

Appendix B

Answers to Selected Problems

Answers to most of the problems can be found in Appendix C, which is available on the web via <http://www.ma.utexas.edu/users/cup/Answers>.

2.1.6 Define $T_{n+1} = \inf\{t > T_n : X_{t+} - X_t \neq 0\} \wedge t$, where t is an instant past which X vanishes. The T_n are stopping times (why?), and X is a linear combination of $X_0 \cdot [0]$ and the intervals $((T_n, T_{n+1}])$.

2.3.6 For $N = 1$ this amounts to $\zeta_1 \leq L_q \zeta_1$, which is evidently true. Assuming that the inequality holds for $N - 1$, estimate

$$\begin{aligned} \zeta_N^2 &= \zeta_{N-1}^2 + (\zeta_N - \zeta_{N-1})(\zeta_N + \zeta_{N-1}) \\ &\leq L_q^2 \sum_{n=1}^N (\zeta_n - \zeta_{n-1})^2 + (\zeta_N - \zeta_{N-1})(\zeta_N + \zeta_{N-1}) \\ &\quad - L_q^2 (\zeta_N - \zeta_{N-1})(\zeta_N - \zeta_{N-1}) \\ &= L_q^2 \sum_{n=1}^N (\zeta_n - \zeta_{n-1})^2 + (\zeta_N - \zeta_{N-1})((1 - L_q^2)\zeta_N + (L_q^2 + 1)\zeta_{N-1}). \end{aligned}$$

Now with $L_q^2 = (q+1)/(q-1)$ we have $1 - L_q^2 = -2/(q-1)$ and $L_q^2 + 1 = 2q/(q-1)$, and therefore

$$(1 - L_q^2)\zeta_N + (L_q^2 + 1)\zeta_{N-1} = \frac{2}{q-1}(-\zeta_N + q\zeta_{N-1}) \leq 0.$$

2.5.17 Replacing F_n by $F_n - p$, we may assume that $p = 0$. Then $S_n \stackrel{\text{def}}{=} \sum_{\nu=1}^n F_\nu$ has expectation zero. Let \mathcal{F}_n denote the σ -algebra generated by F_1, F_2, \dots, F_{n-1} . Clearly S_n is a right-continuous square integrable martingale on the filtration $\{\mathcal{F}_n\}$. More precisely, the fact that $\mathbb{E}[F_\nu | \mathcal{F}_\mu] = 0$ for $\mu < \nu$ implies that

$$\mathbb{E}[S_n^2] = \mathbb{E}\left[\left(\sum_{\nu=1}^n F_\nu\right)^2\right] = \mathbb{E}\left[\sum_{\nu=1}^n F_\nu^2\right] + 2\sum_{1 \leq \mu < \nu \leq n} \mathbb{E}[F_\mu \mathbb{E}[F_\nu | \mathcal{F}_\mu]] \leq n\sigma^2.$$

Therefore $Z_n \stackrel{\text{def}}{=} S_n/n$ has $\|Z_n\|_{L^2} \leq \sigma\sqrt{1/n} \xrightarrow{n \rightarrow \infty} 0$. (Using Chebyscheff's inequality here we may now deduce the weak law of large numbers.) The strong law says that $Z_n \xrightarrow{n \rightarrow \infty} 0$ almost surely and evidently follows if we can show that Z_n converges almost surely to something as $n \rightarrow \infty$. To see that it does we write

$$Z_\nu - Z_{\nu-1} = \frac{F_\nu}{\nu} - \frac{1}{\nu(\nu-1)} \sum_{1 \leq i < \nu} F_i, \quad \nu = 2, 3, \dots,$$

and set
$$\tilde{Z}_n \stackrel{\text{def}}{=} \sum_{1 \leq \nu \leq n} \frac{F_\nu}{\nu}, \quad n = 1, 2, 3, \dots,$$

and $\hat{Z}_1 = 0$,
$$\hat{Z}_n \stackrel{\text{def}}{=} \sum_{1 < \nu \leq n} \frac{S_{\nu-1}}{\nu(\nu-1)}, \quad n = 2, 3, \dots$$

Then
$$Z_n = \tilde{Z}_n - \hat{Z}_n,$$

and it suffices to show that both \tilde{Z} and \hat{Z} converge almost surely. Now \tilde{Z} is an L^2 -bounded martingale and so has a limit almost surely. Namely, because of the cancellation as above,

$$\mathbb{E}[\tilde{Z}_n^2] \leq \sum_{1 \leq \nu \leq n} \frac{\mathbb{E}[F_\nu^2]}{\nu^2} < \sum_{\nu < \infty} \frac{\sigma^2}{\nu^2} < \infty.$$

As to \hat{Z} , since
$$\|\hat{Z}\|_\infty \stackrel{\text{def}}{=} \sum_{2 \leq \nu < \infty} \frac{|S_{\nu-1}|}{\nu(\nu-1)}$$

has
$$\mathbb{E}[\|\hat{Z}\|_\infty] \leq \sum_{2 \leq \nu < \infty} \frac{\|S_{\nu-1}\|_{L^2}}{\nu(\nu-1)} \leq \sum_{2 \leq \nu < \infty} \frac{\sigma\sqrt{\nu-1}}{\nu(\nu-1)} < \infty,$$

the sum defining $\hat{Z}_\infty \stackrel{\text{def}}{=} \lim \hat{Z}_n$ almost surely converges, even *absolutely*.

3.8.10 (i) Both $(Y, Z) \rightarrow [Y, Z](\varpi)$ and $(Y, Z) \rightarrow \langle Y, Z \rangle(\varpi)$ are inner products with associated lengths $S[Z](\varpi), \sigma[Z](\varpi)$. The claims are Minkowski's inequalities and follow from a standard argument.

(ii) Take $U = V = [0, T]$ in theorem 3.8.9 and apply Hölder's inequality.

3.8.18 (i) Choose numbers $\alpha_i > 0$ satisfying inequality (3.7.12):

$$\sum_{i=1}^\infty \langle i\alpha_i \cdot X^i \rangle_{T_0} < \infty$$

(X^i is X stopped at i). With each α_i goes a $\delta_i > 0$ so that $|x - x'| \leq \delta_i$ implies $|F(x) - F(x')| \leq \alpha_i$. Set $T_0^i = 0$ and $T_{k+1}^i \stackrel{\text{def}}{=} \inf\{t > T_k^i : |Z_t - Z_{T_k^i}| \geq \alpha_i\}$. Then set

$${}^iY_t^\eta \stackrel{\text{def}}{=} \sum_{k,\nu} F_\nu^\eta(X_{T_k^i})(X_{T_{k+1}^i \wedge t}^\nu - X_{T_k^i \wedge t}^\nu), \quad \eta = 1, \dots, d.$$

According to theorem 3.7.26, nearly everywhere ${}^iY \xrightarrow{i \rightarrow \infty} (F(X)*X)$, uniformly on bounded time-intervals. Let us toss out the nearly empty set where the limit does not exist and elsewhere select this limit for $F(X)*X$. Now let ω, ω' be such that $X \cdot(\omega)$ and $X \cdot(\omega')$ describe the same arc via $t \mapsto t'$. The times T_k^i may well differ at ω and ω' , in fact $T_k^i(\omega') = (T_k^i(\omega))'$, but the values $X_{T_k^i}(\omega)$ and $X_{T_k^i}(\omega')$ clearly agree. Therefore ${}^iY_t(\omega) = {}^iY_{t'}(\omega')$ for all $n \in \mathbb{N}$ and all $t \geq 0$. In the limit $(F(X)*X)_t(\omega) = (F(X)*X)_{t'}(\omega')$.

