1 Symplectic and Contact Geometry Basics

1.1 Symplectic Geometry

The original motivation for symplectic geometry comes from classical mechanics. The classical phase space for a free particle in n dimensions $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates given q^i and p_j (the q^i's representing the generalized position and p_j's representing the generalized momentum of the particle). Given a Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, where the value $H(q^i, p_j, t)$ is the energy of a particle with position q^i, momentum p_j at time t, Hamilton's equations describe the time evolution of the given particle:

$$\frac{\partial H}{\partial p_j} = \dot{q}^j, \quad \frac{\partial H}{\partial q^i} = -\dot{p}^i$$

We can interpret the preceding equation as one describing a time dependent vector field X on $\mathbb{R}^n \times \mathbb{R}^n$ and package it as

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} = X$$

Recognizing $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ as a skew symmetric matrix, we can realize this equation as

$$dH = \iota_X \omega$$

where ω is the constant form $dq^i \wedge dp_i$. Note that this equation has a unique solution because J is invertible. If we take ω to be the object which takes a Hamiltonian and outputs the corresponding vector field which generates the dynamics of the given Hamiltonian, in wishing to generalize from $\mathbb{R}^n \times \mathbb{R}^n$ to the setting of smooth manifolds, one may make the following definition:

Definition 1.1. A symplectic manifold is a smooth manifold M equipped with a two form $\omega \in \Omega^2_M$ which is non-degenerate and closed, i.e. $\omega_p(u, v) = 0$ for all $v \in T_pM$ implies that $u = 0$ and $d\omega = 0$. The Hamiltonian vector field of a smooth function $H : M \to \mathbb{R}$ is the unique vector field $X_H \in \mathfrak{X}(M)$ such that $\iota_{X_H}\omega = dH$.

Example 1. Our motivating situation of classical mechanics on $\mathbb{R}^n$ yields the example of $(\mathbb{R}^{2n}, \omega = -d(p_i\, dq^i))$ where the equations for the Hamiltonian vector field reproduce Hamilton's equations. We can generalize this example by thinking of $\mathbb{R}^{2n}$ as $T^*\mathbb{R}^n$ and $\theta = p_i\, dq^i$ as the one form which satisfies

$$\alpha^*\theta = \alpha \tag{1}$$

for $\alpha \in \Omega^1(\mathbb{R}^n)$. Equation (1) uniquely determines θ, and such a form is invariant under smooth changes of coordinates in the base. One can then show that for any Q a smooth manifold T^*Q is a symplectic manifold with $\omega = -d\theta$ for θ defined as in Equation (1).

Definition 1.2. A map $f : M \to N$ where (M, ω) and (N, ω') are symplectic manifolds is called a symplectomorphism if it preserves the symplectic form on M, i.e. $f^*\omega' = \omega$. The symplectic manifolds M and N are said to be isomorphic if there exists a symplectomorphism between them which is bijective, (symplectomorphisms must be local diffeomorphisms, so an isomorphism is necessarily a diffeomorphism).

Remark. A few arguments in linear algebra tell us that every symplectic manifold must be even dimensional. One of the first things one should take into account when dealing with symplectic manifolds, is their relatively high amount of flexibility. This is first seen in the following theorem, which tells us that *symplectic manifolds have no local invariants.*

Theorem 1.3. *(Darboux) Let (M^{2n}, ω) be a symplectic manifold. Then there exist local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on M such that the local expression for ω is the standard symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$, i.e. $\omega = dq^i \wedge dp_i$.*

This means that all symplectic manifolds of the same dimension are necessarily locally isomorphic to one another. This contrasts with Riemannian geometry, where the Riemann curvature tensor measures the failure of the manifold to be locally flat. This difference comes from the requirement that a symplectic form be closed, a kind of integrability requirement, which is lacking in the Riemannian case.

An important part of the study of symplectic geometry is the dynamics of the Hamiltonian vector fields. The Hamiltonian vector fields generate automorphisms of a symplectic manifold, as is manifest through the following proposition.

Proposition 1.4. *Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$. The Hamiltonian vector field associated to f, X_f has $\mathcal{L}_{X_f}\omega = 0$ and hence if $\varphi_t : M \to M$ is the flow of X_f then $\varphi_t^*\omega = \omega$.*

Proof. Let f be as above. By Cartan's magic formula

$$\mathcal{L}_{X_f}\omega = d\iota_{X_f}\omega + \iota_{X_f} d\omega = ddH + 0 = 0 \quad \square$$

One other important theorem which shows the flexibility of symplectic manifolds is Moser's theorem

Theorem 1.5. *(Moser) Let M be a compact symplectic manifold with symplectic forms ω_0 and ω_1 connected by a smooth family of symplectic forms ω_t with $t \in [0, 1]$ such that $[\omega_t] = [\omega_0]$ for each t. Then there exists an isotopy $\varphi_t : M \to M$ such that $\varphi_1^*\omega_1 = \omega_0$.*

1.2 Contact Geometry

We now turn to the sister subject of symplectic geometry, the study of an odd dimensional analogue of symplectic manifolds; contact geometry which plays an important roll in symplectic cohomology.

Definition 1.6. Let M^{2n-1} be a smooth manifold. A (vector)-subbundle of $\xi \subset TM$ is called a distribution and is said to be a contact distribution if it is of codimension 1 and is maximally non-integrable, i.e. if $\alpha \in \Omega^1_M$ has $\xi\big|_U = \ker\alpha$ then $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing.

This condition does not depend on the choice of α since any other defining α' must have $\alpha' = f\alpha$ for some nonvanishing $f \in C^\infty(U)$. We then have $f\alpha \wedge (d(f\alpha))^{n-1} = f\alpha \wedge (df \wedge \alpha + fd\alpha)^{n-1} = f^n \alpha \wedge d\alpha^{n-1}$. Note that the preceding equation means that $d\alpha\big|_\xi$ is a non-degenerate form and hence makes ξ into a symplectic vector bundle and guarantees that any manifold which satisfies this condition is necessarily of odd dimension. These reasons are part of why contact geometry is known as the odd dimensional sister to symplectic geometry. A contact manifold in which there exists a globally defined one form $\alpha \in \Omega^1_M$ with $\xi = \ker\alpha$ is called co-orientable. For the rest of this lecture series we will assume that all contact manifolds are cooriented, as these are the ones which naturally occur in symplectic cohomology, and will assume a choice of α to be part of the data of a contact manifold.

Since we now have a distinguished 1 form on our contact manifold, we may study it by studying the dynamics which this form produces. The easiest way to do this is using the Reeb vector field $R \in \mathfrak{X}(M)$ defined by $\iota_R d\alpha = 0$ and $\iota_R\alpha = 1$. The first condition guarantees that R is never lies in $\ker\alpha$ and the second condition makes such a choice unique. We then see that $\mathcal{L}_R\alpha = \iota_R d\alpha + d\iota_R\alpha = 0$ meaning that the Reeb vector field preserves the contact form and $\mathcal{L}_R d\alpha = d\iota_R d\alpha = 0$ means that it acts via symplectomorphism when restricted to $\ker\alpha$.

