1 Introduction

The main motivations for the original study of super manifolds comes from physics, specifically in quantum field theory. The spin statistics theorem tells us that we can put all quantum mechanical particles into two categories, the category being determined by the magnitude of the intrinsic angular momentum of a given particle. Particles with total angular momentum $\hbar^2 j(j+1)$ for j even are called bosons, and quantum field theory tells us that the observable algebra related to bosons should follow a set of commutation relations, much like the commutation relations one sees in non-relativistic quantum mechanics. The category of particles with j odd are called fermions, and the observables related to these particles must satisfy *anticommutation relations* which involve the anticommutator $\{a, b\} = ab + ba$. When one creates a quantum field theory in the scheme of canonical quantization, roughly speaking, one would like to take a classical observable algebra corresponding to functions on the classical phase space equipped with a Poisson bracket and convert it to a non-commutative algebra of observables on a Hilbert space and for the case of bosonic quantization no issues arise, since smooth functions automatically commute, it is unproblematic to take the standard commutation of functions $[a, b] = 0$ and deform it to $\left[\hat{a}, \hat{b}\right] \neq 0$, but since classical smooth functions never anticommute, one can't find a classical space of observables corresponding to fermions. In order to introduce classical observables that can anticommute, we stop modeling spacetime as a smooth manifold $(M, C^\infty(M))$ and think of it as what is called a supermanifold $(M, \mathcal{O}^{m|n})$ where the underlying topological space of M is our regular smooth manifold of space time M, but C^∞ is now a replaced by a sheaf of *commutative super algebras* $\mathcal{O}^{m|n}$ that is locally isomorphic to $C^\infty(\mathbb{R}^m) \otimes \bigwedge^\bullet(\xi^1, \ldots, \xi^n)$ in some sense. This is one of two viewpoints, which lends itself to theoretical study. There is a whole concrete approach, that more resembles classical geometry in its use of charts and definitions of smooth functions. A comprehensive account of the concrete approach can be found in [1] while a more comprehensive coverage can be found [2] and [3]

2 Super Algebra

Before we can reasonably define a super manifold and some of its super-analogues from classical differential geometry, we need to define some of the algebraic structures that we use to model "superspace".

Definition 2.1. Let $\mathbb{V}$ be a k-vector space. It is said to be a super vector space if

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1.$$

The subspace $\mathbb{V}_0$ is called the even subspace and $\mathbb{V}_1$ is called the odd subspace.

A linear map $\phi\colon \mathbb{V} \to \mathbb{W}$ between k-super vector spaces is called a super vector space homomorphism.

Definition 2.2. A super vector space homomorphism $\phi\colon \mathbb{V} \to \mathbb{W}$ is called even if

$$\phi(\mathbb{V}_0) \subset \mathbb{W}_0 \quad \text{and} \quad \phi(\mathbb{V}_1) \subset \mathbb{W}_1$$

and odd if

$$\phi(\mathbb{V}_0) \subset \mathbb{W}_1 \quad \text{and} \quad \phi(\mathbb{V}_1) \subset \mathbb{W}_0$$

more succinctly ϕ is of degree j if

$$\phi(\mathbb{V}_i) \subset \mathbb{W}_{i+j}$$

for $i = 0, 1$ where addition is taken modulo 2.

Definition 2.3. A k-algebra $\mathbb{A}$ is called a super algebra if it is a super vector space and for $a \in \mathbb{A}_i$, the super vector space homomorphism $b \mapsto ab$ is of degree i.

Definition 2.4. A k-super algebra $\mathbb{A}$ is said to be super commutative (or commutative) if $ab = (-1)^{|a||b|} ba$ where $a \in \mathbb{A}_{|a|}$ and $b \in \mathbb{A}_{|b|}$

Example 1. Let $\bigwedge^\bullet V$ be the exterior algebra of a k-vector space V. It is clear that $\bigwedge^\bullet V$ is a graded commutative k-algebra. This descends to a super-commutative, super algebra by defining

$$\left(\bigwedge^\bullet V\right)_0 := \bigoplus_{n \in \mathbb{N}} \bigwedge^{2n} V \quad \text{and} \quad \left(\bigwedge^\bullet V\right)_1 := \bigoplus_{n \in \mathbb{Z}} \bigwedge^{2n+1} V$$

One also calls the unital $\mathbb{R}$ exterior algebra on the generators $\xi^1, \dots, \xi^n$ by $[\xi^1, \dots, \xi^n]$.