(ii) We start with the case $d = 1$. Apply (i) with $f(x) = 2x$ and toss out in addition the nearly empty set of $\omega \in \Omega$ where $[W, W]_t(\omega) \neq t$ for some t . Now if $W \cdot(\omega)$ and $W \cdot(\omega')$ describe the same arc via $t \mapsto t'$, then $[W, W] \cdot(\omega) = W \cdot^2(\omega) - 2(W \cdot * W) \cdot(\omega)$ and $[W, W] \cdot(\omega') = W \cdot^2(\omega') - 2(W \cdot * W) \cdot(\omega')$ describe the same arc also via $t \mapsto t'$. This reads $t = t'$.

In the case $d > 1$ apply the foregoing to the components W^η of \mathbf{W} separately.

3.9.10 (i) Set $M_t \stackrel{\text{def}}{=} \langle \xi | X_{T+t} - X_T \rangle$ and assume without loss of generality that $M_0 \leq a$. This continuous local martingale has a continuous strictly increasing square function $[M, M]_t = \xi_\mu \xi_\nu ([X^\mu, X^\nu]_{T+t} - [X^\mu, X^\nu]_T) \xrightarrow{t \rightarrow \infty} \infty$. The time

transformation $T^\lambda = T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t : [M, M]_t \geq \lambda\}$ of exercise 3.9.8 turns M into a Wiener process. The zero-one law of exercise 1.3.47 on page 41 shows that $T = T^\pm$ almost surely.

(ii) In the previous argument replace T by S . The continuous time transformation $T^\lambda = T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t : [M, M]_t \geq \lambda\}$ turns $M_t \stackrel{\text{def}}{=}} \langle \xi | X_{S+t} - X_S \rangle$ into a standard Wiener process, which exceeds any number, including $a \pm \langle \xi | X_S \rangle$, in finite time (exercise 1.3.48 on page 41). Therefore the stopping times T and T^\pm are almost surely finite. Consider the set \mathcal{X}^- of paths $[S(\omega), \infty) \ni t \mapsto X_t(\omega)$ that have $T(\omega) > \tau$. Each of them stays on the same side of H as $X_S(\omega)$ for $t \in [S(\omega), \tau]$, in fact by continuity it stays a strictly positive distance away from H during this interval. Any other path $X_*(\omega')$ sufficiently close to a path in \mathcal{X}^- will not enter H during $[S(\omega'), \tau]$ either and thus will have $T(\omega') > \tau$: the set \mathcal{X}^- is open.

Next consider the set \mathcal{X}^+ of paths $[S(\omega), \infty) \ni t \mapsto X_t(\omega)$ that have $T^+(\omega) < \tau$. If $X_*(\omega) \in \mathcal{X}^+$, then there exists a $\sigma < \tau$ with $\langle \xi | X_\sigma \rangle > a$: $X_S(\omega)$ and $X_\sigma(\omega)$ lie on different sides of H . Clearly if ω' is such that the path $X_*(\omega')$ is sufficiently close to that of ω , then $X_\sigma(\omega')$ will also lie on the other side of H : \mathcal{X}^+ is open as well. After removal of the nearly empty set $[T \neq T^+]$ we are in the situation that the sets $\{X_*(\omega) : T(\omega) \geq \tau\}$ are open for all τ : T depends continuously on the path.

3.10.6 (i) Let $K^1, K^2 \subset \mathbf{H}$ be disjoint compact sets. There are $\phi_n^i \in C_{00}[\mathbf{H}]$ with $\phi_n^i \downarrow_n K^i$ pointwise and such that ϕ_1^1 and ϕ_1^2 have disjoint support. Then $\phi_n^i * \beta \rightarrow K^i * \beta$ as L^2 -integrators and therefore uniformly on bounded time-intervals (theorem 2.3.6) and in law. Also, clearly $K^1 * \beta$ and $K^2 * \beta$ are independent, inasmuch as $\phi_m^1 * \beta$ and $\phi_n^2 * \beta$ are. In the next step exhaust disjoint relatively compact Borel subsets B^1, B^2 of \mathbf{H} by compact subsets with respect to the Daniell means $B \mapsto \|B \times [0, n]\|_{\beta-2}^*$, $n \in \mathbb{N}$, which are \mathcal{K} -capacities (proposition 3.6.5), to see that $B^1 * \beta$ and $B^2 * \beta$ are independent Wiener processes. Clearly $\nu(B) \stackrel{\text{def}}{=} \mathbb{E}[(B * \beta)_1^2]$ defines an additive measure on the Borels of \mathbf{H} . It is σ -continuous, because it is majorized by $B \mapsto [\|B \times [0, 1]\|_{\beta-2}^*]^2$, even inner regular.

3.10.9 Let I denote the image in $L^2 \stackrel{\text{def}}{=} L^2(\mathcal{F}_\infty^0[\beta], \mathbb{P})$ of $\mathfrak{L}^1[\beta-2]$ under the map $\tilde{X} \mapsto \int_0^\infty \tilde{X} d\beta$, and \bar{I} the algebraic sum $I \oplus \mathbb{R}$. By theorem 3.10.6 (ii), I and then \bar{I} are closed in L^2 , and their complexifications $I_{\mathbb{C}}, \bar{I}_{\mathbb{C}}$ are closed in $L_{\mathbb{C}}^2$. By the Dominated Convergence Theorem for the L^2 -mean both vector spaces are closed under pointwise limits of bounded sequences.

For $h \in \mathcal{E}[\tilde{\mathbf{H}}] \stackrel{\text{def}}{=} \mathcal{E}[\mathbf{H}] \otimes C_{00}(\mathbb{R}_+)$ set $M^h \stackrel{\text{def}}{=} h * \beta$, which is a martingale with square bracket $[M^h, M^h]_t = \int_0^t h^2(\eta, s) \nu(d\eta) ds$. Then let $G^h = \exp(iM^h + [M^h, M^h]/2)$ be the Doléans–Dade exponential of iM^h . Clearly $G_\infty^h = 1 + \int_0^\infty iG^h dM^h$ belongs to $\bar{I}_{\mathbb{C}}$, and so does its scalar multiple $\exp(iM_\infty^h)$. The latter form a multiplicative class \mathcal{M} contained in $\bar{I}_{\mathbb{C}}$ and generating $\mathcal{F}_\infty^0[\beta]$ (page 409). By exercise A.3.5 the vector space $\bar{I}_{\mathbb{C}}$ contains all bounded $\mathcal{F}_\infty^0[\beta]$ -measurable functions. As it is mean closed, it contains all of $L_{\mathbb{C}}^2$. Thus $\bar{I} = L^2$.