Another way in which the connection between symplectic geometry and contact geometry is manifest is the following procedure for creating a symplectic manifold from a contact manifold. Let (M, α) be a contact manifold. Consider the subbundle of $S(M) \subset T^*M$ with fiber given by $S(M)_p = \{\beta \in T^*_pM : \ker\beta = \xi_p\} = \mathbb{R}^*\alpha_p$. In fact $S(M) \cong M \times \mathbb{R}^*$ under the map $(t\alpha_p) \mapsto (p, t)$. One sees that since $\alpha^*\theta = \alpha$, under the map $\psi : S(M) \to M \times \mathbb{R}^*$ we have $\psi^*\theta = t\alpha$. Then $\psi^*\omega = d(t\alpha) = dt \wedge \alpha + t\,d\alpha$. Note that $\psi^*\omega\big|_{\ker\alpha} = dt \wedge \alpha\big|_{\ker\alpha} + td\alpha\big|_{\ker\alpha} = td\alpha\big|_{\ker\alpha}$, meaning that $\psi^*\omega$ is non-degenerate when restricted to $\ker\alpha$. The contact condition requires that $dt \wedge \alpha$ is nondegenerate when restricted to the kernel of $d\alpha$ and hence $\psi^*\omega$ actually yields a symplectic form on $S(M)$. This space, $S(M)$ is called the symplectization of (M, α) and we can study M by studying the symplectic geometry of $S(M)$.

Example 2. One of the motivating examples for definitions in contact geometry is that of an energy hyper surface of a symplectic manifold. Let (M, ω) be a symplectic manifold and $H \in C^\infty(M)$ with E a regular value. We have $M_E = H^{-1}(E)$ a codimension one submanifold of M. If we choose a primitive for ω (now we necessarily assume that M is not compact and/or is with boundary, as there are no exact compact symplectic manifolds without boundary of positive dimension), θ, we may give M_E the distribution defined by $\ker\theta\big|_{M_E}$.

If X is the symplectic dual to θ, i.e. $\iota_X \omega = \theta$ (this vector field is called the Liouville field) then, assuming that $\theta\big|_{M_E}$ is nonvanishing, we see that $\theta \wedge \omega^{n-1} = \frac{1}{n}\iota_X \omega^n$ which never vanishes meaning that θ makes M_E a contact manifold. Using Gray's stability theorem (a contact analog of Moser's theorem), one can show that all such contact structures we may give M_E in this manner are isomorphic given an orientability assumption.

Example 3. The previous example, applying to every exact symplectic manifold, furnishes many simple contact manifolds. Consider $(M, \omega) = (\mathbb{C}^n, \sum dx^i \wedge dy^i)$, given coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ and $H = \frac{1}{2}\|z\|^2$, the energy hyper surfaces are copies of S^{2n-1}. If we take the primitive $\theta = \frac{1}{2}(x^i\, dy^i - y^i\, dx^i)$, which has symplectic dual $X = \frac{1}{2}\left(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}\right)$ the radial vector field on $\mathbb{C}^n$. This means that $\iota_X \omega^n$, up to a constant multiple yields the standard volume form on S^{2n-1} and $\theta\big|_{H^{-1}(E)}$ yields a contact structure. A short calculation shows that the Reeb vector field is given by $R = \frac{1}{n}\left(\frac{1}{y^i}\frac{\partial}{\partial x^i} - \frac{1}{x^i}\frac{\partial}{\partial y^i}\right)$ as

$$\iota_R \frac{1}{2}\sum \left(x^i dy^i - y^i dx^i\right) = \sum_{i=1}^{n} \frac{1}{n} = 1$$
$$\iota_R \sum dx^i \wedge dy^i = \frac{1}{n}\sum \left(\frac{1}{y^i}\, dy^i + \frac{1}{x^i}\, dx^i\right).$$

We can see that when restricted to M_E, the Reeb vector field corresponds with a multiple of the standard $U(1)$ action on an odd sphere and the space of orbits of this action is given by $\mathbb{C}P^{n-1}$.

1.3 Liouville Domains

We will now define the principal object of study for the rest of the course, symplectic manifolds with contact type boundary, or shortly, Liouville domains. In order to motivate the final form of the definition, we start by stating a few relevant facts about compact symplectic manifolds.

Let (M^n, μ) be a compact oriented manifold with volume form μ. If $\mu = d\alpha$ for $\alpha \in \Omega_M^{n-1}$, Stokes' theorem tells us that

$$\mathrm{Vol}_\mu M = \int_M d\alpha = \int_{\partial M} \alpha.$$

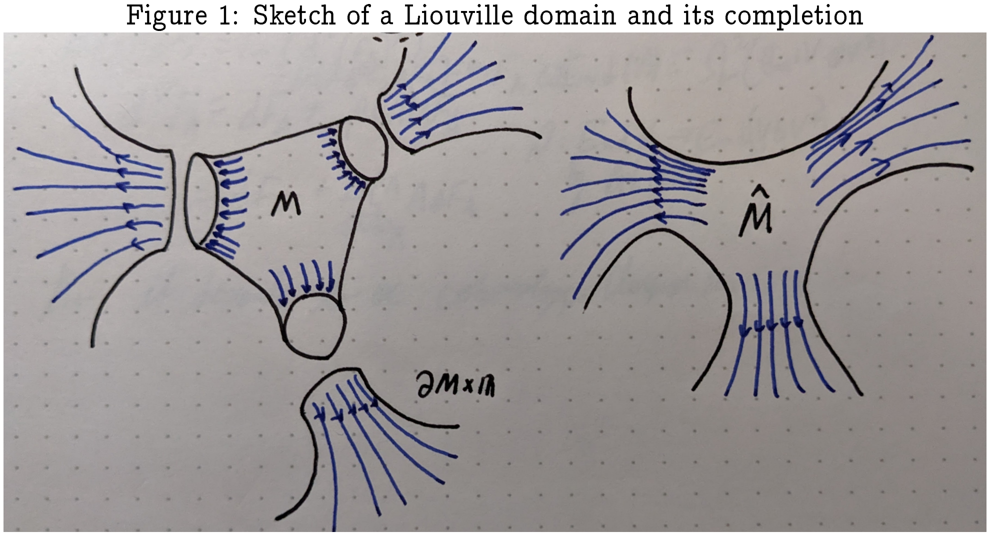
If M is closed, the right hand side of this equation is necessarily 0 and hence M is given 0 volume by this form. But a local calculation shows that the integral of a volume form is necessarily positive over contracitble neighborhood meaning that this equality may not hold if M is closed. If we assume that ω is a symplectic form on M, $\omega^{n/2}$ is a volume form, and if $\omega = d\theta$ then $\omega^{n/2} = d\left(\frac{1}{n/2-1}\theta \wedge \omega^{n/2-2}\right)$ meaning that a closed symplectic manifold may not be exact, i.e. it's symplectic form does not have a primitive. For reasons we may touch upon later, compactness of the underlying symplectic manifold is important for proving the compactness of the moduli space of solutions to the Floer equations, which are the objects counted by any Floer Homology. Compactness of this moduli space is essential for the well definedness of chain map of a Floer Homology theory. Requiring your symplectic form to be exact allows one to define the Floer complex without having to worry about the issue of sphere bubbling, a phenomenon in Floer theories which also causes issues with compactness.

