = B_1 \sqcup \dots \sqcup B_m$. A theorem of Tarski then asserts that G is paradoxical if and only if it is nonamenable.

While there is a wealth of nonamenable groups, if we take dual perspective then it turns out that discrete groups in general tend to behave like Dedekind-finite objects. For instance, the group von Neumann algebra $L(G)$, which along with the reduced group C*-algebra may be viewed as a generalized Pontryagin dual, admits a faithful normal tracial state and hence is always finite in the sense of not admitting infinite projections. Connes's embedding problem asks whether such tracial von Neumann algebras always admit matrix models of a certain type ("microstates"), and is a continuous analogue of the question of whether every discrete group is sofic. Both may be seen as translations of the problem of whether Dedekind finiteness for a set implies ordinary finiteness, as they ask for some kind of structural description in terms of finite or finite-dimensional approximation.

Another version of Dedekind finiteness for a group is the following notion of surjunctivity.

Definition 2.1. A discrete group G is *surjunctive* if, for every $k \in \mathbb{N}$, if one considers the left shift action $G \curvearrowright \{1, \dots, k\}^G$ then every continuous G-equivariant injective map from $\{1, \dots, k\}^G$ to itself is surjective.

Gottschalk's surjunctivity problem asks the following.

Problem 2.2. Is every countable discrete group surjunctive?

Gromov gave an affirmative answer for sofic groups, and in fact this was the motivation for his introduction of the concept of soficity (the terminology being later coined by Weiss). As pointed out by Gromov, surjunctivity can be demonstrated in the amenable case in a natural ways using topological entropy, which is a dynamical invariant originally introduced for single transformations by Adler, Konheim, McAndrew in the early 1960s. For an action $G \curvearrowright X$ of an amenable group on a compact space, entropy gives expression to the idea of the (logarithmic) average cardinality of X under the action to within finer and finer degrees of resolution. This "average cardinality" strictly decreases when passing from $G \curvearrowright \{1,\dots,k\}^G$ to a proper subshift. We thereby conclude that G is surjunctive, for the restriction of the shift action to the image X of any continuous G-equivariant injective map from $\{1,\dots,k\}^G$ to itself has the same entropy as the shift itself and hence $X = \{1,\dots,k\}^G$. We now make this notion of entropy more precise.

2.2. Topological entropy for actions of amenable groups. Given a countable amenable group G and a continuous action $G \curvearrowright X$ on a compact metrizable space with compatible metric ρ, we take a Følner sequence $\{F_n\}$ (i.e., each F_n is a nonempty finite subset of G and $|sF_n \Delta F_n|/|F_n| \to 0$ as $n \to \infty$) and define the *topological entropy* of the action as

$$h_{\mathrm{top}}(X,G) = \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{1}{|F_n|} \log \mathrm{sep}(F_n,\varepsilon) \tag{18}$$

where $\mathrm{sep}(n,\varepsilon)$ denotes the maximum cardinality of a set $E \subseteq X$ with the property that $\max_{s\in F_n} \rho(sx,sy) \geq \varepsilon$ for all distinct $x,y \in E$ (in which case we say that E is *(F_n,ε)-separated*). Thus for every $\varepsilon > 0$ we are measuring the asymptotic exponential growth as $n \to \infty$ of the number of partial orbits over F_n that one can distinguish up to within ε. One can show that the above does not depend on the choice of Følner sequence and metric ρ. This partial orbit approach to entropy is due to Bowen and Dinaburg independently and is equivalent to the orginal open cover definition of Adler, Konheim, McAndrew [ref???].

In fact, using the Følner property it is straightforward to check that one still gets the same value in (2.8) if ρ is merely assumed to be a continuous pseudometric which is dynamically generating in the sense that for all distinct $x,y \in X$ there is an $s \in G$ such that $\rho(sx,sy) > 0$. This is useful as it allows us to immediately compute the entropy of the shift action $G \curvearrowright \{1,\dots,k\}^G$ to be $\log k$, as we can use the dynamically generating pseudometric

$$\rho((x_s),(y_s)) = \begin{cases} 0 & \text{if } x_e = y_e \\ 1 & \text{if } x_e \neq y_e \end{cases} \tag{19}$$

and observe that the maximum cardinality of an (F_n, ε)-separated is precisely $k^{|F_n|}$. The following result then shows that G is surjunctive, for if $\varphi : \{1, \dots, k\}^G \to \{1, \dots, k\}^G$ is a continuous injective G-equivariant map then G acts on the image of φ with the same entropy as the shift itself, so that φ is surjective.

