Appendix C

Answers

Acknowledgements

I am grateful to **Milan Lukic** <lmilan@shell.core.com> for pointing out several errors (now corrected): Equation (*) on page 3 was wrong, and the answer on page 7 to exercise 1.3.10 on page 29 contained a typo (Z_t instead of Z_T). Also, the answers on page 9 to exercises 1.3.34–36 were garbled.

Roger Sewell, <rfs@cambridgeconsultants.com>, found many errata, both in the main text and in the answers. I tried to give credit at every instance, but I am not sure I succeeded; I know that on occasion I managed to introduce a new mistake in response to his suggestions. I want to state here that I am profoundly grateful for his faithful reports of errata he found.

Answers

1.1.1 See [9, page 41], or [5, page 293 ff.]: Consider a physical system, for instance an ear of corn, member of a (vast) corn field. An ***observable*** is a quantity about the ear that can be measured by a well–defined procedure. For example, the girth G, the number N of kernels, the length L, the weight W of our ear of corn are observables, measurable by applying a tape measure, by counting, or by weighing on a scale, respectively. The observables form an algebra and vector lattice $\mathcal{E}$ in the obvious way; for instance, $G+3W$, LN, and $G \wedge W$ are the observables having the procedures "measure girth and weight and add three times the latter to the former," "multiply the length by the number of kernels," and "take the smaller of girth and weight," respectively. In fact, for a technical reason that will become transparent later let us consider complex observables ($G+iW$ etc). They clearly form a commutative algebra $\mathcal{A}$ over the compex field $\mathbb{C}$. (Note the implicit requirement that different observables can be measured simultaneously – no quantum effects here.) For every observable $Z = X + iY$ let Z^* denote the observable whose value is the complex conjugate $x - iy$ whenever a measurement of Z produces $x+iy$. Clearly $(Z_1Z_2)^* = Z_2^*Z_1^*$, $Z^{**} = Z$, and $(zZ)^* = \overline{z}Z^*$ for $z \in \mathbb{C}$ and $Z, Z_1, Z_2 \in \mathcal{A}$. Let $\|Z\|$ be the supremum of the possible values of $|Z| = |X+iY| \stackrel{\text{def}}{=} \sqrt{X^2+Y^2}$, taken over the ensemble to be represented (the corn field). Then restrict attention to the commutative normed $\mathbb{C}$-algebra $\mathcal{A}$ of those observables Z that have $\|Z\| < \infty$. The evident submultiplicativity $\|Z_1Z_2\| \le \|Z_1\| \cdot \|Z_2\|$ makes $\mathcal{A}$ a ***normed algebra***. Its completion $\overline{\mathcal{A}}$ is easily seen to be a Banach algebra with a multiplicative unit I (the observable that produces the value 1 upon all measurements), where ***involution*** $Z \mapsto Z^*$ and norm $\| \ \|$ satisfy $\|Z_1Z_2\| \le \|Z_1\| \cdot \|Z_2\|$ and $\|Z^*Z\| = \|Z\|^2$. A Banach algebra with unit and involution as above is known as a ***unital C^*-algebra***.

Another common example of a commutative unital C^*-algebra is the space $C_{\mathbb{C}}(\Omega)$ of continuous functions on a compact Hausdorff space Ω, with pointwise addition, multiplication, involution = complex conjugation, and equipped with the supremum norm. This example is typical. Namely, ***every commutative unital C^*-algebra $\overline{\mathcal{A}}$ is of the form $C_{\mathbb{C}}(\Omega)$ for some compact Hausdorff space Ω.*** Let us take this fact for granted right now – its proof, which will also clarify the precise meaning of "is of the form," can be found further down. Given this fact, the algebra and vector lattice closed under chopping $\mathcal{E}$ of real bounded observables is then identified with a dense subset $\widehat{\mathcal{E}}$ of the real part $C_{\mathbb{R}}(\Omega)$ of $C_{\mathbb{C}}(\Omega)$, for some compact Hausdorff Ω.