If we wish to have both of these properties which insure convenient compactness results, one may ask for their symplectic manifold of study to be both compact and exact. But now the manifold must necessarily be with boundary! Now, in order to have our cake and eat it too we must work with manifolds with boundary and deal with how dynamics may occur there. In order to control the dynamics on the boundary, we will combine symplectic geometry with contact geometry, by requiring that the primitive we choose for our given symplectic form yields a contact structure on the boundary.

Definition 1.7. Let M^{2n} be a smooth compact manifold. For $\theta \in \Omega^1_M$, the pair (M, θ) is called a symplectic manifold with contact type boundary (or simply a Liouville domain) if $d\theta$ is a symplectic form on M and the Liouville vector field X is outward facing along ∂M, i.e. $\iota_X(\omega^n)\big|_{\partial M}$ is a volume form. This condition means that $\theta\big|_{\partial M}$ imbues ∂M with a contact structure.

The condition on the Liouville vector field, means that $\mathcal{L}_X\omega = \omega$ and hence under the flow of X, $\varphi_t^*\omega = \exp(t)\omega$. The requirement that X face outward along the boundary of M means that there is a neighborhood of ∂M which is trivially foliated by copies of ∂M and in fact, there is a neighborhood of ∂M which is symplectomorphic to $(-\epsilon, 0] \times \partial M$ seen as a submanifold with boundary of $S(M)$. By gluing M and $(-\epsilon, \infty) \times \partial M$ along $(-\epsilon, 0] \times \partial M$, we yield an exact symplectic manifold with a complete Lioville flow. The result of this gluing is written as $\hat{M}$ and is referred to as the symplectic completion of M.

Figure 1: Sketch of a Liouville domain and its completion



Because of the gluing process, far from M we may control the dynamics on $\hat{M}$ by considering hamiltonians which either level off sufficiently far from M or those which grow with a particular rate as the t coordinate goes to ∞. These allow us to do clever accounting and later solve many pesky compactness issues that often arise in the non compact setting.

Example 4. Consider the closed unit ball $\bar{B}^{2n} \subset \mathbb{C}^n$ with the restriction of the exact symplectic form $\omega = d\frac{1}{2}\sum \left(x^i dy^i - y^i dx^i\right)$. Because $\partial\bar{B}^{2n}$ is a hypersurface of constant energy, we can conclude that $\bar{B}^{2n}$ is a Liouville domain. Since we can extend the Liouville vector field to all of $\mathbb{C}^n$, we can conclude that $\hat{\bar{B}}^{2n} \cong \mathbb{C}^n$.

Definition 1.8. A Lioville isomorphism between two domains $(M, \theta), (N, \lambda)$ is a diffeomorphism $\varphi : M \to N$ such that $\varphi^*\lambda = \theta + df$ for f a compactly supported function on M. Such isomorphisms preserve the Liouville flow outside of the support of f (e.g. at infinity) and are symplectomorphisms. One finds that the following Moser/Gray theorem holds for Liouville isomorphisms as well:

Theorem 1.9. *If θ_t is a smooth family of Liouville structures on M, θ_0 and θ_1 are Liouville isomorphic.*

2 Floer Theory

2.1 Motivations

Remark. Much of the content of these notes and the rest of the series come from Paul Seidel's *A Biased View of Symplectic Cohomology* with parts coming from Alexandru Oancea's *A Survey of Floer Homology for Manifolds with Contact Type Boundary.*

As symplectic geometry arises as the study of generalized classical mechanics, it is only natural to attempt to relate the dynamics on symplectic manifolds to their topology. One of the great conjectures of early symplectic geometry is Arnold's conjecture:

Conjecture 2.1. *(Arnold) Let (M, ω) be a closed symplectic manifold. If $H \in C^\infty(M \times \mathbb{R})$, and $P(H)$ denotes the set of fixed points of the time 1 hamiltonian flow,*

$$|P(H)| \geq \mathrm{rk}(H^*(M, \mathbb{Q}))$$

Much of the work in symplectic geometry in the latter half of the 20th century was devoted to the solution of this problem and we will sketch the method by which it has been proved for many cases as this will give us new tools to study many classes of symplectic manifolds. As our final setting will be dealing with exact symplectic manifolds, we will present the set up for the exact setting. For the remainder of this section, let $(M, \omega = d\theta)$ be an exact symplectic manifold. The main motivation of Floer theory is to set up a homology theory whose chain complex is generated by the 1 periodic orbits of a given Hamiltonian, and to (at least in the compact case) show that corresponding homology groups are isomorphic to the homology of M by way of Morse homology. Since the number of generators of a freely generated cochain complex is bounded below by its total rank, the existence of such an isomorphism proves the Arnold conjecture. This conjecture concerns closed symplectic manifolds, but our object of interest for this series is in Liouville domains, which are expressly non-compact.

2.2 Floer Basics

In Darboux coordinates,

$$A_H(x) = \int_0^1 \Big(p_i(t)\dot{q}^i(t) - H(q(t), p(t), t)\Big)\,dt\,.$$

In the proper setup, we can find the extrema of the action functional by solving the corresponding Euler-Lagrange equations. Taking the Lagrangian to be $L = p_i\dot{q}^i - H$, we have $\frac{\partial L}{\partial q^i} = \frac{\partial H}{\partial q^i}$, $\frac{\partial L}{\partial \dot{q}^i} = p_i$, $\frac{\partial L}{\partial p_i} = \dot{q}^i - \frac{\partial H}{\partial p_i}$, and $\frac{\partial L}{\partial \dot{p}_i} = 0$. Then $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \dot{p}_i$ and $\frac{d}{dt}\left(\frac{\partial}{\partial L}(\dot{p}_i)\right) = 0$. More rigorously, one has

$$dA_{H,\gamma}(u) = \int_0^1 \langle dH_{\gamma(t)}, u\rangle dt + \int_0^1 \omega(\dot{\gamma}(t), u(t))\,dt = \int_0^1 \omega(X_{H_t} - \dot{\gamma}(t), u)\,dt\,.$$