PROPOSITION 2.3. Let $G \curvearrowright X$ be the restriction of the shift action on $\{1, \dots, k\}^G$ to a proper closed G-invariant subset. Then $h_{\mathrm{top}}(X, G) < \log k$.

PROOF. Since X is a proper, closed, and G-invariant, we can find a finite set $E \subseteq G$ and an $f \in \{1, \dots, k\}^E$ such that for each $t \in G$ the function $s \mapsto f(t^{-1}s)$ on tE is not the restriction of an element in X. Now let F be any finite subset of G. Observe that for any $s \in G$ the number of $t \in G$ such that $sE \cap tE \neq \emptyset$ is at most $|EE^{-1}|$, and so there is a set $F' \subseteq F$ such that $sE \cap tE \neq \emptyset$ for all distinct $s, t \in F'$ and $|F'|$ is the greatest integer less than $|F|/|E|^2$. Working with the restriction of the pseudometric defined by (19), which is dynamically generating on X, we thus have, given $0 < \varepsilon < 1$,

$$\mathrm{sep}(F, \varepsilon) \leq (k^{|E|} - 1)^{|F'|} k^{|F \setminus F'E|}.$$

Letting F range across a Følner sequence $\{F_n\}$, we then get

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \log \mathrm{sep}(F_n, \varepsilon) \leq \frac{1}{|E|^2} \log(k^{|E|}-1) + \left(1 - \frac{1}{|E|}\right) \log k < \log k,$$

establishing the result. □

2.3. Sofic topological entropy. If we now have an action of a sofic group G, then instead of considering partial orbits over a Følner set as the local models for the dynamics we externalize the set-up and model the dynamics with respect to a sofic approximation. Fixing a sofic approximation sequence for G, we then measure the exponential growth of the observable number of models relative to the size of the finite sets on which the sofic approximations live. Because of the nature of a sofic approximation sequence, this averaging is asymptotically G-invariant, which enables us to compare the values for any two dynamically generating pseudometrics as in the amenable case and hence provides us with a computable invariant. Now however the asymptotic growth might depend on the choice of sofic approximation sequence, and so in general we may obtain a collection of numerical invariants.

To make the definition of sofic topological entropy precise we proceed as follows. Let $G \curvearrowright X$ be a continuous action of a countable sofic group on a compact metrizable space. Let $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$ be a sofic approximation sequence for G, so that

(i) $\lim_{i\to\infty} \frac{1}{d_i} \big|\{k \in \{1, \ldots, d_i\} : \sigma_{i,st}(k) = \sigma_{i,s}\sigma_{i,t}(k)\}\big| = 1$ for all $s, t \in G$,

(ii) $\lim_{i\to\infty} \frac{1}{d_i} \big|\{k \in \{1, \ldots, d_i\} : \sigma_{i,s}(k) \neq \sigma_{i,t}(k)\}\big| = 1$ for all distinct $s, t \in G$.

To avoid pathologies we also assume that $d_i \to \infty$ as $i \to \infty$, which is automatic if G is infinite. Let ρ be a continuous pseudometric on X. For a $d \in \mathbb{N}$, we define on the set of all maps $\{1, \ldots, d\} \to X$ the pseudometrics

$$\rho_2(\varphi, \psi) = \left(\frac{1}{d} \sum_{v=1}^{d} (\rho(\varphi(v), \psi(v)))^2 \right)^{1/2},$$
$$\rho_\infty(\varphi, \psi) = \max_{v=1,\ldots,d} \rho(\varphi(v), \psi(v)).$$

Definition 2.4. Let F be a nonempty finite subset of G and $\delta > 0$, and let $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$. We define $\mathrm{Map}(\rho, F, \delta, \sigma)$ to be the set of all maps $\varphi : \{1, \ldots, d\} \to X$ such that $\rho_2(\varphi\sigma_s, \alpha_s\varphi) < \delta$ for all $s \in F$, where α_s denotes the transformation $x \mapsto sx$ of X.