The probabilistic aspect in this story comes from the interest in the ***average*** of the various observables: what is the average weight of an ear of corn, and how can one estimate it without harvesting the whole field and toting it to the scales? A little reflection shows that the average is a linear positive functional on the observables $\mathcal{E}$, with the average of the unit observable I

being 1. Corresponding to it is a positive linear functional $\mathbb{E} : \widehat{\mathcal{E}} \to \mathbb{R}$ with $\mathbb{E}[1] = 1$. There is an immediate extension by continuity to all of $C_{\mathbb{R}}(\Omega)$. On $C_{\mathbb{R}}(\Omega)$, then, the average is (represented as) a positive Radon measure $\mathbb{E}$ of total mass one. Such is automatically σ-additive. The extension theory of page 394 ff. applies and produces a σ-additive extension, called the ***expectation*** and again denoted by $\mathbb{E}$. Its restriction to the integrable subsets $\mathcal{F}$ of Ω (see notation A.1.4) is the ***probability*** $\mathbb{P}$. The laws of large numbers allow the identification of $\mathbb{P}[F]$ with a limiting frequency (and of $\mathbb{E}$ as a limiting average). Most often $(\Omega, \mathcal{F}, \mathbb{P})$ is taken as the basic mathematical model for the probabilistic analysis, possibly because people might find the frequency of events intuitively more appealing than the average of measurements, and despite the difficulty of justifying the *ad hoc* requirement of σ-additivity of $\mathbb{P}$.

The latter is gone from the model $(\mathcal{E}, \mathbb{E})$.

The structure theorem for a unital commutative C^*-algebra $\overline{\mathcal{A}}$ is left to be established. There will be several steps.

(i) If $Z \in \overline{\mathcal{A}}$ is invertible, then $Z + H$ is invertible with inverse

$$(Z+H)^{-1} = Z^{-1} \sum_{k=0}^{\infty} (-HZ^{-1})^k \ , \qquad (*)$$

provided $\|H\| < \|Z^{-1}\|$; simply multiply the evidently convergent sum on the right or left with $Z + H$, obtaining I in both cases. The invertible elements of $\overline{\mathcal{A}}$ therefore form an open set $\mathcal{G}$. Since a proper ideal $\mathcal{I}$ is disjoint from $\mathcal{G}$ so is its closure $\overline{\mathcal{I}}$, which therefore is proper as well.

(ii) An element $X \in \overline{\mathcal{A}}$ is ***self–adjoint*** if $X^* = X$. An ideal $\mathcal{I}$ is self–adjoint if it equals $\mathcal{I}^* \stackrel{\text{def}}{=} \{X^* : X \in \mathcal{I}\}$. By Zorn's lemma, a proper self–adjoint ideal is contained in a maximal proper self–adjoint ideal $\mathcal{M}$, which by (i) is closed. The quotient $\dot{\mathcal{A}} \stackrel{\text{def}}{=} \mathcal{A}/\mathcal{M}$ is in the obvious way a C^*-algebra and clearly contains no proper self–adjoint ideal. In fact, every non–zero element $\dot{Z} \in \dot{\mathcal{A}}$ is invertible: if not, then the self–adjoint ideal $\dot{\mathcal{A}} \cdot \dot{Z}^* \dot{Z}$, which would not contain the unit $\dot{I}$, equals $\{\dot{0}\}$, whence $\dot{Z}^*\dot{Z} = \dot{0}$ and $\dot{Z} = \dot{0}$. In other words, $\dot{\mathcal{A}}$ is a field. We shall see soon that "the C^*-field" $\dot{\mathcal{A}}$ equals $\mathbb{C}$.

(iii) From the submultiplicativity of the norm, $\|Z^n\|^{1/n} \le \|Z\|$, so that $\nu(Z) \stackrel{\text{def}}{=} \inf\{\|Z^n\|^{1/n} : n \in \mathbb{N}\}$ exists. It is not hard to see that $\nu(Z) = \lim_{n\to\infty} \|Z^n\|^{1/n}$. Indeed, given an $\epsilon > 0$, find an $N \in \mathbb{N}$ with $\|Z^N\|^{1/N} < \nu(Z) + \epsilon$. For $n > N$ there are q, r with $n = qN + r$ and $0 \le r < N$. Then $\|Z^n\|^{1/n} \le \|Z^N\|^{q/n}\|Z\|^{r/n} \le (\nu(Z)+\epsilon)^{qN/n}\|Z\|^{r/n} \xrightarrow[n\to\infty]{} \nu(Z)+\epsilon$; hence $\limsup_{n\to\infty} \|Z^n\|^{1/n} \le \nu(Z) + \epsilon \quad \forall\, \epsilon > 0$ and $\limsup_{n\to\infty} \|Z^n\|^{1/n} = \nu(Z)$. Note that, for a self–adjoint element $X \in \overline{\mathcal{A}}$, $\|X^2\| = \|X^*X\| = \|X\|^2$, hence $\|X\| = \|X^{2^k}\|^{2^{-k}} \xrightarrow[k\to\infty]{} \nu(X)$, whence finally $\nu(X) = \|X\|$.