Regarding a path in $\mathcal{L}M$ with end points x_- and x_+ as a map $\nu : \mathbb{R}\times S^1 \to M$ with $\lim_{t\to\pm\infty}\nu = x_\pm$. Since $dA_{H,\gamma}(u) = \int_0^1 \omega(X_H - \dot{\gamma}, u)\,dt$ if J is a compatible almost complex structure ($\omega(J\cdot,\cdot)$ is a Riemannian metric), we can write this as $dA_{H,\gamma}(u) = \int_0^1 g(J(X_H - \dot{\gamma}), u)\,dt$ and hence $\nabla A_{H,\gamma}(t) = J(X_{H_t} - \dot{\gamma}(t))$, so a solution to the gradient flow must satisfy $\partial_s u = J(X_{H_t} - \partial_t u)$ or

$$\partial_s u + J\partial_t u - J(X_{H_t}) = 0 \tag{2}$$

This equation is known as Floer's equation. The solution theory for such an equation is not well behaved everywhere on $\mathcal{L}M$, but has nice properties when connecting critical points of A_H. Note: for $H_t = 0$, we recover the Cauchy-Riemann equation. For this reason the Floer equation is often also referred to as the perturbed Cauchy-Riemann equation. We define $\tilde{\mathcal{M}}_{x_-,x_+}((H,J))$ to be the space of solutions to the Floer equation connecting x_- and x_+, we will refer to this as the space of Floer solutions. $\mathbb{R}$ acts freely on $\tilde{\mathcal{M}}_{x_-,x_+}((H,J))$ by translations on the domain when $x_- \neq x_+$, since these translated solutions are the same in some sense, we work with the quotient $\mathcal{M}_{x_-,x_+}(H,J) := \tilde{\mathcal{M}}_{x_-,x_+}((H,J))/\mathbb{R}$ and refer to these unparametrized flows as Floer trajectories.

Definition 2.2. Let (M,ω) be a symplectic manifold with J a compatible almost complex structure, and $H : M\times S^1 \to \mathbb{R}$ a time dependent Hamiltonian with discrete critical points. We define the Symplectic cochain group of (M,ω,J,H) to be $SC^*(M,J,H) := \langle \mathrm{crit} A_H\rangle$, the free vector space over $\mathbb{Z}/2\mathbb{Z}$ generated (as a direct sum) by 1 periodic orbits of H_t.

The space of maps $\mathbb{R}\times S^1 \to M$ connecting x_- to x_+ has a natural vector bundle living over it, given by $\xi_u := \Gamma(S^1\times\mathbb{R}, u^*TM)$. The Floer equation defines a section $\sigma : u \mapsto$

$(\partial_s u + J\partial_t u - JX_{H_t})$ and hence the space of Floer solutions is the intersection of im σ with 0_ξ. As such, assuming the relevant transversality results hold, we can calculate the expected dimension of a component by calculating the Fredholm index of the linearization of σ at a given solution by way of a generalized implicit function theorem. For this reason, when calculating the differential we only consider components of $\tilde{\mathcal{M}}_{x_-,x_+}((H,J))$ of dimension 1. Denote the local dimension of $\tilde{\mathcal{M}}_{x_-,x_+}((H,J))$ at u by $\mu(u)$.

Because the space of compatible almost complex structures is contractible, the isomorphism type of TM as a complex vector bundle is an invariant of M as a symplectic manifold. As such, it makes sense to speak of the Chern classes of a symplectic manifold, denoted by $c_1(M)$. If we assume that $c_1(M)$ is zero, this implies that the (complex) determinant line of TM, $\bigwedge_{\mathbb{C}}^n TM$ is trivial. Fix a trivialization $\psi : \bigwedge_{\mathbb{C}}^n TM \to \mathbb{C}$. If $\varphi_t : M \to M$ is the Hamiltonian flow of H, then $\det(T\varphi_t)\big|_x$ for a 1 periodic orbit of the flow defines a path γ in $U(1)$. We define the index of x, $\iota(x)$ to be the intersection number of γ with $1 \in U(1)$. In the case where $c_1(M) = 0$, if u is a Floer trajectory from x_- to x_+ that $\mu(u) = \iota(x_+) - \iota(x_-)$. This means that we can define a $\mathbb{Z}$ grading on $SC^*(M, \omega, J, H)$ in such cases. We can define $\delta : SC^*(M, \omega, H, J) \to SC^{*-1}(M, \omega, H, J)$ by

$$\delta(x_-) = \sum_{\iota(x_+)-\iota(x_-)=1} |\mathcal{M}_{x_-,x_+}(H,J)| x_+.$$

In order to prove that δ makes SC^* a chain complex, it is necessary to prove two things. First of all, one must show that $\mathcal{M}_{x_-,x_+}(H,J)$ is compact so that the count $|\mathcal{M}_{x_-,x_+}(H,J)|$ is finite and that for $\iota(y) = \iota(x_-) + 2$, the closure of moduli space $\mathcal{M}_{x_-,y}(H,J)$ can be written as

$$\mathcal{M}_{x_-,y}(H,J) \bigcup_{\iota(x_+)\iota(x_-)=1} \mathcal{M}_{y,x_+}(H,J) \times \mathcal{M}_{x_+,x_-}(H,J),$$

i.e. as the union of honest trajectories and broken trajectories. The latter condition ensures that $\delta^2 = 0$ since

$$\delta^2 x_- = \sum_{\iota(y)-\iota(x_-)=2, \iota(x_+)-\iota(x_-)=1} |\mathcal{M}_{x_-,x_+} \times \mathcal{M}_{x_+,y}| y$$

, $\mathcal{M}_{x_-,x_+} \times \mathcal{M}_{x_+,y} = \partial \overline{\mathcal{M}_{x_-,y}}$, and the count of the boundary components of a 1 manifold with boundary is necessarily 0.

3 Symplectic Cohomology

3.1 First Defintion

For the rest of the lecture we will restrict ourselves to a class of Hamiltonians which are adapted to the Liouville domain setting. A Hamiltonian $H : \hat{M} \times \mathbb{R} \to \mathbb{R}$ is said to have slope λ if for $r = \exp(t) \gg 0$, where t is the time coordinate on the cylindrical ends of $\hat{M}$, $H = \lambda r$, and more generally we consider Hamiltonians which for large r are given by $H = h(r)$ for some smooth $h : \mathbb{R} \to \mathbb{R}$. One has $dH = h'(r)\,dr$ and hence far away $X_H = h'(r)R$ where R is the Reeb vector field on M. We will refer to Hamiltonians which take the form $H = h(r)$ for large r as being contact type at infinity or simply of contact type. By studying the 1-periodic orbits of X_H for $h' < a$ we can study the Reeb orbits with period less than $1/a$. For a general hamiltonian and almost complex structure, $\mathcal{M}_{x_-,x_+}$ may not be compact and a sequence of solutions to Floer's equation may be allowed to "run away" and limit to two separate solutions connecting x_- to infinity and infinty to x_+. If we restrict to particular Hamiltonians and complex structures of contact type, (these are ones for which $dr \circ J = -\theta$ near infinty), $\mathcal{M}_{x_-,x_+}$ is now compact by the application of a weak maximum principle.