Definition 2.5. Let F be a nonempty finite subset of G and $\delta > 0$. For $\varepsilon > 0$ set

$$h_\Sigma^\varepsilon(\rho, F, \delta) = \limsup_{i\to\infty} \frac{1}{d_i} \log N_\varepsilon(\mathrm{Map}(\rho, F, \delta, \sigma_i), \rho_\infty),$$
$$h_\Sigma^\varepsilon(\rho) = \inf_F \inf_{\delta>0} h_\Sigma^\varepsilon(\rho, F, \delta),$$
$$h_\Sigma(\rho) = \sup_{\varepsilon>0} h_\Sigma^\varepsilon(\rho),$$

where F in the second line ranges over the nonempty finite subsets of G. If $\mathrm{Map}(\rho, F, \delta, \sigma_i)$ is empty for all sufficiently large i, we set $h_\Sigma^\varepsilon(\rho, F, \delta) = -\infty$.

Using Stirling's approximation one can show that if we substitute ρ_2 is for ρ_∞ in the first line of the above definition then after taking the infimum over F and δ and the supremum over ε we end up with the same quantity. In fact in the proof Proposition 2.7 we will use the ρ_2 pseudometric to measure separation in the space $\mathrm{Map}(\rho, F, \delta, \sigma)$, as we need to combine it with approximate equivariance, which must be expressed in the ρ_2 pseudometric due to the nature of a sofic approximation. The reason for using ρ_∞ above is that it facilitates the computation of the entropy for examples like the shift action. It is also consistent with the use of (F_n, ε)-separated sets in the amenable case.

LEMMA 2.6. Let ρ and ρ' be continuous pseudometrics on X and suppose that ρ' is dynamically generating. Let F be a nonempty finite subset of G and $\delta > 0$. Then there is a nonempty finite subset F' of G and $\delta' > 0$ such that $\mathrm{Map}(\rho', F', \delta', \sigma) \subseteq \mathrm{Map}(\rho, F, \delta, \sigma)$ for any sufficiently good sofic approximation $\sigma : G \to \mathrm{Sym}(d)$.

PROOF. As ρ is dynamically generating, a simple compactness argument shows that there exist a nonempty finite set $F'' \subseteq G$ and a $\delta'' > 0$ such that if $\rho'(sx, sy) < \delta''$ for all $s \in F''$ then $\rho(x, y) < \delta/2$. Put $F' = F'' \cup (F''F)$. Given a $\delta' > 0$, a map $\sigma : G \to \mathrm{Sym}(d)$ for some $d \in \mathbb{N}$, and a $\varphi \in \mathrm{Map}(\rho', F', \delta', \sigma)$, we observe that the set of all $v \in \{1, \dots, d\}$ such that both

$$\rho'(s_1 s_2 \varphi(a), \varphi((s_1 s_2)a)) < \sqrt{\delta'} \quad \text{and} \quad \rho'(s_1 \varphi(s_2 a), \varphi(s_1(s_2 a))) < \sqrt{\delta'}$$

for all $s_1 \in F''$ and $s_2 \in F$ has cardinality at least $(1 - 2|F''||F|\delta')d$, and so if $2\sqrt{\delta'} < \delta''$ and σ is a good enough sofic approximation so that

$$|\{a \in \{1, \dots, d\} : (s_1 s_2)a = s_1(s_2 a) \text{ for all } s_1 \in F'', s_2 \in F\}| \geq (1-\delta')d.$$

we will have

$$\begin{aligned} &|\{a \in \{1, \dots, d\} : \rho(s\varphi(a), \varphi(sa)) < \delta/2 \text{ for all } s \in F\}| \\ &\qquad \geq |\{a \in \{1, \dots, d\} : \rho'(s_1 s_2 \varphi(a), s_1 \varphi(s_2 a)) < 2\sqrt{\delta'} \text{ for all } s_1 \in F'', s_2 \in F\}| \\ &\qquad \geq (1 - (1 + 2|F''||F|)\delta')d. \end{aligned}$$

This shows that $\varphi \in \mathrm{Map}(\rho, F, \delta, \sigma)$ whenever δ' is small enough independently of d and σ, establishing the lemma. □

PROPOSITION 2.7. Let ρ and ρ' be dynamically generating continuous pseudometrics on X. Then $h_\Sigma(\rho) = h_\Sigma(\rho')$.