The preceding example will be one of the elementary building blocks of super manifolds, we will eventually codify super manifolds as spaces equipped with a sheaf of super algebras that are locally isomorphic to $C^\infty(\mathbb{R}^m) \otimes [\xi^1, \dots, \xi^n]$.

We can define super modules over a super algebra as follows:

Definition 2.5. Let $\mathbb{A}$ be a k-super algebra and M a $\mathbb{A}$ module. M is said to be a $\mathbb{A}$-super module if M is a super vector space and

$$\mathbb{A}_i M_j \subset M_{j+i} \quad \text{for} \quad i, j = 0, 1$$

One such super $\mathbb{A}$ module that we will be useful in generalizing the tangent bundle is the super derivations of $\mathbb{A}$.

Definition 2.6. A super vector space homomorphism $\delta \colon \mathbb{A} \to \mathbb{A}$ for $\mathbb{A}$ a super commutative super algebra is said to be a super derivation if

$$\delta(A_1 A_2) = \delta(A_1) A_2 + (-1)^{|A_1||f|} A_1 \delta(A_2)$$

for all $A_1, A_2 \in \mathbb{A}$. This generalizes the notion of derivations of commutative algebras, like for $C^\infty(M)$.

Proposition 2.7. *Let $\mathbb{A}$ be a super commutative super k-algebra, the space of super derivations of $\mathbb{A}$,* $\mathrm{Der}(\mathbb{A})$ *is a super $\mathbb{A}$ module.*

Proof. It is clear that $\mathrm{Der}(\mathbb{A})$ is a vector space, so it suffices to show that left multiplication by $\mathbb{A}$ preserves the super derivation property. Since, by definition $\mathrm{Der}(\mathbb{A}) = \mathrm{Der}(\mathbb{A})_0 \oplus \mathrm{Der}(\mathbb{A})_1$ where $\mathrm{Der}(\mathbb{A})_i$ is the space of i-degree derivations, it suffices to consider elements of fixed degree. for $A \in \mathbb{A}_i, A_1 \in \mathbb{A}_{j_1}, A_2 \in \mathbb{A}_{j_2}$ and $\delta \in \mathrm{Der}(\mathbb{A})_l$ we have

$$\begin{aligned}(A\delta)(A_1A_2) &= A(\delta(A_1A_2)) = A(\delta(A_1)A_2 + (-1)^{j_1 l}A_1\delta(A_2)) \\ &= (A\delta(A_1))A_2 + (-1)^{j_1 l}AA_1\delta(A_2) \\ &= (A\delta(A_1))A_2 + (-1)^{j_1 l}(-1)^{j_1 i}A_1(A\delta(A_2)) \\ &= (A\delta(A_1))A_2 + (-1)^{j_1(l+i)}A_1(A\delta(A_2))\end{aligned}$$

So $A\delta$ is a derivation of degree $l+i$ and $\mathrm{Der}(\mathbb{A})$ is a super $\mathbb{A}$-module. □

Just as the derivations of a commutative algebra form a Lie algebra, we want to generalize this notion to the derivations of a super commutative super algebra, motivating the following definition.

Definition 2.8. Let $\mathbb{U}$ be a super vector space. A bilinear map $[\cdot,\cdot]\colon \mathbb{U} \times \mathbb{U} \to \mathbb{U}$ is said to define a super Lie algebra structure on $\mathbb{U}$ if for homogeneous elements $u, v \in \mathbb{U}$ we have $[u,v] \in \mathbb{U}_{|u|+|v|}$,

$$[u,v] = -(-1)^{|u||v|}[v,u],$$

i.e. $[\cdot,\cdot]$ is super-skew symmetric and $[\cdot,\cdot]$ satisfies the super-Jacobi identity:

$$(-1)^{|u||w|}[[u,v],w] + (-1)^{|w||v|}[[w,u],v] + (-1)^{|u||v|}[[v,w],u] = 0$$

for homogeneous elements u, v, w.