Proposition 3.1. *Let* $u : \mathbb{R} \times S^1 \to \hat{M}$ *be a solution to Floer's equation contained in a region where* $H = h(r)$ *with* $h' > 0$*. The function* $\rho = r \circ u$ *obeys a weak maximum principle, i.e. if* $D \subset \mathbb{R} \times S^1$ *is precompact, the maximum of* $\rho\big|_{\bar{D}}$ *occurs on the boundary of* D*.*

Proof. One has

$$\begin{aligned}
\Delta\rho\, ds \wedge dt &= -dd^c\rho \\
&= -\, d(d(u^*r) \circ J) \\
&= -\, d\big(-u^*\theta + \rho h'(\rho)\, dt\big) \\
&= u^*\omega - h'(\rho)d\rho \wedge dt - \rho dh'(\rho) \wedge dt \\
&= \omega(\partial_s u, \partial_t u) ds \wedge dt - h'(\rho)\partial_s\rho\, ds \wedge dt - \rho h''(\rho)\partial_s\rho\, ds \wedge dt \\
&= (\omega(\partial_s u, \partial_t u - X_H) - \rho h''(\rho)\partial_s\rho))\, ds \wedge dt \\
&= (|\partial_s u|^2 - \rho h''(\rho)\partial_s\rho) ds \wedge dt
\end{aligned}$$

Because $|\partial_s u|^2 \geq 0$ we have $\Delta\rho + \rho h''(\rho)\partial_s\rho \geq 0$ and hence ρ satisfies the weak maximum principle. □

This maximum principle tells us that Floer trajectories connecting two one periodic orbits of a given H as above, never leave a compact subset of $\hat{M}$ and hence there are no issues arising from compactness in this way. We are now able to give the first definition of symplectic cohomology

Definition 3.2. Let M, θ be a Liouville domain. The symplectic cohomology of M with respect to a contact type Hamiltonian H, such that $H = h(r)$ at infinity and $h'(r) \to \infty$ as $r \to \infty$ is the Floer cohomology $SH^*(M, H, J)$ with respect to J a complex structure of contact type.

This definition depends on the choice of Hamiltonian and almost complex structure, so in order to yield a well defined invariant one must show that all such choices lead to isomorphic groups. This is done through the use of continuation maps. For two given choices $(H_-, J_-), (H_+, J_+)$, by interpolating H_s, J_s which are as above for each s, we can get a chain map $\psi : SH^*(M, H_-, J_-) \to SH^*(M, H_+, J_+)$ by counting solutions to a modified Floer equation

$$\partial_s u + J_s(\partial_t u - X_{H_s}) = 0.$$

where $\lim_{s \to \pm\infty} u = x_\pm \in SC^*(M, H_\pm, H_\pm)$. Since the time symmetry is broken by the interpolation we do not quotient the relevant moduli space, and look at solutions which connect trajectories which have equal index. Directly this map has

$$\psi(x_-) = \sum_{x_+ \in SC^*(H_+, J_+), \iota(x_+) = \iota(x_-)} |\mathcal{M}_{\text{interp}}(x_-, x_+, J_s, H_s)| x_+.$$

Furthermore, through substantial analytical work one can show that ψ does not depend on the choice of H_s and J_s and that such a map is an isomorphism, meaning that $SH^*(M) := SH^*(M, H, J)$ is well defined. This definition can be somewhat unwieldy since it produces the whole of symplectic cohomology at once, and its chain complex is necessarily infinitely generated if ∂M has closed Reeb orbits.

3.2 Second Definition

One way to deal with the lack of finite dimensional data in our original definition is to build the symplectic cohomology as a direct limit of finitely generated cohomology groups.

Let $\lambda > 0$ such that λ is not a multiple of a period of any Reeb orbit on ∂M. If we take a Hamiltonian of contact type, where for sufficiently large r we have $H^\lambda = \lambda r + c$. This means that $X_H = \lambda R$ for r large and the one periodic orbits of this correspond to λ periodic Reeb orbits. Considering Hamiltonians of these forms guarantee that $SC^*(M, H, J)$ is generated by 1 periodic orbits of H lying in a compact neighborhood of M and the Reeb orbits with period less than λ (a finite set under the proper transversality conditions). Define

$$SH^*(M)^{<\lambda} := SH^*(M, H^\lambda, J)$$

If $\lambda_+ < \lambda_-$ are two such values, λ_s is a family interpolating from λ_- to λ_+ and H_s a corresponding family of interpolating Hamiltonians. By studying the interpolation map given by this family H_s, we can first show that $SC^*(M)^{<\lambda}$ does not depend on our choice of H^λ by yielding a quasisomorphism between any two such chain complexes. One also yields a map

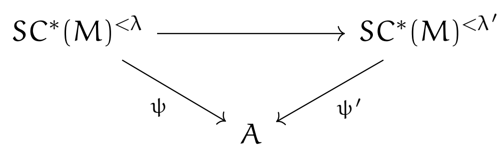
$$SC^*(M)^{<\lambda_+} \to SC^*(M)^{<\lambda_-}$$

Remark. In order to set up the moduli space for the continuation maps in this case, we may only look at $\lambda_+ < \lambda_-$ as the maximum principle we relied on only works for families of Hamiltonians with $\partial_s H_s \leq 0$. In the compact case, the continuation maps work for generic Hamiltonians.

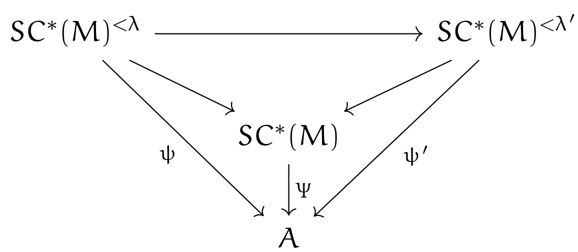
If we consider the ordered category $\mathcal{C}$ with objects given by $\lambda \in \mathbb{R}^+$ not a multiple of a Reeb period inheriting morphisms as a subcategory of the $(\mathbb{R}^+, <)$, the continuation maps yield a functor $F : \mathcal{C} \to \text{GrAb}$, the symplectic cohomology is the limit of this functor

$$SC^*(M) := \varinjlim F$$

(or more appropriately, the symplectic cohomology is the target of the morphisms given by the limit of the functor). That is, $SC^*(M)$ is the the graded abelian group (equipped with maps $SC^*(M)^{<\lambda} \to SC^*(M)$) for which given maps $\psi_1 : SC^*(M)^{<\lambda} \to A, \psi_2 : SC^*(M)^{<\lambda'} \to A$, with $\lambda < \lambda'$ such that the following diagram commutes



there is a unique map $\Psi : SC^*(M) \to A$ making the following diagram commute



By finding a system of Hamiltonians which interpolate between H_- with unbounded growth at infinity and H_+ with slope λ, the interpolation maps yield chain maps

$$CF^*(H^\lambda) \to CF^*(H).$$

By choosing the interpolating Hamiltonians, one can make this map the inclusion of a sub complex meaning that taking the limit and cohomology yields an isomorphism $SC^*(M) \to SC^*(H)$. Hence, the definitions coincide.