PROOF. By symmetry it suffices to show that $h_\Sigma(\rho) \leq h_\Sigma(\rho')$. Let $0 < \varepsilon < 1$, and write κ for the minimum of ε^2 and $1/m$ where m is the minimum cardinality of a $(\rho, \varepsilon/2)$-spanning subset of X. Using the fact that ρ is dynamically generating we can find a finite set $K \subseteq G$ and a $\kappa' > 0$ such that, for all $x, y \in X$, if $\rho(sx, sy) < \sqrt{3\kappa'}$ for all $s \in K$ then $\rho'(x, y) < \kappa/\sqrt{2}$.

Take a finite set $F \subseteq G$ containing K and a $\delta > 0$ with $\delta \leq \kappa'$ such that $h^\kappa_\Sigma(\rho, F, \delta) \leq h^\kappa_\Sigma(\rho) + \varepsilon$. Since ρ' is dynamically generating, by Lemma 2.6 there are a nonempty finite set $F' \subseteq G$ and a $\delta' > 0$ such that $\mathrm{Map}(\rho', F', \delta', \sigma) \subseteq \mathrm{Map}(\rho, F, \delta, \sigma)$ for any good enough sofic approximation $\sigma : G \to \mathrm{Sym}(d)$. Given such a σ, let $\varphi, \psi \in \mathrm{Map}(\rho', F', \delta', \sigma)$ be such that $\rho_\infty(\varphi, \psi) < \kappa'$ and let us show that $\rho'_2(\varphi, \psi) < \kappa$. For each $s \in K$ we have, writing α_s for the transformation $x \mapsto sx$ of X and noting that the ρ_∞ distance dominates the

ρ_2 distance,

$$\rho_2(\alpha_s\varphi, \alpha_s\psi) \le \rho_2(\alpha_s\varphi, \varphi\sigma_s) + \rho_\infty(\varphi\sigma_s, \psi\sigma_s) + \rho_2(\psi\sigma_s, \alpha_s\psi) < \delta + \kappa' + \delta \le 3\kappa'.$$

Consequently there is a set $W \subseteq \{1, \ldots, d\}$ of cardinality at least $(1-3\kappa'|K|)d$ such that for every $v \in W$ we have $\rho(s\varphi(v), s\psi(v)) < \sqrt{3\kappa'}$ for every $s \in K$, which implies by our choice of κ' that $\rho'(\varphi(v), \psi(v)) < \kappa/\sqrt{2}$. Since we may assume that X has ρ'-diameter at most one and that κ' was chosen small enough to ensure that $3\kappa'|K| < (\kappa/\sqrt{2})^2$, we deduce that

$$\rho_2'(\varphi, \psi) \le \sqrt{(\kappa/\sqrt{2})^2 + 3\kappa'|K|} < \kappa,$$

as desired. It follows that the maximum cardinality of a (ρ_∞, κ')-separated subset of $\mathrm{Map}(\rho, F, \delta, \sigma)$ is at least as large as the maximum cardinality of a (ρ_2', κ)-separated subset of $\mathrm{Map}(\rho', F', \delta', \sigma)$.

Next we estimate the maximum number of $(\rho_\infty', \varepsilon)$-separated elements in the open (ρ_2', κ)-ball of a given $\varphi \in \mathrm{Map}(\rho', F', \delta', \sigma)$. Every element in this ball agrees with φ to within $\sqrt{\kappa}$ on a subset of $\{1, \ldots, d\}$ of cardinality at least $(1-\kappa)d$. As $\kappa \le \varepsilon^2$, it follows that the maximum cardinality of a $(\rho_\infty', \varepsilon)$-separated subset of the open (ρ_2', κ)-ball around φ is at most $\sum_{j=0}^{\lfloor \varepsilon d \rfloor} \binom{d}{j} \varepsilon^{-j}$, which by Stirling's approximation is bounded above, for all d sufficiently large, by $e^{\beta d}\varepsilon^{-\varepsilon d}$ for some $\beta > 0$ not depending on d with $\beta \to 0$ as $\varepsilon \to 0$.

Combining the observations in the above two paragraphs, we get

$$N_\varepsilon(\mathrm{Map}(\rho', F', \delta', \sigma), \rho_\infty') \le e^{\beta d}\varepsilon^{-\varepsilon d} N_\kappa(\mathrm{Map}(\rho, F, \delta, \sigma), \rho_\infty).$$

Since $(\beta - \varepsilon \log \varepsilon) \to 0$ as $\varepsilon \to 0$, we conclude that $h_\Sigma(\rho) \le h_\Sigma(\rho')$. □

In view of the above proposition we can now define sofic topological entropy as follows.