Proposition 2.9. *Let $\mathbb{A}$ be a commutative super k-algebra. The space of super derivations* $\mathrm{Der}(\mathbb{A})$ *of $\mathbb{A}$ forms a super Lie algebra under the super commutator bracket:*

$$[\delta,\gamma] = \delta\gamma - (-1)^{|\delta||\gamma|}\gamma\delta$$

Proof. This proposition is easily verified using the same techniques used to prove that the space of derivations of a commutative algebra form a Lie algebra. □

Proposition 2.10. *Let $\mathbb{A}$ and $\mathbb{B}$ be super k-algebras. Their tensor product $\mathbb{A} \otimes \mathbb{B}$ is a super k-algebra with*

$$(\mathbb{A} \otimes \mathbb{B})_l = \bigoplus_{i+j=l} \mathbb{A}_i \otimes \mathbb{B}_j$$

and multiplication defined as

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (-1)^{|B_1||A_2|}(A_1A_2 \otimes B_1B_2)$$

and if $\mathbb{A}$ and $\mathbb{B}$ are super commutative $\mathbb{A} \otimes \mathbb{B}$ is super commutative too.

Proof. Since the tensor product of k-algebras is again a k-algebra, it suffices to show that multiplication satisfies the correct grading. For $A_1, A_2 \in \mathbb{A}$ and $B_1, B_2 \in \mathbb{B}$ all of definite degree we have

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (-1)^{|B_1||A_2|}(A_1A_2 \otimes B_1B_2)$$

and hence $|(A_1 \otimes B_1)(A_2 \otimes B_2)| = |A_1| + |A_2| + |B_1| + |B_2| \mod 2 = |A_1 \otimes B_1| + |A_2 \otimes B_2| \mod 2$

Assume that $\mathbb{A}$ and $\mathbb{B}$ are commutative super k-algebras. For A_1, A_2, B_1 and B_2 as above, we have

$$\begin{aligned}(A_1 \otimes B_1)(A_2 \otimes B_2) &= (-1)^{|B_1||A_2|}(A_1A_2 \otimes B_1B_2) \\ &= (-1)^{|B_1||A_2|}((-1)^{|A_1||A_2|}A_2A_1 \otimes (-1)^{|B_1||B_2|}B_2B_1) \\ &= (-1)^{|B_1||A_2|+|A_1||A_2|+|B_1||B_2|}(A_2A_1 \otimes B_2B_1)\end{aligned}$$

Working from the other direction, we have

$$\begin{aligned}(-1)^{|A_1\otimes B_1||A_2\otimes B_2|}(A_2 \otimes B_2)(A_1 \otimes B_1) &= (-1)^{(|A_1|+|B_1|)(|A_2|+|B_2|)+|B_2||A_1|}(A_2A_1 \otimes B_2 \otimes B_1) \\ &= (-1)^{|A_1||A_2|+|A_1||B_2|+|B_1||A_2|+|B_1||B_2|+|B_2||A_1|}(A_2A_1 \otimes B_2B_1) \\ &= (-1)^{|A_1||A_2|+|B_1||A_2|+|B_1||B_2|}(A_2A_1 \otimes B_2B_1) \\ &= (A_1 \otimes B_1)(A_2 \otimes B_2)\end{aligned}$$

In this calculation, we see that in order for tensor products to preserve commutativity, we have to include the appropriate sign rule in our multiplication. □

3 Super Manifolds

Now that we have described the basic algebraic structures of superalgebra, we can now describe super manifolds. In the study of smooth manifolds, the classical view is that a smooth manifold is a Hausdorff second countable topological space M equipped with a maximal smooth atlas $\mathcal{A}$, however one can take a different, more algebraic approach that allows one to study non-commutative geometry. To each smooth manifold $(M, \mathcal{A})$ we can associate a sheaf of commutative algebras C^∞ which associates to each $U \subset M$ open the algebra of smooth functions on U, $C^\infty(U)$. For $(U, \psi) \in \mathcal{A}$, we have $C^\infty\big|_U \cong C^\infty\big|_{\psi(U)}$ where $C^\infty\big|_{\psi(U)}$ is the restriction of the sheaf of smooth functions on $\mathbb{R}^{\dim M}$ to $\psi(U)$. The existence of such a sheaf with this local isomorphism property is equivalent to the existence of a smooth manifold structure, inspiring us to define super manifolds as analogous spaces with a sheaf of "functions" which is locally isomorphic to the super algebra $C^\infty(\mathbb{R}^n) \otimes \bigwedge^\bullet \mathbb{R}^m$.