3.3 Calculation for the closed ball

Proposition 3.3. *Consider the Liouville domain* $M = \bar{B}^{2n}_r \subset \mathbb{C}^n$ *equipped with the standard symplectic form with Liouville form given by* $\frac{1}{2}\sum_j (y^j dx^j - x^j dy^j)$. *One has* $SH^*(M) = 0$.

Proof. We proceed with the calculation using the limit definition of cohomology. We begin by choosing the family of Hamiltonians $H_k(z) = \frac{\tau_k}{2}\|z\|^2$ where $2\pi k < \tau_k < 2\pi(k+1)$, these are admissible since $\hat{M} = \mathbb{C}^n$ with coordinate given by $\frac{1}{2}\|z\|^2$ we can deal with the flow on this ambient space. This choice of τ_k guarantees that the only one periodic orbits of

X_{H_k} are the constant orbits at 0. We can thus conclude that $SC^*(M)^{<\tau_k} = \langle 0 \rangle$ with the appropriate grading.

One has $X_{H_k} = -i\nabla H_k = -iz$ and hence the Hamiltonian flow is $z \mapsto \exp(-i\tau_k t)z$. We see that the linearization of this flow at $z = 0$ is simply $z \mapsto \exp(-i\tau_k t)$ and hence by the choice of τ_k, $\iota(0_k) = nk$. This means that each of the continuation maps $SC^*(M)^{<\tau_k} \to SC^*(M)^{<\tau_{k'}}$ are 0 and hence the resulting limit defining the symplectic cohomology yields $SH^*(M) = 0$. □

4 Growth Rates and Affine Varieties

4.1 Growth Rates

Recall that if ∂M contains a single periodic Reeb orbit, that $SC^*(M)$ is necessarily infinitely generated. This hints at a greater inconvenience in the fact that the symplectic cohomology itself $SH^*(M)$ is often infinitely generated and difficult to compute. The groups $SH^*(M)^{<\tau}$ are always finite dimensional. These groups are not Liouville invariants as compressing the neck of the completion in a linear fashion will cause more orbits to be included in these groups, but we can yield interesting information by measuring the rate at which these groups grow.

Definition 4.1. Take $r(M,\tau) = \operatorname{rank}(\operatorname{Im}\psi_\tau)$ where $\psi_\tau : SH^*(M)^{<\tau} \to SH^*(M)$ is the universal morphism. Define the growth rate of a Liouville domain to be $\Gamma(M) := \lim_{\tau\to\infty} \frac{r(M,\tau)}{\log(\tau)}$ allowing for infinite values. This measures the polynomial growth rate of the cohomology and yields a somewhat more tractable invariant.

Remark. We can use the growth rate to detect whether or not the symplectic cohomology vanishes since in such a case $r(M,\tau) := 0$ meaning that $\Gamma(M) = 0$.

4.2 Affine Varieties

We now pivot towards an application of symplectic cohomology which relates to algebraic geometry, logarithmic geometry, and homological mirror symmetry.

Consider Y a smooth projective variety. This means that $Y \subset \mathbb{CP}^n$ for some n is cut out as a submanifold by some number of homogeneous polynomials. Let $L \to Y$ be a holomorphic line bundle over Y, D a divisor of Y, and s a holomorphic section for which $s^{-1}(0) = D$. One can think of a divisor as the vanishing set of a section of a holomorphic line bundle over Y. Given a hermitian metric on L, $\|\cdot\|$, one has $dd^c h = 4iF$ away from D where F is the Chern connection associated compatible with $\|\cdot\|$. In this way if $4iF$ is symplectic (i.e. $Y\backslash U$ is Kähler) we can yield a Liouville domain (under the right assumptions on D) by looking at a sublevel set of h.

Let X now be an smooth affine algebraic variety, that is, a submanifold of $\mathbb{C}^n$ cut out by some number of polynomial equations. Resolution of singularities tells us that X

admits a nice compactification by a smooth projective variety, i.e. there is a holomorphic map $\varphi : X \to Y$ for some smooth projective variety where tautological line L over Y is ample (i.e. its curvature is represented by a Kähler form) and $\operatorname{Im}\varphi = Y \backslash D$ for some normal crossings divisor $D = s^{-1}(0)$ for a section s of L. A normal crossings divisor is one which at critical points looks like the intersection of some number of codimension 1 hyper planes. The discussion from the preceding paragraph means that we can associate to X a Liouville domain given by the sublevel set of $h := -\log \|s\|$. A few facts about resolutions of singularity insure that the isomorphism type of this Liouville domain does not depend on the choice of resolution.

This means that to any affine algebraic variety we can associate new invariants, the symplectic cohomology: $SH^*(X) := SH^*(h^{-1}((-\infty, C])$ and the corresponding growth rate.

One can bound the growth rate of the symplectic cohomology associated to an affine variety by finding a bound on the growth rates of the generators of the corresponding cochain complex. This can be done by adapting the Liouville structure of the associated Liouville domain to explicitly describe the Reeb dynamics as we describe more explicitly in the following section.

Proposition 4.2. *Let X be a smooth affine algebraic surface. $\Gamma(X) \leq 2$.*

In order to prove this, we must prove lemmata relating to the following procedure

a) Construct local Hamiltonian T^1 actions on X near D.

b) Deform the primitive of ω near the degenerate points of D so that it is preserved under the local T^2 action and that the Liouville flow commutes with the T^2 action

c) Make the boundary of our Liouville domain symmetric under the T^1 actions.

d) Make the Reeb flow on the boundary symmetric

We begin by proving that there is a metric $\|\cdot\|'$ on $\mathcal{L}$ such that near any crossing the Kähler form is standard. Let $\|\cdot\|$ be a norm on ω for which the Kähler form is $\omega = -dd^c \log \|s\|$. Lemma 1.7 from Seidel's paper "A long exact sequence for symplectic Floer cohomology" tells us that there is a Kähler form ω' on Y for which in a neighborhood of each crossing point of D, there is a contractable open neighborhood of each crossing for which ω' is the standard form on $\mathbb{C}^2$ with respect to the chart in which $D = \{x^1x^2 = 0\}$ near a crossing point and agrees with ω near the boundary of this open neighborhood. This means that $\omega - \omega'$ is supported on such a ball and hence we may write $\omega - \omega' = dd^c\psi$ for some smooth function ψ. Taking $\|\cdot\|' = e^{\psi}\|\cdot\|$ we then have $-dd^c \log \|s\|' = -dd^c \log \|s\| - dd^c\psi = \omega - dd^c\psi = \omega'$. By deforming $\|\cdot\|$ by $e^{t\psi}$ we see that the symplectic form defined here is isomorphic to the standard one on Y.