Definition 2.8. We define the topological entropy $h_\Sigma(X, G)$ with respect to Σ as the common value of $h_\Sigma(\rho)$ over all dynamically generating continuous pseudometrics ρ on X.

2.4. Surjunctivity for sofic groups via entropy. We now apply our notion of sofic topological entropy (Definition 2.8) to give a proof of surjunctivity for countable sofic groups. In the following two propositions, G is a countable sofic group.

Proposition 2.9. Let $k \in \mathbb{N}$ and let $G \curvearrowright \{1, \ldots, k\}^G$ be the left shift action. Let Σ be a sofic approximation sequence. Then $h_\Sigma(\{1, \ldots, k\}^G, G) = \log k$.

Proof. As we did before in the amenable case, consider on $\{1,\dots,k\}^G$ the canonical dynamically generating pseudometric

$$\rho((x_s),(y_s)) = \begin{cases} 0 & \text{if } x_e = y_e \\ 1 & \text{if } x_e \neq y_e. \end{cases} \tag{20}$$

Let F be a finite subset of G containing e and let $\delta > 0$. Let $\sigma : G \to \mathrm{Sym}(d)$ be a sofic approximation which is good enough for purposes to be described.

For each $\omega \in \{1,\dots,k\}^d$ choose a $\varphi_\omega : \{1,\dots,d\} \to X$ such that $\varphi_\omega(v)_{s^{-1}} = \omega(\sigma_s(v))$ for all $v \in \{1,\dots,d\}$ and $s \in F$. For such a φ_ω we then have, for every $s \in F$ and $v \in V$ satisfying $\sigma_e\sigma_s(v) = \sigma_s(v)$,

$$\varphi(\sigma_s(v))_e = \omega(\sigma_e\sigma_s(v)) = \omega(\sigma_s(v)) = \varphi_\omega(v)_{s^{-1}} = s\varphi_\omega(v)$$

Thus if σ is a good enough sofic approximation so that $\sigma_e v = v$ for all v in a subset of $\{1,\dots,d\}$ of proportional size close enough to 1, we will have $\rho_2(\varphi\sigma_s, s\varphi_\omega) < \delta$ for all $s \in F$. Since the maps φ_ω for $\omega \in \{1,\dots,k\}^d$ are distinct and 1-separated with respect to ρ, for every $0 < \varepsilon < 1$ we get $N_\varepsilon(\mathrm{Map}(\rho,F,\delta,\sigma)) \geq k^d$. The reverse inequality is clear from the definition of ρ, and so we conclude that $h_\Sigma(\{1,\dots,k\}^G, G) = \log k$. □

Proposition 2.10. Let $k \in \mathbb{N}$ and let $G \curvearrowright X$ be the restriction of the left shift action $G \curvearrowright \{1,\dots,k\}^G$ to some proper closed G-invariant set. Let Σ be a sofic approximation sequence. Then $h_\Sigma(X,G) < \log k$.

Proof. Let ρ be the continuous pseudometric on X defined as in the case of the full shift above by (20), and note that it is dynamically generating. Since X is a proper closed G-invariant subset of $\{1,\dots,k\}^G$, there are a nonempty finite set $F \subseteq G$ and a map $f : F^{-1} \to \{1,\dots,k\}$ such that f is not the restriction of an element of X to F^{-1}. Let $0 < \delta < 1/2$, and let $\sigma : G \to \mathrm{Sym}(d)$ be a sofic approximation which is good enough so that the set A of all $v \in \{1,\dots,d\}$ such that the function $s \mapsto \sigma_s(v)$ from F to $\{1,\dots,d\}$ fails to be injective satisfies $|A| < \delta d$.

Take a maximal $(\rho_\infty,\varepsilon)$-separated set $M \subseteq \mathrm{Map}(\rho,F,\delta,\sigma)$. Write $\mathcal{C}$ for the collection of all sets $B \subseteq \{1,\dots,d\}$ such that $|B| = \lceil 2\delta d \rceil$ and $A \subseteq B$. Let $B \in \mathcal{C}$, and define Ω_B to be the set of all $\varphi \in M$ which are F-equivariant on $\{1,\dots,d\} \setminus B$, i.e., $\varphi\sigma_s(v) = s\varphi(v)$ for all $v \in \{1,\dots,d\} \setminus B$ and $s \in F$. Take a maximal set $V \subseteq \{1,\dots,d\} \setminus B$ such that $\sigma(F)v \cap \sigma(F)w = \emptyset$ for all distinct $v,w \in V$. By the pigeonhole principle,

$$|V| \geq \frac{|\{1,\dots,d\} \setminus B|}{|\sigma(F)^{-1}\sigma(F)|} \geq \frac{(1-2\delta)d}{|F|^2}. \tag{21}$$