4.3.10 Use exercise 3.7.19, proposition 3.7.33, and exercise 3.7.9.

4.5.8 Inequality (4.5.9) and the homogeneity argument following it show that for any bounded previsible $X \geq 0$

$$\mu^{(\sigma)}(X) \leq (\mu^{(\rho)}(X))^{\frac{\tau-\sigma}{\tau-\rho}} \cdot (\mu^{(\tau)}(X))^{\frac{\sigma-\rho}{\tau-\rho}} \leq (\mu^{(\rho)} \vee \mu^{(\tau)})(X).$$

4.5.9 Since $z \mapsto (z + \zeta)^p - z^p$ increases, inequality (4.5.16) can with the help of theorem 2.3.6 be continued as

$$\|Z_\infty\|_{L^p}^p \leq (C_p^* \dagger Z \dagger_{\mathcal{I}^p} + \zeta[Z])^p - C_p^{*p} \dagger Z \dagger_{\mathcal{I}^p}^p.$$

Replace Z by $X*Z$ and take the supremum over X with $|X| \leq 1$ to obtain

$$\|Z\|_{\mathcal{I}^p}^p \leq (C_p^* \|Z\|_{\mathcal{I}^p} + \zeta[Z])^p - C_p^{*p} \|Z\|_{\mathcal{I}^p}^p,$$

or
$$(1 + C_p^{*p})^{1/p} \|Z\|_{\mathcal{I}^p} \leq C_p^* \|Z\|_{\mathcal{I}^p} + \zeta[Z]$$

and
$$\|Z\|_{\mathcal{I}^p} \leq c_p^\diamond \zeta[Z]$$

with
$$c_p^\diamond \leq \left((1 + C_p^{*p})^{1/p} - C_p^* \right)^{-1} \leq 0.6 \cdot 4^p.$$

4.5.21 Let $T = \inf\{t : A_t \geq a\}$ and $S = \inf\{t : Y_t \geq y\}$. Both are stopping times, and T is predictable (theorem 3.5.13); there is an increasing sequence T_n of finite stopping times announcing T . On the set $[Y_\infty > y, A_\infty < a]$, S is finite, $Y_S \geq y$, and the T_n increase without bound. Therefore

$$[Y_\infty > y, A_\infty < a] \leq \frac{1}{y} \cdot \sup_n Y_{S \wedge T_n}$$

and so
$$\begin{aligned} \mathbb{P}[Y_\infty > y, A_\infty < a] &\leq \frac{1}{y} \cdot \sup_n \mathbb{E}[Y_{S \wedge T_n}] \\ &\leq \frac{1}{y} \cdot \sup_n \mathbb{E}[A_{S \wedge T_n}] \leq \frac{1}{y} \cdot \mathbb{E}[A_\infty \wedge a]. \end{aligned}$$

Applying this to sequences $y_n \downarrow y$ and $a_n \uparrow a$ yields inequality (4.5.30). This then implies $\mathbb{P}[Y = \infty, A \leq a] = 0$ for all $a < \infty$; then $\mathbb{P}[Y = \infty, A < \infty] = 0$, which is (4.5.31).

4.5.24 Use the characterizations 4.5.12, 4.5.13, and 4.5.14. Consider, for instance, the case of $\mathbf{Z}^{(q)}$. Let ${}^q\mathbf{X}, {}^q\check{H}$ be the quantities of exercise 4.5.14 and its answer for $'\mathbf{Z}$. Then $\check{H}(\mathbf{y}, s) \stackrel{\text{def}}{=} {}^q\check{H} \circ C(\mathbf{y}, s) = {}^q\check{H}(C\mathbf{y}, s) = \langle {}^q\mathbf{X}_s | C\mathbf{y} \rangle = \langle C^T {}^q\mathbf{X}_s | \mathbf{y} \rangle$, where $C^T : \ell^\infty(d) \rightarrow \ell^\infty(d)$ denotes the (again contractive) transpose of C . By exercise 4.5.14, the Doléans–Dade measure $'\mu$ of $|\check{H}|^q * j_{\mathbf{Z}}$ is majorized by that of $\Lambda^{(q)}[\mathbf{Z}]$. But $'\mu$ is the Doléans–Dade measure of $\Lambda^{(q)}['\mathbf{Z}]$! Indeed, the compensator of $|\check{H}|^q * j_{\mathbf{Z}} = |{}^q\check{H} \circ C|^q * j_{\mathbf{Z}} = |{}^q\check{H}|^q * C[j_{\mathbf{Z}}] = |{}^q\check{H}|^q * j_{'\mathbf{Z}}$ is $\Lambda^{(q)}['\mathbf{Z}]$. The other cases are similar but easier.

5.2.2 Let $S < T^\mu$ on $[T^\mu > 0]$. From inequality (4.5.1)

$$\begin{aligned} \|\Delta * \mathbf{Z}|_S^*\|_{L^p} &\leq C_p^\diamond \cdot \max_{\rho=1, p^\diamond} \left\| \left(\int_0^S |\Delta|^\rho d\Lambda \right)^{1/\rho} \right\|_{L^p} \\ &\leq C_p^\diamond \cdot \max_{\rho=1, p^\diamond} \left\| \left(\int_0^\mu \delta^\rho d\lambda \right)^{1/\rho} \right\|_{L^p} = \delta \cdot C_p^\diamond \max_{\rho=1, p^\diamond} \mu^{1/\rho}. \end{aligned}$$

Letting S run through a sequence announcing T^μ , multiplying the resulting inequality $\|\Delta * \mathbf{Z}|_{T^\mu}^*\|_{L^p} \leq \delta \cdot C_p^\diamond \max_{\rho=1, p^\diamond} \mu^{1/\rho}$ by $e^{-M\mu}$, and taking the supremum over $\mu > 0$ produces the claim after a little calculus.

5.2.18 (i) Since $e^{pmW_t - p^2m^2t/2} = \mathcal{E}_t[pmW]$ is a martingale of expectation one

we have
$$|\mathcal{E}_t[mW]|^p = e^{pmW_t - pm^2t/2} = \mathcal{E}_t[pmW] \cdot e^{(p^2-p)m^2t/2},$$

$$\mathbb{E}[|\mathcal{E}_t[mW]|^p] = e^{(p^2-p)m^2t/2}, \quad \text{and} \quad \|\mathcal{E}_t[mW]\|_{L^p} = e^{m^2(p-1)t/2}.$$

Next, from $e^{|x|} \leq e^x + e^{-x}$ we get

$$e^{|mW_t|} \leq e^{mW_t} + e^{-mW_t} = e^{m^2t/2} \times (\mathcal{E}_t[mW] + \mathcal{E}_t[-mW]),$$

$$e^{|mW|_t^*} \leq e^{m^2 t/2} \times (\mathcal{E}_t^*[mW] + \mathcal{E}_t^*[-mW]),$$

$$\text{and } \left\| e^{|mW|_t^*} \right\|_{L^p} \leq e^{m^2 t/2} \times (\|\mathcal{E}_t^*[mW]\|_{L^p} + \|\mathcal{E}_t^*[-mW]\|_{L^p})$$

$$\text{by theorem 2.5.19: } \leq e^{m^2 t/2} \times 2p' \cdot e^{m^2(p-1)t/2} = 2p' \cdot e^{m^2 p t/2}.$$