Lemma 4.3. *Let $K \subset D$ be a smooth component. There is a neighborhood U of K in Y which carries a T^1 action fixing K.*

Proof. Since K is a smooth component, it is a symplectic submanifold of (Y, ω). Let $\pi : P \to K$ be the unitary frame bundle of the normal bundle to K, N_K (equipped with the restriction of the Kähler metric). Denote the action of $U(1)$ on $P, \mathbb{C}$ and $P \times \mathbb{C}$ by φ, ψ, and Φ respectively. If $\mu : \mathbb{C} \to i\mathbb{R}$ is the moment map of the standard $U(1)$ action on $\mathbb{C}$ and α is a connection one form on P, $\tilde{\beta} = \langle \pi_1^*\alpha, \pi_2^*\mu \rangle$ is a $U(1)$ invariant form on $P \times \mathbb{C}$ as seen below

$$\Phi_g^*\langle \pi_1^*\alpha, \pi_2^*\mu \rangle = \langle \Phi_g^*\pi_1^*\alpha, \Phi_g^*\pi_2^*\mu \rangle = \langle \pi_1^*\varphi_g^*\alpha, \pi_2^*\psi_{g^{-1}}\mu \rangle = \beta$$

$\tilde{\beta}$ then descends to a form β on N as does $\omega_{\mathbb{C}}$ the standard symplectic form on $\mathbb{C}$. Because $d\beta$ is zero at $(p, 0)$ in $P \times \mathbb{C}$ we conclude that there is a neighborhood U_K of the zero section in N where $\beta + \omega_{\mathbb{C}} + \pi^*\omega$ defines a symplectic form which coincides with ω when restricted to the zero section. μ descends to a moment map for an action of $U(1)$ on N_K under the constructed symplectic form, which preserves the 0 section. By the symplectic submanifold theorem, there is a symplectomorphism $\tilde{U}_K \to V_K$, where V_K is an open neighborhood of K and K is fixed. Pushing the T^1 action on N_K forward by this symplectomorphism then yields the desired action.

Since we have taken ω to be the standard form on a neighborhood of $b \in K_1 \cap K_2$, on the normal bundle of K_1 in a neighborhood of b we may take α' the trivial connection with respect to the given trivialization. Then $\alpha - \alpha'$ is a T^1 equivariant $i\mathbb{R}$ valued 1-form on P and $\alpha + t(\alpha' - \alpha)$ is a connection form for $t \in [0, 1]$. Taking a cutoff function φ which is identically 0 away from b and identically 1 near b and applying the preceding construction on the connection form $\alpha'' = \alpha + \varphi(\alpha' - \alpha)$ yields a T^1 action which interpolates between the starting action away from b and the standard linear action near b. This means that our T^1 actions glue together near crossing points into T^2 actions given by the standard linear ones in the standard coordinates. □

Lemma 4.4. *There is a smooth function λ on X such that the form $\theta' = \theta + d\lambda$ is T^2 invariant near any order crossing.*

Proof. Near a crossing of order j, we may assume $s(x^1, x^2) = (x^1)^{w_1}(x^2)^{w_2}$ and the metric takes the form $\|s\|^2 = \exp(2\psi)|s|^2$ with respect to our trivialization. Since $\omega = dd^c(-\log\|s\|)$ we have $\omega = dd^c(\frac{1}{2}|x|^2) = dd^c(-\psi - d^c \log |x^1|^{w_1} |x^2|^{w_2}) = dd^c(-\psi)$ as a quick calculation shows that $d^c \log |s|$ is closed. We then have $d^c\psi + d^c\frac{1}{2}|x|^2$ closed and hence exact by the dd^c-lemma. Let λ' be a primitive for $d^c\psi + d^c\frac{1}{2}|x|^2$ and take φ a radially symmetric bump function. We define $\lambda := \varphi\lambda'$. Because

$$\theta = -d^c h = -d^c\psi - \sum w_i d^c \log \left|x^i\right|,$$

near a crossing point we have $\theta' = -\sum w_i d^c \log |x^i| + d^c\frac{1}{2}|x|^2$ and $\theta + d\lambda$ is symmetric under the T^2 action. □

Near a crossing, the Liouville vector field is given by $Z' = -\sum w_i\left(\frac{2i}{x^i}\frac{\partial}{\partial x^i} - \frac{2i}{\bar{x}^i}\frac{\partial}{\partial \bar{x}^i}\right) + Z$ where Z is the standard Liouville field. It is clear that this has $dh(Z) < 0$ since $(Z - Z')h =$

$2i\sum_j w_j((x^j)^2-(\bar{x}^j)^2)-(Z-Z')\psi$ which is arbitrarily small as $h\to 0$. Then the sublevel set $\{h(x)\le\epsilon\}$ is a Liouville domain under θ'. This means that the family of forms given by $\theta+td\lambda$ are Liouville structures and hence for $t=0$ and $t=1$ define isomorphic Liouville structures on $M=\{h(x)\le\epsilon\}$ by the following theorem (Gray's Stability).

Theorem 4.5. *Let M be a smooth manifold with boundary such that θ_t defines a continuous family of Liouville structures on M for $0\le t\le 1$. There exists a Liouville isomorphism between (M,θ_0) and (M,θ_1).*

Lemma 4.6. *There exists a subdomain M' of M which is Liouville isomorphic to M and invariant under the local T^1 actions.*

Proof. For K a smooth component of D let m be a moment map for the local T^1 action such that $m\big|_K=0$. For some $\epsilon>0$ we have $dm(Z')(x)<0$ for $m(x)<\epsilon$. Let ρ be the local circle action near K. The linearization of ρ along K is given by the standard circle action on N_K by construction. We can then conclude that when K is given locally by $\{x^1=0\}$ we have $m(x)=\frac{1}{2}\left|x^1\right|^2+h(x)$ where $h(x)$ is of order 3 or higher. Consequently $\nabla m(x)=u^1\frac{\partial}{\partial u^1}+v^1\frac{\partial}{\partial v^1}+\nabla h$ where $x^1=u^1+iv^1$. Away from a crossing, we have

$$dm(Z')=dm(Z)=\nabla m(h)=\left(u^1\frac{\partial}{\partial u^1}+v^1\frac{\partial}{\partial v^1}+\nabla h\right)(-\log\left|x^1\right|+\psi).$$

In the limit $x^1\to 0$ we have $\nabla h(-\log\left|x^1\right|+\psi)\to 0$ and

$$(u^1\frac{\partial}{\partial u^1}+v^1\frac{\partial}{\partial v^1})(-\log\left|x^1\right|)=-1$$

for any x^1 meaning that there is some $\epsilon>0$ such that $(Z\cdot m)(x)<0$ for $m(x)<\epsilon$ and x far from a crossing.