We begin by describing the local model of a super manifold:

Definition 3.1. The space $\mathbb{R}^n$ equipped with the standard topology and the sheaf of super algebras $\mathcal{O}^{m|n}$ with

$$\Gamma(V, \mathcal{O}^{m|n}) := C^\infty(V) \otimes [\xi^1, \dots, \xi^n]$$

where $C^\infty(V) = C^\infty(V)_0 \oplus 0$

Definition 3.2. A super manifold of dimension (m, n) is a topological space M and a sheaf of super algebras $\mathcal{O}$ with an open cover $\{U_i\}_{i\in I}$ such that

$$\mathcal{O}|_{U_i} \cong \mathcal{O}^{m|n}|_{V_i}$$

for $V_i \subset \mathbb{R}^m$ homeomorphic to U_i.

In the following proposition, we show that underlying every super manifold of dimension (m, n) is a smooth manifold of dimension m.

Proposition 3.3. *Let $(M, \mathcal{O})$ be a super manifold with dimension (m, n). Let $\mathfrak{N}$ be the sheaf of ideals generated by nilpotent elements of $\mathcal{O}$. The quotient sheaf $(M, \mathcal{O}/\mathfrak{N})$ is locally isomorphic to $(\mathbb{R}^m, C^\infty)$ and hence $(M, \mathcal{O}/\mathfrak{N})$ is a smooth manifold.*

Proof. Since $\mathcal{O}$ is locally isomorphic to $C^\infty \otimes [\xi^1, \dots, \xi^n]$ it suffices to show that $C^\infty \otimes [\xi^1, \dots, \xi^n]/\mathfrak{N}_{\mathcal{O}^{m|n}} \cong C^\infty$. We start by seeing that $[\xi^1, \dots, \xi^n] = \mathbb{R} \oplus (\langle \xi^1, \dots \xi^n \rangle)$. Given $\alpha \in \langle \xi^1, \dots, \xi^n \rangle$, of homogeneous degree l, α^{n+1} has degree $l(n+1) > n$ and hence $\alpha^{n+1} = 0$. Since $\mathbb{R}$ has no non-trivial nilpotents, the nilpotent ideal of $[\xi^1, \dots, \xi^n]$ is $\langle \xi^1, \dots \xi^n \rangle$.

Because $\mathbb{R}$ has no non-trivial nilpotents, $C^\infty(U)$ has no nontrivial nilpotents for any open set $U \subset \mathbb{R}^m$. This means that the nilpotent ideal of $C^\infty(U) \otimes [\xi^1, \dots, \xi^n]$ is $\langle 1 \otimes \xi^1, \dots, 1 \otimes \xi^n \rangle$ We then have, for every U,

$$\begin{aligned} \frac{C^\infty(U) \otimes [\xi^1, \dots, \xi^n]}{\langle 1 \otimes \xi^1, \dots, 1 \otimes \xi^n \rangle} &= \frac{C^\infty(U) \otimes (\mathbb{R} \oplus (\langle \xi^1, \dots, \xi^n \rangle))}{\langle 1 \otimes \xi^1, \dots, 1 \otimes \xi^n \rangle} \\ &= \frac{C^\infty(U) \oplus \langle 1 \otimes \xi^1, \dots, 1 \otimes \xi^n \rangle}{\langle 1 \otimes \xi^1, \dots, 1 \otimes \xi^n \rangle} \\ &\cong C^\infty(U) \end{aligned}$$

Since this can be carried out for arbitrary open U, and our isomorphisms respect restriction to open sets, this defines an isomorphism of sheaves $(\mathbb{R}^{m|n}, \mathcal{O}^{m|n}/\mathfrak{N}) \to (\mathbb{R}^m, C^\infty)$.