(ii) We do this with $\|\cdot\|$ denoting the ℓ^1 -norm on \mathbb{R}^d . First,

$$\left\| e^{|mZ^*|_t} \right\|_{L^p} = e^{|m|t} \times \prod_{\eta} \left\| e^{|W^{\eta*}|_t} \right\|_{L^p}$$

$$\begin{aligned} \text{by independence of the } W^{\eta}: & \leq e^{|m|t} \times \left(2p' \cdot e^{m^2 p t/2} \right)^{d-1} \\ & = (2p')^{d-1} \times e^{(|m|+(d-1)m^2 p/2) \cdot t}. \end{aligned}$$

$$\text{Thus } \left\| e^{|mZ^*|_t} \right\|_{L^p} \leq A_{p,d} \times e^{M_{d,m,p} \cdot t}. \quad (1)$$

$$\begin{aligned} \text{Next, } \left\| |Z^*|_t^r \right\|_{L^p} &= \left\| |Z^*|_t \right\|_{L^{rp}}^r \leq \left(t + \sum_{\eta} \left\| |W^{\eta*}|_t \right\|_{L^{rp}} \right)^r \\ &= \left(t + (d-1) \cdot \left\| |W|_t^* \right\|_{L^{rp}} \right)^r \end{aligned}$$

$$\begin{aligned} \text{by theorem 2.5.19: } & \leq \left(t + (d-1)(rp)' \cdot \left\| |W|_t \right\|_{L^{rp}} \right)^r \\ & \leq 2^{r'} \left(t^r + (d-1)^r (rp)^{r'} \cdot \left\| |W|_t \right\|_{L^{rp}}^r \right) \end{aligned}$$

$$\text{by exercise A.3.47 with } \sigma = \sqrt{t} := 2^{r'} \left(t^r + (d-1)^r (rp)^{r'} \Gamma_{p,r} \cdot t^{r/2} \right).$$

$$\text{Thus } \left\| |Z^*|_t^r \right\|_{L^p} \leq B_r t^r + B_{d,r,p} t^{r/2}. \quad (2)$$

Applying Hölder's inequality to (1) and (2), we get

$$\begin{aligned} \left\| |Z^*|_t^r \cdot e^{|mZ^*|_t} \right\|_{L^p} &\leq \left(B_r t^r + B_{d,r,p} t^{r/2} \right) \times \left(A_{2p,d} e^{M_{d,m,p} t} \right) \\ &= t^{r/2} A_{2p,d} \left(B_{d,r,2p} + B_r t^{r/2} \right) \times e^{M_{d,m,2p} t} : \end{aligned}$$

we get, for suitable $B' = B'_{d,p,r}$, $M' = M'_{d,m,p,r}$, the desired inequality

$$\left\| |Z^*|_t^r \cdot e^{|mZ^*|_t} \right\|_{L^p} \leq B' \cdot t^{r/2} e^{M' t}.$$

5.3.1 $\mathfrak{S}_{p,M}^{*n}$ is naturally equipped with the collection \mathfrak{N}° of seminorms $\left\| \cdot \right\|_{p^\circ, M^\circ}$, where $2 \leq p^\circ < p$ and $M^\circ > M$. \mathfrak{N}° forms an increasing family with pointwise limit $\left\| \cdot \right\|_{p,M}$. For $0 \leq \sigma \leq 1$ set $u^\sigma \stackrel{\text{def}}{=} u + \sigma(v-u)$ and $X^\sigma \stackrel{\text{def}}{=} X + \sigma(Y-X)$. Write F for F_η , etc. Then the remainder $F[v, Y] - F[u, X] - D_1 F[u, X] \cdot (v-u) - D_2 F[u, X] \cdot (Y-X)$ becomes, as in example A.2.48,

$$RF[u, X; v, Y] = \int_0^1 \left(Df(u^\sigma, X^\sigma) - Df(u, X) \right) \cdot \begin{pmatrix} v-u \\ Y-X \end{pmatrix} d\sigma.$$

With $R^\sigma f \stackrel{\text{def}}{=} Df(u^\sigma, X^\sigma) - Df(u, X)$, $1/p^\circ = 1/p + 1/r$, and $\|R\|_{pp^\circ}$ denoting the operator norm of a linear operator $R : \ell^p(k+n) \rightarrow \ell^{p^\circ}(n)$ we get

$$|RF[u, X; v, Y]_{T^\lambda}^*|_{p^\circ} \leq C \cdot r_\lambda \cdot (|v-u| + \|Y-X\|_{T^\lambda}^*),$$

where

$$r_\lambda \stackrel{\text{def}}{=} \sup_{t < T^\lambda} \sup_{0 \leq \sigma \leq 1} \|R^\sigma f\|_{pp^\circ},$$

and where C is a suitable constant depending only on k, n, p, p° . Now r_λ is a bounded random variable and converges to zero in probability as $|v-u| + \|Y-X\|_{p, M}^* \rightarrow 0$. (Use that the T^λ of definition (5.2.4) on page 283 are bounded; then X_t^σ ranges over a relatively compact set as $[0 \leq \sigma \leq 1]$ and $0 \leq t \leq T^\lambda$.) In other words, the uniformly bounded increasing functions $\lambda \mapsto \|r_\lambda\|_r$ converge pointwise – and thus uniformly on compacta – to zero as $|v-u| + \|Y-X\|_{p, M}^* \rightarrow 0$. Therefore the first factor on the right in

$$\|RF[u, X; v, Y]\|_{p^\circ, M^\circ} \leq C \sup_\lambda e^{(M-M^\circ)\lambda} \|r_\lambda\|_r \cdot (|v-u| + \|Y-X\|_{p, M}^*)$$

converges to zero as $|v-u| + \|Y-X\|_{p, M}^* \rightarrow 0$, which is to say $\|RF[u, X; v, Y]\|_{p^\circ, M^\circ} = o(|v-u| + \|Y-X\|_{p, M}^*)$.

5.4.19 (ii) For fixed μ and δ set $k \stackrel{\text{def}}{=} \lceil \mu/\delta \rceil$, and $\lambda_i \stackrel{\text{def}}{=} i\delta$ and $T_i \stackrel{\text{def}}{=} T^{\lambda_i}$ for $i = 0, 1, \dots, k$. Then $\lambda_{k-1} < \mu \leq \lambda_k$. Let Δ_i^* denote the maximal function of the difference of the global solution at T_i , which is $X_{T_i} = \Xi[C, \mathbf{Z}]_{T_i}$, from its Ξ' -approximate X'_{T_i} . Consider an $s \in [T^{\lambda_i}, T^{\lambda_{i+1}}]$.