(iv) The ***resolvent*** of $Z \in \overline{\mathcal{A}}$ is the function $\mathbb{C} \ni z \mapsto (zI - Z)^{-1}$, defined on the open (by (i)) set υ_Z of $z \in \mathbb{C}$ for which the inverse exists. The compact complement σ_Z of υ_Z is called the ***spectrum*** of Z. From the straightforward

resolvent identity

$$(zI - Z)^{-1} - (z'I - Z)^{-1} = -(z - z')(zI - Z)^{-1}(z'I - Z)^{-1}$$

it is clear that $z \mapsto (zI - Z)^{-1}$ is not only continuous, but even complex differentiable (analytic) on υ_Z. The observation that $(zI - Z)^{-1} \xrightarrow[z\to\infty]{} 0$ proves that *the spectrum σ_Z is not empty*: if it were, any continuous linear functional on $\overline{\mathcal{A}}$ applied to $(zI - Z)^{-1}$ would produce an entire function that vanishes at infinity; such must vanish identically, and by the Hahn–Banach theorem we would arrive at the impossible consequence $(zI - Z)^{-1} = 0 \quad \forall z$. We apply this in the C^*-field $\dot{\mathcal{A}} = \overline{\mathcal{A}}/\mathcal{M}$ of (ii): Let $0 \neq \dot{Z} \in \dot{\mathcal{A}}$ and $z \in \sigma_{\dot{Z}}$. Then $z\dot{I} - \dot{Z}$, not being invertible, must be zero: $\dot{Z} = zI$.
(v) By $(*)$, $(zI - Z)^{-1} = (1/z) \cdot \sum_{k=0}^{\infty} z^{-k} Z^k$. The time–honored root test, suitably adapted to series in $\overline{\mathcal{A}}$, shows that the convergence radius of the series (in $1/z$) on the right hand side is $\big(\nu(Z)\big)^{-1}$. That is to say, the circle $\big[|1/z| = \big(\nu(Z)\big)^{-1}\big] = \big[|z| = \nu(Z)\big]$ must contain a singularity of $(zI - Z)^{-1}$. From this we conclude that $\nu(Z)$ equals the ***spectral radius*** $\rho(Z)$ of Z, which is defined as $\rho(Z) = \max\{|z| : z \in \sigma_Z\}$. For a self–adjoint element $X = X^* \in \overline{\mathcal{A}}$, therefore,

$$\|X\| = \nu(X) = \rho(X) . \tag{$**$}$$

(vi) Let Ω denote the collection of all linear multiplicative $\big(\omega(Z_1 Z_2) = \omega(Z_1)\omega(Z_2)\big)$ functionals $\omega : \overline{\mathcal{A}} \to \mathbb{C}$ that take I to 1 and respect the involution $\big(\omega(Z^*) = \overline{\omega(Z)}\big)$, the ***$*$-characters***. Such ω has norm 1. Indeed, let $\|Z\| < 1$. Then $\|Z^*Z\| < 1$. Define a_n by $\sqrt{1-x} = \sum a_n x^n$ for $|x| < 1$, and set $Y \stackrel{\text{def}}{=} \sum a_n (Z^*Z)^n$. Then $I - Z^*Z = Y^*Y$ and so $\omega(I) - \omega(Z^*Z) = \omega(Y^*Y) > 0$ and $|\omega(Z)| < 1$. Given the topology of pointwise convergence, Ω is a compact Hausdorff space (see exercise A.2.13). For $Z \in \overline{\mathcal{A}}$ set