For each $\varphi \in M$ associate an $\omega_\varphi \in \{1, \ldots, k\}^d$ given by $\omega_\varphi(v) = \varphi(v)_e$, and notice that this coding of elements of M is injective by the definition of ρ. For every $v \in V$ the function $s \mapsto \sigma_{s^{-1}}(v)$ from F^{-1} to $\{1, \ldots, d\}$ is injective since $V \cap A = \emptyset$, and the composition of this function with ω_φ for any $\varphi \in M$ cannot be equal to f since $\omega_\varphi(\sigma_s(v)) = (s\varphi(v))_e = \varphi(v)_{s^{-1}}$ for every $s \in F$. These exclusions of f mean that for every $v \in V$ there at most $k^{|F|} - 1$ elements among the restrictions of the codes ω_φ to $\sigma(F)v$, and so

$$|\Omega_B| \leq k^{d-|F||V|}(k^{|F|} - 1)^{|V|} = (k^{|F|})^{d/|F|-|V|}(k^{|F|} - 1)^{|V|}.$$

By (1) we see that this upper bound is at most $k^{(1-\varepsilon)d}$ for some $\varepsilon > 0$ not depending on the choice of $B \in \mathcal{C}$.

Observe next that the number of sets $B \subseteq \{1, \ldots, d\}$ such that $|B| = \lceil 2\delta d \rceil$ has cardinality $\binom{d}{\lceil 2\delta d \rceil}$, which by Stirling's formula is bounded above by $e^{\eta d}$ for some $\eta > 0$ not depending on d, with $\eta \to 0$ as $\delta \to 0$. Since $|B \setminus A| \geq \delta d$, every element of M lies in Ω_B for some $B \in \mathcal{C}$, so that

$$|M| \leq \left| \bigcup_{B \in \mathcal{C}} \Omega_B \right| \leq |\mathcal{C}| k^{(1-\varepsilon)d} \leq e^{\eta d} k^{(1-\varepsilon)d}.$$

and hence

$$h_\Sigma^\varepsilon(\rho, F, \delta) \leq \eta + (1 - \varepsilon) \log k.$$

This last upper bound is strictly smaller that $\log k$ if δ is sufficiently small, and so we conclude that $h_\Sigma(X, G) = \sup_{\varepsilon > 0} h_\Sigma^\varepsilon(X, G) < \log k$. $\square$

THEOREM 2.11. Every countable sofic group is surjunctive.

PROOF. Let G be a sofic group, and let $G \curvearrowright \{1, \ldots, k\}^G$ be the left shift action for some $k \in \mathbb{N}$. If $\psi : \{1, \ldots, k\}^G \to \{1, \ldots, k\}^G$ is an injective G-equivariant continuous map, then this gives a conjugacy between the left shift action and the restriction of the left shift action to the image of ψ, and therefore the G-action on the image of ψ has entropy $\log k$ with respect to any sofic approximation sequence by Proposition 2.9. Thus ψ is surjective by Proposition 2.10, establishing the surjunctivity of G. $\square$

Notational Index.

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$, $\mathbb{Z}_2$,...	abelian groups
Γ	a discrete group
$K < \Gamma$	K is a subgroup of Γ
$N \triangleleft \Gamma$	H is a normal subgroup of Γ
$\phi, \theta, \ldots$	homomorphisms or approximations
$g, h, k, \ldots$	elements of a group
$\mathbb{F}_n$	the free group of rank n
$d(\cdot, \cdot)$	distance defined on $\Gamma \times \Gamma$
$l(\cdot)$	length function
$S(n)$	symmetric group on n elements
σ_i	a permutation, element of $S(n)$
$SL_n(\mathbb{Z})$	
$\omega_i, \omega_i(g_1, \ldots, g_n)$	a word on generators $g_1 \ldots g_n$
$l_p(\Gamma)$	
n, k	usually in $\mathbb{N}$
$\lvert \cdot \rvert$	cardinality of a set

CHAPTER 7

Sofic dimension

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