Let (U_i, ϕ) be a family of isomorphisms $\mathcal{O}|_{U_i} \cong \mathcal{O}^{m|n}|_{V_i}$. Because $\mathfrak{N}(\mathcal{O}|_{U_i}) \cong \mathfrak{N}(\mathcal{O}^{m|n}|_{V_i})$ we have $\mathcal{O}|_{U_i}/\mathfrak{N}(\mathcal{O}|_{U_i}) \cong \mathcal{O}^{m|n}|_{V_i}/\mathfrak{N}(\mathcal{O}^{m|n}|_{V_i}) \cong (\mathcal{O}^{m|n}/\mathfrak{N})|_{V_i} \cong C^\infty|_{V_i}$ then $(M, \mathcal{O}/\mathfrak{N})$ is a smooth manifold. □

We will hereby refer to $\mathcal{O}/\mathfrak{N}(\mathcal{O})$ as C^∞.

Definition 3.4. Let $(M, \mathcal{O})$ and $(N, \mathcal{S})$ be super manifolds. A morphism of super manifolds $f: (M, \mathcal{O}) \to (N, \mathcal{S})$ is a continuous map $f: M \to N$ and a morphism of sheaves of super algebras $f^*: f_*(\mathcal{O}) \to \mathcal{S}$ where the push forward sheaf $f_*(\mathcal{O})$ is defined by $f_*(\mathcal{O})(V) = \mathcal{O}(f^{-1}(V))$ for all $V \subset N$ open.

Remark. This generalizes the notion of smooth maps between manifolds. We can see that a morphism of super manifolds defines a smooth map between the underlying manifolds as follows. Since the image of a nilpotent under a ring homomorphism is nilpotent, we have $f^*(\mathfrak{N}(f_*\mathcal{O})) \subset \mathfrak{N}(\mathcal{S})$. Then the map $f^*: f_*\mathcal{O} \to S$ induces a map $f^*: f_*\mathcal{O}/\mathfrak{N}(f_*\mathcal{O}) \to S/\mathfrak{N}(S)$ and since $f_*\mathcal{O}/\mathfrak{N}(f_*\mathcal{O}) \cong f_*(\mathcal{O}/\mathfrak{N}(\mathcal{O}))$ we get a morphism $(M, C^\infty) \to (N, C^\infty)$.

Just as the space of vector fields of a smooth manifold is the space of derivations of its algebra of smooth functions, we easily come up with the following definition:

Definition 3.5. Let $(M, \mathcal{O})$ be a super manifold. We say that the space of vector fields on M is $\mathfrak{X}(M) = \mathrm{Der}(\mathcal{O}(M))$, the space of super derivations of $\mathcal{O}(M)$, and $\mathfrak{X} = \mathrm{Der}(\mathcal{O})$ is the sheaf of local vector fields on M.

Since $\mathcal{O}$ is locally isomorphic to $C^\infty(M) \otimes [\xi^1, \ldots, \xi^n]$ we can easily find local frames for $\mathfrak{X}(M)$. Let (U, ψ) be a neighborhood with $\mathcal{O}(U) \cong \mathcal{O}^{m|n}(V)$. The typical vector fields $\frac{\partial}{\partial x^i}$ which form the basis of derivations of $C^\infty(V)$ give a generating set for the even derivations. We will call $\frac{\partial}{\partial x^i} \otimes 1 := \frac{\partial^E}{\partial x^i}$ and $1 \otimes \iota_{\xi_j} := \frac{\partial^O}{\partial \xi^j}$. We can then pull these back to M to get local frames there.

Definition 3.6. We can define differential forms on a super manifold $(M, \mathcal{O})$ by taking $\Omega^k(M) := \mathcal{A}^k(\mathfrak{X}(M), \mathcal{O}(M))$ where

$$\mathcal{A}^k(\mathfrak{X}(M), \mathcal{O}(M)) := \{\omega \colon \mathfrak{X}(M)^k \to \mathcal{O}(M) | \omega \text{ is alternating and } \mathcal{O}(M) \text{linear}\}$$

One can also define a wedge product and exterior differential on $\Omega^\bullet(M)$ in the same way as their classical counterparts leading to a theory of de Rham cohomology on super manifolds. However, this theory is isomorphic to the classical $(m, 0)$ theory.

One can define many structures analogously to those from classical differential geometry, by taking the sheaf of functions, algebro-geometric view of the classical structures.

References

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