$$\widehat{Z}(\omega) \stackrel{\text{def}}{=} \omega(Z) .$$

This defines a continuous function $\widehat{Z} : \Omega \to \mathbb{C}$. The map $Z \mapsto \widehat{Z}$ is clearly linear, multiplicative, turns involution on $\overline{\mathcal{A}}$ into complex conjugation on $C_{\mathbb{C}}(\Omega)$, and has $\widehat{I} = 1$ and $\|\widehat{Z}\|_\infty \leq \|Z\|$. It is left to be shown that it is in fact an isometry. To this end let $Z \in \overline{\mathcal{A}}$ and $z = \|Z\|$. Then by $(**)$ we have $|z|^2 \in \sigma_{Z^*Z}$, and the proper self–adjoint ideal $\mathcal{I} \stackrel{\text{def}}{=} \overline{\mathcal{A}} \cdot (|z|^2 I - Z^*Z)$ is contained in a maximal proper self–adjoint ideal $\mathcal{M}$, which is closed and gives rise to a bijective $*$-character $\dot{\omega} : \dot{\mathcal{A}} \stackrel{\text{def}}{=} \overline{\mathcal{A}}/\mathcal{M} \to \mathbb{C}$. Let ω be the composition of $\dot{\omega}$ with the quotient map $\overline{\mathcal{A}} \to \dot{\mathcal{A}}$. At this point $\omega \in \Omega$ clearly $|\widehat{Z}(\omega)|^2 = \omega(Z^*Z) = |z|^2$, so that $\|\widehat{Z}\|_\infty = |\widehat{Z}(\omega)| = \|Z\|$.
$Z \mapsto \widehat{Z}$ furnishes the desired linear isometric multiplicative involution–preserving identification of $\overline{\mathcal{A}}$ with $C_{\mathbb{C}}(\Omega)$.
1.2.4 (ii): For $u \in \mathbb{N}$ let W^u be a countable uniformly dense subset of $\{w \in C[0,u] : w_0 = 0\}$. Identify every path w in W^u with that path in $\mathscr{C}$

which agrees with w on $[0,u]$ and is constant thereafter. The collection $\bigcup_u W^u$ is plainly dense in $\mathscr{C}$ in the topology of uniform convergence on compacta.

1.2.10 $\mathbb{E}[W_t-W_s|\mathcal{F}_s^0[W_.]]=\mathbb{E}[W_t-W_s]=0$. As $\mathbb{E}[-2W_tW_s|\mathcal{F}_s^0[W_.]]=-2W_s^2$, $\mathbb{E}\left[W_t^2-W_s^2|\mathcal{F}_s^0[W_.]\right]=\mathbb{E}\left[(W_t-W_s)^2\,|\mathcal{F}_s^0[W_.]\right]=\mathbb{E}\left[(W_t-W_s)^2\right]=t-s$.

1.2.11 (i) $\Longrightarrow$ (ii): The independence of the increments gives

$$\begin{aligned}\mathbb{E}\big[M_t^z-M_s^z|\mathcal{F}_s^0[X.]\big]&=\mathbb{E}\Big[\big(e^{z(X_t-X_s)-z^2(t-s)/2}-1\big)e^{zX_s-z^2s/2}\big|\mathcal{F}_s^0[X.]\Big]\\&=\mathbb{E}\Big[\big(e^{z(X_t-X_s)-z^2(t-s)/2}-1\big)\Big]\mathbb{E}\Big[e^{zX_s-z^2s/2}\Big]\,.\end{aligned}$$

Now
$$\begin{aligned}\mathbb{E}\Big[e^{zX_s-z^2s/2}\Big]&=\frac{1}{\sqrt{2\pi\,s}}\int e^{zx-z^2s/2}e^{-x^2/2s}\,dx\\&=\frac{1}{\sqrt{2\pi\,s}}\int e^{-(x-zs)^2/2s}\,dx=1\,,\end{aligned}$$

so $\mathbb{E}\big[M_t^z-M_s^z|\mathcal{F}_s^0[X.]\big]=0$.