$$\begin{aligned} X'_s - X_s &= \Xi'[X'_{T_i}, \mathbf{Z}_s - \mathbf{Z}_{T_i}] - \Xi'[X_{T_i}, \mathbf{Z}_s - \mathbf{Z}_{T_i}] \\ &\quad + \Xi'[X_{T_i}, \mathbf{Z}_s - \mathbf{Z}_{T_i}] - \Xi[X_{T_i}, \mathbf{Z}_s - \mathbf{Z}_{T_i}], \end{aligned}$$

$$5.4.17 \text{ gives } \|\Delta_{i+1}^*\|_{L^p} \leq \|\Delta_i^*\|_{L^p} \times e^{L'\delta} + (\|X_{T_i}^* + 1\|_{L^p}) \times (\underline{M}\delta)^r e^{\underline{M}\delta},$$

$$\text{which implies } |\Delta_k^*| \leq (\|X_{T_k}^* + 1\|_{L^p}) \times (\underline{M}\delta)^r e^{\underline{M}\delta} \cdot \sum_{0 \leq i < k} e^{iL'\delta}$$

$$\text{for } X \in \mathfrak{S}_{p, M}, \text{ as } \lambda_k = k\delta: \leq (\|X\|_M^* e^{M\lambda_k} + 1) \times (\underline{M}\delta)^r e^{\underline{M}\delta} \cdot \frac{e^{L'k\delta} - 1}{e^{L'\delta} - 1}$$

$$\text{by (5.2.23), as } \delta \rightarrow 0: \leq \frac{2}{1-\gamma} \left(\|C\|_M^* + 1 \right) e^{M\lambda_k} \times (\underline{M}\delta)^r e^{\underline{M}\delta} \cdot k e^{L'\lambda_k}$$

$$\begin{aligned} \text{since } k = \lambda_k/\delta: &\leq \text{const} (\|C\|_M^* + 1) \underline{M}^r e^{\underline{M}\delta} \delta^{r-1} \times \lambda_k \cdot e^{(M+L')\lambda_k} \\ &\leq \overline{B} \cdot (\|C\|_{L^p} + 1) \times \delta^{r-1} \cdot e^{\overline{M}\lambda_k} \end{aligned}$$

for suitable $\overline{B} = \overline{B}[f; \Xi']$ and $\overline{M} = \overline{M}[f; \Xi'] > M+L'$.

5.5.7 Let $\mathbb{P}, \overline{\mathbb{P}} \in \mathfrak{P}$. Thanks to the uniform ellipticity (5.5.17) there exist bounded functions h^η so that $f_0 = -\sum_\eta f_\eta \cdot h^\eta$. Then $M \stackrel{\text{def}}{=} \mathbf{h} * \mathbf{W}$ is a martingale under both \mathbb{P} and $\overline{\mathbb{P}}$, and by exercise 3.9.12 so is the Doléans–Dade exponential G' of M . The Girsanov theorem 3.9.19 asserts that $\mathbf{W}' \stackrel{\text{def}}{=} \mathbf{W}_t + \int_0^t \mathbf{h}_s ds$ is a Wiener process under the probabilities \mathbb{P}' and $\overline{\mathbb{P}}'$ that on every \mathcal{F}_t agree with $G'_t \mathbb{P}$ and $G'_t \overline{\mathbb{P}}$, respectively. Now X satisfies equation (5.5.20) with \mathbf{W}' replacing \mathbf{W} , and therefore by assumption $\mathbb{P}' = \overline{\mathbb{P}}'$. This clearly implies $\mathbb{P} = \overline{\mathbb{P}}$.

5.5.13 Let $s \leq t \leq t'$. Then by Itô's formula

$$\begin{aligned} u(t'-t, X_t) &= u(t'-s, X_s) - \int_s^{t'} \dot{u}(t'-\sigma, X_\sigma) d\sigma \\ &\quad + \int_s^{t'} u_{;\nu}(t'-\sigma, X_\sigma) dX_\sigma^{x\nu} + \int_s^{t'} \mathcal{A}u(t'-\sigma, X_\sigma) d\sigma \\ &= u(t'-s, X_s) + \int_s^{t'} u_{;\nu}(t'-\sigma, X_\sigma) dX_\sigma^{x\nu}. \end{aligned}$$

Taking the conditional expectation under \mathcal{F}_s exhibits $t \mapsto u(t'-t, X_t)$ as a bounded local martingale.

5.5.14 It suffices to consider equation (5.5.20). Recall that \mathfrak{P} is the collection of all probabilities on \mathcal{C}^n under which the process \mathbf{W}_t of (5.5.19) is a standard Wiener process. Let $\mathbb{P}, \bar{\mathbb{P}} \in \mathfrak{P}$. From $\phi_0, \phi_1, \dots, \phi_k \in C_b^\infty(\mathbb{R}^n)$ and $0 = t_0 < t_1 < \dots < t_k$ make the function $\Phi \stackrel{\text{def}}{=} \phi_0(X_0^x) \cdot \phi_1(X_{t_1}^x) \cdots \phi_k(X_{t_k}^x)$ on the path space \mathcal{C}^n . Their collection forms a multiplicative class that separates the points of \mathcal{C}^n . Since path space is polish and consequently every probability on it is tight (proposition A.6.2), or simply because the functions Φ generate the Borel σ -algebra on path space, $\mathbb{P} = \bar{\mathbb{P}}$ will follow if we can show that $\mathbb{E} = \bar{\mathbb{E}}$ on the functions Φ (proposition A.3.12). This we do by induction in k . The case $k = 0$ is trivial. Note that the equality $\mathbb{E}[\Phi] = \bar{\mathbb{E}}[\Phi]$ on Φ made from smooth functions ϕ_i persists on functions Φ made from continuous, even bounded Baire, functions ϕ_i , by the usual sequential closure argument. To propel ourselves from k to $k+1$ let u denote a solution to the initial value problem $\dot{u} = \mathcal{A}u$ with $u(0, x) = \phi_{k+1}(x)$ and write $\Phi \stackrel{\text{def}}{=} \phi_0 \circ X_0^x \cdot \phi_1 \circ X_{t_1}^x \cdots \phi_k \circ X_{t_k}^x$. Then, with $t = t' = t_{k+1}$ and $s = t_k$, exercise 5.5.13 produces

$$\begin{aligned} \mathbb{E}[\phi_{k+1}(X_{t_{k+1}}^x) | \mathcal{F}_{t_k}] &= \mathbb{E}[u(0, X_{t_{k+1}}^x) | \mathcal{F}_{t_k}] \\ &= \mathbb{E}[u(t_{k+1}-t_k, X_{t_k}^x)] \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}[\Phi \cdot \phi_{k+1}(X_{t_{k+1}}^x)] &= \mathbb{E}[\Phi \cdot u(t_{k+1}-t_k, X_{t_k}^x)] \\ &= \bar{\mathbb{E}}[\Phi \cdot u(t_{k+1}-t_k, X_{t_k}^x)] \end{aligned} \tag{*}$$

by the same token:

$$= \bar{\mathbb{E}}[\Phi \cdot \phi_{k+1}(X_{t_{k+1}}^x)].$$

At (*) we used the fact that the argument of the expectation is a k -fold product of the same form as Φ , so that the induction hypothesis kicks in.

A.2.23 Let U_n be the set of points x in F that have a neighborhood $V_n(x)$ whose intersection with E has ρ -diameter strictly less than $1/n$. The U_n clearly are open and contain E . Their intersection is E . Indeed, if $x \in \bigcap U_n$, then the sets $E \cap V_n(x)$ form a Cauchy filter basis in E whose limit must be x .

A.3.5 The family of all complex finite linear combinations of functions in $\mathcal{M} \cup \{1\}$ is a complex algebra \mathcal{A} of bounded functions in \mathcal{V} that is closed under complex conjugation; the σ -algebra it generates is again \mathcal{M}^Σ . The real-valued functions in \mathcal{A} form a real algebra \mathcal{A}_0 of bounded functions that again generates \mathcal{M}^Σ . It is a multiplicative class contained in the bounded monotone class \mathcal{V}_0 of real-valued functions in \mathcal{V} . Now apply theorem A.3.4 suitably.