We now consider the case near a crossing. Taking m_1 and m_2 to be the moment maps for the two T^1 actions near a crossing, given in local coordinates by $m_j(x)=\frac{1}{2}\left|x^j\right|^2$, let $\kappa(s,t)$ be a smooth function such that $\kappa(s,0)=\kappa(0,t)=0$ and $\frac{\partial}{\partial s}\kappa>0$ if $t>0$ and $\frac{\partial}{\partial t}\kappa>0$ if $t>0$. e.g. if

$$f_a(x)=\begin{cases}0 & x\le 0\\ \exp(-1/ax) & x>0\end{cases}$$

and

$$g_{a,b}(x)=b\int_0^x\frac{f_a(x)}{f_a(x)+f_a(1-x)}\,dx,\quad h_{a,b}(x)=-g_{a,b}(1-x)+g_{a,b}(1)$$

then $h_{a,b}(s)h_{a,b}(t)$ satisfies the preceding relations for all $a,b>0$. For $\left|x^1\right|^2$ sufficiently large, there are values of a and b such that $k(m_1,m_2)=m_2$ and vice verse for m_2. This means that if we take ∂M to be the complement of the regions where m_i and k are less than ϵ we yield a new Liouville domain which is invariant under the T^1 actions. (Corollary 2.3 Oancea) Two open subsets of $\hat{M}$ with outward facing Liouville vector field have isomorphic symplectic cohomology. □

Now in order to finally control the Reeb dynamics on the resulting Liouville domain, we must make the Reeb flow symmetric under the T^1 actions.

Lemma 4.7. *There is a Liouville structure $\hat{\theta}$ on M' which is invariant under the local T^1 actions which is isomorphic to (M', θ').*

Proof. By construction, θ' is T^1 invariant near each of the singular points of D. Away from these singular points, one has $\rho_t^* Z$ and Z point outward and hence the integral

$$\tilde{\theta} = \frac{1}{2\pi}\int_0^{2\pi} \rho_t^*\theta \, dt$$

yields a well defined Liouville form on union of the domains of definition of the T^1 actions. Let $\varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}$ be a smooth function which is identically 2π on $[0, \epsilon]$ for some $\epsilon > 0$ and which is identically 0 on $[\delta, \infty)$ where $m^{-1}(\delta) \neq \varnothing$ for all m the normalized moment map of the T^1 action near a component of D. The form

$$\hat{\theta} = \frac{1}{\varphi \circ m}\int_0^{\varphi\circ m} \rho_t^*\theta' \, dt$$

glues with θ' to yield a Liouville form with the desired property. By considering a family of such forms parametrized by the maximum value of φ, we see that Gray's theorem insures that (M', θ') is Liouville isomorphic to $(M', \hat{\theta})$. □

We now investigate the Reeb flow of $\partial M'$ using our newly constructed invariant contact structure. Let ξ denote the fundamental vector field of the T^1 action on the neighborhood of D. We denote $\hat{\theta}$ by θ. Because θ is invariant under the action, we conclude that $\mathcal{L}_\xi\theta = 0$. By Cartan's magic formula $dm = -d\iota_\xi\theta$. We can then conclude that $\iota_\xi\theta$ and m differ by a locally constant function and $\iota_\xi\theta'$ is locally constant on the level sets of M. Then when we restrict ξ to $\partial M'$, away from singular points of D, we have $\iota_\xi\theta'\big|_{\partial M} = (\iota_\xi\theta')\big|_{\partial M}$ constant. Since $\iota_\xi\omega = dm$, $\iota_\xi(\omega\big|_{\partial M}) = dm\big|_{\partial M} = 0$. We then conclude that up to scaling, ξ coincides with the Reeb vector field away from the singular points of D.

Near a singular point of D with fundamental vector fields given by ξ_1 and ξ_2, we have $d(\kappa(m_1, m_2))\,\big|_{\partial M} = (\partial_s\kappa dm_1 + \partial_t\kappa dm_2)\big|_{\partial M} = 0$, so then $\Xi = \partial_s\kappa\xi_1 + \partial_t\kappa\xi_2$ has

$$\iota_\Xi d\theta\Big|_{\partial M'} = 0.$$

A short calculation shows that $\iota_{\xi_i}\theta = m_i$. We have

$$\iota_\Xi\theta = (\partial_s\kappa)m_1 + (\partial_t\kappa)m_2.$$

Because $m_i, \partial_s\kappa, \partial_t\kappa$ take nonnegative values, this quantity vanishes iff $m_1 = m_2 = 0$ if we take ϵ small enough. Define $\alpha = \partial_s\kappa m_1 + \partial_t\kappa m_2$ and we have $\frac{\Xi}{\alpha}$ the Reeb vector field.

Near the singular points of D we conclude that the Reeb vector field restricts to a flow on the T^2 orbits. On a given torus, the Reeb flow is given by

$$\left(\frac{\partial_s \kappa}{(\partial_s \kappa)m_1 + (\partial_t \kappa)m_2}, \frac{\partial_t \kappa}{(\partial_s \kappa)m_1 + (\partial_t \kappa)m_2}\right)$$

For large m_2, we have $\partial_t \kappa \to 0$ yielding $(1/m_1, 0)$ and for large m_1 we yield $(0, 1/m_2)$.

We can bound the dimension of $SH^*(X)^{<\tau}$ by bounding the dimension of $SC^*(X)^{<\tau}$. As τ increases, new contributions to $\dim SC^*(X)^{<\tau}$ are from the periodic Reeb orbits on M with period less than τ. The boundary components near smooth parts of D have Reeb orbits given by a T^1 action, i.e. every point lies on a periodic orbit. Let Q be the domain of ∂M on which only one T^1 acts at a time. In this case, there is a pertubation to θ so that the periodic orbits of the Reeb flow on Q are given by critical points of some Morse function on Q. The contribution from each critical point is linear in τ, since if $\gamma : S^1 \to \partial M$ is a periodic orbit with period τ', the path given by traversing γ n times yields a periodic orbit with period $n\tau'$. This gives the following lemma:

Lemma 4.8. *Let X be a smooth affine algebraic surface. If there exists a resolution of singularities $X \to Y$ with $X \cong Y \backslash D$ such that D is smooth, $\Gamma(X) \leq 1$.*

We now turn out attention to the regions near singular points of D. Since the Reeb flow restricts to the orbits of the T^2 action, we must count the periodic tori of this flow. Assume that an orbit with velocity (ξ^1, ξ^2). Assuming that neither ξ^1 or ξ^2 is zero and that $\xi^1 > \xi^2$, then $\frac{\xi^1}{\xi^2} = \frac{p}{q}$ for some integers p and q in lowest terms. The first periodic orbit of this torus will occur with $\tau = \frac{q}{\xi^2}$ since this has $\tau\xi^2 = q$ and $\tau\xi^1 = p$. Then all periodic tori of lowest period less than τ are given by $q < \tau\xi_1$. This means that the number of distinct periodic tori grows linearly and hence the total number of periodic tori grows quadratically as desired.