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LNM stands for Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, New York

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Index of Notations

$1_A = A$	the indicator function of A	365
$\Phi[\mu] = \mu \circ \Phi^{-1}$	the image of the measure μ under Φ	405
$\mathcal{A}[\mathcal{F}]$	the \mathcal{F} -analytic sets	432
\mathcal{A}_∞	the algebra $\bigcup_{0 \leq t < \infty} \mathcal{F}_t$	22
$\mathcal{A}_{\infty\sigma}$	the countable unions of sets in \mathcal{A}_∞	35
$\mu \star \nu$	the convolution of μ and ν	413
\mathcal{V}^*	dual of the topological vector space \mathcal{V}	381
$X \ast Z$	the indefinite Itô integral	133
Z^*	the maximal process of Z	26
ϕ^*, μ^*	ϕ, μ reflected through the origin	410
\mathcal{E}_e^*	the universal completion of \mathcal{E}	407
\mathbf{B}	the base space $[0, \infty) \times \Omega$	22
$\check{\mathbf{B}}$	product of auxiliary space with base space \mathbf{B}	109
$\check{\eta} = (\eta, \varpi)$	$= (\eta, s, \omega)$, the typical point of $\check{\mathbf{B}} \stackrel{\text{def}}{=} \mathbf{H} \times \mathbf{B}$	172
$\mathcal{B}^*(E), \mathcal{B}^\bullet(E)$	the Baire & Borel sets or functions of E	391
$b(p)$	$= \frac{2}{\pi} \int_0^\infty \xi^{p-1} \sin \xi \, d\xi$	458
$\llbracket 0, t \rrbracket$	$\stackrel{\text{def}}{=} \mathbb{R}_*^d \times \llbracket 0, t \rrbracket$	181
$\llbracket 0, T \rrbracket$	$\stackrel{\text{def}}{=} \mathbf{H} \times \llbracket 0, T \rrbracket$	172
A^c	the complement of A	373
$C_b(E)$	the continuous bounded functions on E	376
$C_0(\mathbf{B})$	the continuous functions vanishing at infinity	366
$C_{00}(\mathbf{B})$	the continuous functions with compact support	370
$\widehat{\mu}^\Gamma$	the characteristic function of μ for Γ	410
C_b^k	the bounded functions with k bounded cont. partials	281
$C^k(D)$	the k -times continuously diff'ble functions on D	372
\mathcal{C}^d	the path space $C_{\mathbb{R}^d}[0, \infty)$	20
$\mathcal{C} = \mathcal{C}^1$	the path space $C[0, \infty)$	14
$\delta^{\eta\theta}$	the Kronecker delta	19
δ_s	the Dirac measure, or point mass, at s	398

ΔX	the jump process of $X \in \mathfrak{D}$ ($\Delta X_0 \stackrel{\text{def}}{=} 0$)	25
$D^\lambda F[u]$	the λ^{th} (weak) derivative of F at u	305
dF	the measure with distribution function F	406
$d\dot{F} = dF $	its variation	406
$\mathfrak{D} = \mathfrak{D}[\mathcal{F}\cdot]$	the càdlàg adapted processes	24
$\mathscr{D}, \mathscr{D}^d, \mathscr{D}_E$	canonical space of càdlàg paths	66
\mathbb{E}^x	$\mathbb{E}^x[F] = \int F d\mathbb{P}^x$	351
$\mathbb{E}, \mathbb{E}^{\mathbb{P}}$	expectation under the prevailing probability, \mathbb{P}	32
$\mathbb{E}[f \Phi], \mathbb{E}[f \mathcal{Y}]$	the conditional expectation of f given Φ, \mathcal{Y}	407
$\mathcal{E} = \mathcal{E}[\mathcal{F}\cdot]$	the elementary stochastic integrands	46
\mathcal{E}_1	the unit ball $\{X \in \mathcal{E} : -1 \leq X \leq 1\}$ of \mathcal{E}	51
\mathcal{E}_+^\uparrow	the pointwise suprema of sequences in \mathcal{E}	88
$\mathcal{E}^\sigma, (\mathcal{E}^d)^\sigma$	the sequential closure of $\mathcal{E}, \mathcal{E}^d$	392
\mathcal{E}_{00}	the \mathcal{E} -confined functions in \mathcal{E}	369
$\overline{\mathcal{E}}_{00}$	the confined uniform closure of \mathcal{E}_{00}	370
\mathcal{E}_{00}^σ	the \mathcal{E} -confined functions in \mathcal{E}^σ	393
$\mathcal{E}^{\mathbb{P}} = \mathcal{E}[\mathcal{F}\cdot^{\mathbb{P}}]$	the elementary integrands of the natural enlargement	57
ϵ	denoting measurability, as in $f \in \mathcal{F}/\mathcal{G}$	391
ϵ	“is member of” or “is measurable on”	23
$=$	denotes near equality and indistinguishability	35
${}^0\mathbf{F}$	coupling coefficients adjusted so that ${}^0\mathbf{F}[0] = 0$	272
0C	initial condition adjusted correspondingly	272
\mathcal{F}_+	the positive elements of \mathcal{F}	363
\mathcal{F}_t	the past or history at time t	21
\mathcal{F}_T	the past at the stopping time T	28
$\mathcal{F}\cdot$	the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq \infty}$	21
$\mathcal{F}\cdot_-$	the left-continuous version of $\mathcal{F}\cdot$	123
$\mathcal{F}\cdot_+$	the right-continuous version of $\mathcal{F}\cdot$	37
$\mathcal{F}\cdot^0[Z]$	the basic filtration of the process Z	23
$\mathcal{F}\cdot[Z]$	the natural filtration of the process Z	39
$\mathcal{F}\cdot^0[\mathscr{D}_E], \mathcal{F}\cdot^0_+[\mathscr{D}_E]$	basic, canonical filtration of path space	66
$\mathcal{F}\cdot[\mathscr{D}_E]$	the natural filtration of path space	67
\mathcal{F}_δ	the intersections of countable subfamilies of \mathcal{F}	432
$\mathcal{F}_\infty, \mathcal{F}_\infty^*$	the σ -algebra $\bigvee_{0 \leq t < \infty} \mathcal{F}_t$, its universal completion	22
\mathcal{F}_{T-}	the strict past of T	120
$\mathfrak{F}[\ \cdot\ ^*]$	the processes finite for $\ \cdot\ ^*$	97
\mathcal{F}_σ	the unions of countable subfamilies of \mathcal{F}	432
$\mathcal{F}_{\sigma\delta}$	the intersections of countable subfamilies of \mathcal{F}_σ	432
$\mathcal{F}\cdot^{\mathbb{P}}, \mathcal{F}\cdot^{\mathfrak{P}}$	the \mathbb{P}, \mathfrak{P} -regularization of $\mathcal{F}\cdot$	37

$\mathcal{F}_+^{\mathfrak{F}}$	the natural enlargement of a filtration	39
\mathcal{G}_δ	the intersections of countable subfamilies of \mathcal{G}	441
\mathcal{G}	the open sets of the topological space at hand	441
\mathcal{K}_σ	the unions of countable subfamilies of \mathcal{K}	441
\mathcal{K}	the compact sets of the topological space at hand	441
\tilde{F}	a predictable envelope of F .	125
$S^\Delta \stackrel{\text{def}}{=} S \dot{\cup} \{\Delta\}$	the one-point compactification of S	374
$\check{H} \stackrel{\text{def}}{=} H \times [0, \infty)$	the product of auxiliary space with time	177
h_0	prototypical sure Hunt function $\mathbf{y} \mapsto \mathbf{y} ^2 \wedge 1$	180
h'_0	$\mathbf{y} \mapsto \int_{\{ \zeta \leq 1\}} e^{i\langle \zeta \mathbf{y} \rangle} - 1 ^2 d\zeta$, another one	182
$\eta_{p,q}(\mathcal{I})$	a factorization constant	192
$\int \mathbf{X} d\mathbf{Z}$	the elementary integral for vectors	56
$\int X dZ$	the elementary integral	47
$\int_A F = \int A \cdot F$	the integral over the set A of the function F	105
$\int \mathbf{X} d\mathbf{Z}$	the (extended or Itô) integral for vectors	110
$\int X dZ$	the (extended or Itô) stochastic integral	99
$\int X dZ$	the (extended) stochastic integral revisited	134
$\int_0^T \mathbf{X} d\mathbf{Z}$	its value at $T \in \mathfrak{T}$	134
$\mathbf{X} * \mathbf{Z}$	the indefinite Itô integral for vectors	134
$\int X \delta Z$	the Stratonovich integral	169
$X \circ Z$	the indefinite Stratonovich integral	169
$\int_0^T G dZ \stackrel{\text{def}}{=} \int G dZ^T$		131
$\int_{S^+}^T G dZ \stackrel{\text{def}}{=} \int_0^T G \cdot (S, \infty) dZ$		131
\mathcal{J}_Z	the jump measure of \mathbf{Z}	181
$H * \mathcal{J}_Z$	the indefinite integral of H against jump measure	181
L^∞	the essentially bounded measurable functions	448
$L^p = L^p(\mathbb{P})$	the space of p ; \mathbb{P} -integrable functions	33
$L^0, L^0(\mathcal{F}_t, \mathbb{P})$	(classes of) measurable a.s. finite functions	33
M_∞	the limit of the martingale M at infinity	75
$\ \cdot \ _p = \ \cdot \ _{\ell^p}$	$ x _p \stackrel{\text{def}}{=} (\sum_\nu x^\nu ^p)^{1/p}$, $0 < p < \infty$	364
$\ \cdot \ _\infty = \ \cdot \ _{\ell^\infty}$	$ x _\infty \stackrel{\text{def}}{=} \sup_\nu x^\nu $	364
$\ \cdot \ $	any of the norms $\ \cdot \ _p$ on \mathbb{R}^n	364
$\langle \mathbf{x} \mathbf{y} \rangle$	the inner product of vectors \mathbf{x} and \mathbf{y}	238
ℓ^p	the vectors or sequences $x = (x^\nu)$ with $ x _p < \infty$	364
$\ell^0 \stackrel{\text{def}}{=} \mathbb{R}^{\mathbb{N}}$	the Fréchet space of scalar sequences	364
\mathcal{L}	the left-continuous paths with finite right limits	24
$\mathfrak{L} = \mathfrak{L}[\mathcal{F}_\cdot]$	the collection of adapted maps $Z : \Omega \rightarrow \mathcal{L}$	24
$\mathfrak{L}^1[\mathfrak{F} \cdot \mathfrak{F}^*] \& \mathfrak{L}^1[Z-p]$	the $\mathfrak{F} \cdot \mathfrak{F}^*$ - & Z - p -integrable processes	99

$\mathcal{L}^1[\zeta-p] = \mathcal{L}^1[\llbracket \cdot \rrbracket_{\zeta-p}^*]$	the ζ - p -integrable processes	175
$\Lambda^{(q)}[\mathcal{Z}]$	THE previsible controller for \mathcal{Z}	238
$\mathfrak{M}^*(E) \ \& \ \mathfrak{M}^*(E)$	the σ -additive & order-continuous measures on E	421
$\mathfrak{M}^*[\mathcal{E}]$	the σ -additive measures on \mathcal{E}	406
\bigvee	$\bigvee \mathcal{F} =$ supremum or span of the family \mathcal{F}	22
$\wedge \ \& \ \vee$	$a \vee b \ \& \ a \wedge b$: smaller & larger of a, b	364
$\ \cdot \ _E$	the quasinorm on the quasinormed space E	381
$\ \cdot \ _{\mathcal{E}}$	the sup-norm on the space \mathcal{E} of functions	188
$\ \mathcal{I} \ = \ \mathcal{I} \ _{L(E,F)}$	$= \sup\{\ \mathcal{I}(x) \ _F : x \in E, \ x \ _E \leq 1\}$	381
$\ u \ = \ u \ _{L(E,F)}$	$= \sup\{\ u(x) \ _F : x \in E, \ x \ _E \leq 1\}$	381
M^g	the martingale $M^g = \mathbb{E}[g \mathcal{F}]$	72
$\ f \ _p^* = \ f \ _{p;\mathbb{P}}^*$	its mean $\ f \ _{L^p(\mathbb{P})}^* \stackrel{\text{def}}{=} (\int^* f ^p d\mathbb{P})^{1/p}$	452
$\ f \ _p = \ f \ _{L^p(\mathbb{P})}$	$\stackrel{\text{def}}{=} (\int f ^p d\mathbb{P})^{1/p}$	33
$\llbracket \cdot \rrbracket_{Z-p}$	the semivariation for $\llbracket \cdot \rrbracket_p$	53
$\ \cdot \ _{Z-p}^*$	the Daniell extension of $\ \cdot \ _{Z-p}$	88
$\llbracket f \rrbracket_p^* = \llbracket f \rrbracket_{p;\mathbb{P}}^*$	its mean $\llbracket f \rrbracket_{L^p(\mathbb{P})}^* \stackrel{\text{def}}{=} (\int^* f ^p d\mathbb{P})^{1/p \wedge 1}$	452
$\llbracket f \rrbracket_p = \llbracket f \rrbracket_{L^p(\mathbb{P})}$	$= \llbracket f \rrbracket_{L^p(\mathbb{P})} \stackrel{\text{def}}{=} (\int f ^p d\mathbb{P})^{1/p \wedge 1}$	33
$\llbracket \cdot \rrbracket^*$	a subadditive mean	95
$\llbracket \zeta^{h,t} \rrbracket_{\mathcal{I}^p} = \llbracket \zeta^{h,t} \rrbracket_{\mathcal{I}^p[\mathbb{P}]}$	integrator (quasi)norms of the random measure ζ	173
$\llbracket \mathcal{Z} \rrbracket_{\mathcal{I}^p} = \llbracket \mathcal{Z} \rrbracket_{\mathcal{I}^p[\mathbb{P}]}$	integrator (quasi)norms of the integrator \mathcal{Z}	55
$\llbracket \cdot \rrbracket_{Z-p}^* = \llbracket \cdot \rrbracket_{Z-p;\mathbb{P}}^*$	THE Daniell mean	88
$\llbracket \mathcal{Z} \rrbracket_{\mathcal{I}^p} = \llbracket \mathcal{Z} \rrbracket_{\mathcal{I}^p[\mathbb{P}]}$	$\stackrel{\text{def}}{=} \sup\{\ \int X dZ \ _{L^p(\mathbb{P})} : X \in \mathcal{E}, X \leq 1\}$	55
$\llbracket \mathcal{Z} \rrbracket_{\mathcal{I}^p} = \llbracket \mathcal{Z} \rrbracket_{\mathcal{I}^p[\mathbb{P}]}$	$\stackrel{\text{def}}{=} \sup\{\ \int \mathbf{X} d\mathbf{Z} \ _{L^p(\mathbb{P})} : \mathbf{X} \in \mathcal{E}, \mathbf{X} \leq 1\}$	56
$\ \cdot \ _{[\alpha]}^*$	the corresponding mean	452
$\ \cdot \ _{[\alpha]}$	$\ f \ _{[\alpha]} \stackrel{\text{def}}{=} \inf\{\lambda > 0 : \mathbb{P}[f > \lambda] \leq \alpha\}$	34
$\ \cdot \ _{Z-[\alpha]}$	the corresponding semivariation	53
$\ \cdot \ _{Z-[\alpha]}^*$	the Daniell extension of $\ \cdot \ _{Z-[\alpha]}$	88
$\ \cdot \ _{[\alpha]}$	the integrator size according to $\ \cdot \ _{[\alpha]}$	55
$\ \cdot \ _{p,M}, \ \cdot \ _{p,M}^*$	Picard Norms	283
μ_Z	the Doléans–Dade measure of \mathcal{Z}	222
$\llbracket f \rrbracket_0 = \llbracket f \rrbracket_{0;\mathbb{P}}$	the metric $\inf\{\lambda : \mathbb{P}[f \geq \lambda] \leq \lambda\}$ on L^0	34
\mathcal{F}^μ	μ -completion of \mathcal{F}	414
$O(\cdot), o(\cdot)$	big O and little o	388
(Ω, \mathcal{F})	the underlying filtered measurable space	21
ϖ	the typical point (s, ω) of $\mathbb{R}_+ \times \Omega$	22
\mathfrak{P}	the pertinent probabilities	32
\mathcal{P}_{00}	the bdd. predictable processes with bdd. carrier	128
\mathcal{P}_b	the bounded predictable processes	135
$\mathfrak{P}[Z]$	the probabilities for which Z is an L^0 -integrator	61

$\check{\mathcal{P}} \stackrel{\text{def}}{=} \mathcal{B}^\bullet(\mathbf{H}) \otimes \mathcal{P}$	$= (\mathcal{E}[\mathbf{H}] \otimes \mathcal{E}[\mathcal{F}_\bullet])^\sigma$, predictable random functions	172
$\mathfrak{P}^*(E) = \mathfrak{M}_{1,+}^*(E)$	the probabilities on $C_b(E)$	421
$\mathfrak{P}^\bullet(E) = \mathfrak{M}_{1,+}^\bullet(E)$	the order-continuous probabilities on E	421
$p^\diamond = p^\diamond[\mathbf{Z}]$	p if \mathbf{Z} jumps, 2 otherwise	238
$1^\diamond = 1^\diamond[\mathbf{Z}]$	2 if \mathbf{Z} is a martingale, 1 otherwise	238
$\mathcal{P} = \mathcal{P}[\mathcal{F}_\bullet]$	the predictable processes or σ -algebra	115
$\mathcal{P}^\mathbb{P}$	the processes previsible with \mathbb{P}	118
\mathbb{R}_+	the positive reals, i.e., the reals ≥ 0	363
\mathbb{R}_*^d	punctured d -space $\mathbb{R}^d \setminus \{0\}$	363
T_A	the reduction of $T \in \mathfrak{T}$ to $A \in \mathcal{F}_T$	31
0_A	the stopping time 0 reduced to $A \in \mathcal{F}_0$	48
$\bar{\rho}(r, s)$	the arctan metric on $\overline{\mathbb{R}}$	364
$\overline{\mathbb{R}}$	the extended reals $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$	363
$\sigma, \mathcal{E}^\sigma, \mathcal{E}_{\mathbb{R}}^\sigma$	σ denotes sequential closure, of \mathcal{E}	392
$\mathcal{V}, {}^j\mathcal{V}$	continuous & jump part of finite variation process V	69
$\tilde{c}Z, {}^rZ$	continuous martingale part of Z , rest $Z - \tilde{c}Z$	235
$\tilde{s}Z \& {}^lZ$	small-jump martingale part & large-jump part of Z	237
$\hat{v}Z$	a continuous finite variation part of Z	237
pZ	the part of Z supported on a sparse previsible set	235
qZ	the quasi-left-continuous rest $Z - {}^pZ$	235
$[Z, Z], [{}^cZ, {}^cZ] \& [{}^jZ, {}^jZ]$	square bracket, its continuous & jump parts	148
$[Y, Z], [{}^cY, {}^cZ] \& [{}^jY, {}^jZ]$	square bracket, its continuous & jump parts	150
$\mathfrak{S}_{p,M}^{*n}$	processes with finite Picard Norm $\ \cdot \ _{p,M}^*$	283
$S[Z] \& \sigma[Z]$	square function & continuous square function of Z	148
\mathcal{S}	Schwartz space of C^∞ -functions of fast decay	269
$\sigma(C_b^*(E), C_b(E))$	the topology of weak convergence of measures	421
Z^T	$Z_s^T = Z_{T \wedge s}$ is the process Z stopped at $T \in \mathfrak{T}$	28
$Z^{\mathcal{S}}$	the \mathcal{S} -scalæfication of Z	139
$\mathfrak{T} = \mathfrak{T}[\mathcal{F}_\bullet]$	the \mathcal{F}_\bullet -stopping times	27
$[T] = [T, T]$	the graph of the stopping time T	28
$T^\bullet : \lambda \mapsto T^\lambda$	THE time transformation for Z	239
$X_{-} \& X_{+}$	left- & right-continuous version of X	24
V^μ	the Doléans–Dade process of μ	222
$ z , \mu $	variation of distribution function z , or measure μ	45
$ dz = dz $	the variation measure of the measure dz	45
$\widehat{J}_Z \& \widetilde{J}_Z$	compensator & compensatrix of jump measure	232
\mathbb{W}	Wiener measure	16
\underline{W}	Wiener process as a $C[0, \infty)$ -valued random variable	14
\dot{f}	the equivalence class of f mod negligible functions	13

$Z_\infty = Z_{\infty-}$	the limit (possibly $\pm\infty$) of Z at ∞	27
$\ll\cdot$ & $\approx\cdot$	local absolute continuity & local equivalence	40
\ll	denoting absolute continuity	407
$Z_\cdot(\omega)$	the path $s \mapsto Z_s(\omega)$	23
\Rightarrow	denoting weak convergence of measures	421
Z_T	the random variable $\omega \mapsto Z_{T(\omega)}(\omega)$.	28
Z_s	the function $\omega \mapsto Z(s, \omega)$	23
$\zeta^T = \llbracket 0, T \rrbracket \zeta$	the random measure ζ stopped at T	173
$\hat{\zeta}, \tilde{\zeta}$	previsible & martingale parts of random measure ζ	231
\hat{Z}, \tilde{Z}	previsible & martingale parts of the integrator Z	221

Symbols

Index

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