(ii) $\Longrightarrow$ (iii) is obvious; and a computation as above shows that if (iii) is satisfied, then
$$\mathbb{E}\Big[e^{i\alpha(X_t-X_s)+\alpha^2(t-s)/2}\big|\mathcal{F}_s^0[X.]\Big]=1\,.$$
This clearly implies that the increment X_t-X_s is independent of all previous increments and that its characteristic function is $e^{-\alpha^2(t-s)/2}$. This identifies X_t-X_s as a normal random variable with mean zero and variance $t-s$.

1.2.12 The Borel functions ϕ for which this equation holds form a vector space that contains the constants and is closed under pointwise limits of bounded sequences. Thanks to exercise A.3.5 it suffices to establish the claim for functions ϕ of the form $\phi(y)=\exp(i\alpha y)$. These form a multiplicative class that generates the Borel σ-algebra on $\mathbb{R}$. For such ϕ the right-hand side can be evaluated; by exercise A.3.45 on page 419 it equals

$$\exp\big(-\alpha^2(t-s)/2\big)\cdot\exp(i\alpha W_s)\,. \qquad (*)$$

For any $\mathcal{F}_s^0[W_.]$-measurable bounded random variable F

$$\begin{aligned}\mathbb{E}\left[e^{i\alpha W_t}F\right]&=\mathbb{E}\left[e^{i\alpha(W_t-W_s)}Fe^{i\alpha W_s}\right]=\mathbb{E}\left[e^{i\alpha(W_t-W_s)}\right]\mathbb{E}\left[Fe^{i\alpha W_s}\right]\\&=e^{-\alpha^2(t-s)/2}\cdot\mathbb{E}\left[Fe^{i\alpha W_s}\right].\end{aligned}$$

Integrating $(*)$ against F yields the same. Thus the two $\mathcal{F}_s^0[W_.]$-measurable random variables of the statement are a.s. the same.

1.2.14 (ii): To study the joint law of the increments

$$t_1\cdot W_{1/t_1}-t_0\cdot W_{1/t_0},\ldots,t_n\cdot W_{1/t_n}-t_{n-1}\cdot W_{1/t_{n-1}}$$

use the characteristic function:

$$\begin{aligned}
&\mathbb{E}\Big[e^{i\sum_{k=1}^{n}\alpha_k\cdot t_k W_{1/t_k}-t_{k-1}W_{1/t_{k-1}}}\Big]\\
&=\mathbb{E}\Big[e^{i\alpha_1\cdot t_1 W_{1/t_1}-t_0 W_{1/t_0}}\Big]\times\mathbb{E}\Big[e^{i\sum_{k=2}^{n}\alpha_k\cdot t_k W_{1/t_k}-t_{k-1}W_{1/t_{k-1}}}\Big]\\
&=\mathbb{E}\Big[e^{i\alpha_1\cdot t_0 W_{1/t_1}-t_0 W_{1/t_0}}\cdot e^{i\alpha_1(t_1-t_0)W_{1/t_1}}\Big]\times\mathbb{E}\Big[\dots\Big]\\
&=\mathbb{E}\Big[e^{i\alpha_1\cdot t_0 W_{1/t_1}-t_0 W_{1/t_0}}\Big]\cdot\mathbb{E}\Big[e^{i\alpha_1(t_1-t_0)W_{1/t_1}}\Big]\times\mathbb{E}\Big[\dots\Big]\\
&=e^{-\alpha_1^2\cdot t_0^2\cdot(\frac{1}{t_0}-\frac{1}{t_1})}\cdot\mathbb{E}\Big[e^{-\alpha_1^2\cdot\frac{(t_1-t_0)^2}{t_1}}\Big]\times\mathbb{E}\Big[\dots\Big]\\
&=e^{-\alpha_1^2\cdot(t_1-t_0)}\times\mathbb{E}\Big[\dots\Big]
\end{aligned}$$

leads by induction to

$$\mathbb{E}\Big[e^{i\sum_{k=1}^{n}\alpha_k\cdot t_k W_{1/t_k}-t_{k-1}W_{1/t_{k-1}}}\Big]=e^{-\sum_{k=1}^{n}\alpha_k^2\cdot(t_k-t_{k-1})}\ ,$$

which is the characteristic function of the joint distribution of the increments $W_{t_1}-W_{t_0},\dots,W_{t_n}-W_{t_{n-1}}$ of a standard Wiener process. We conclude that $W'_t\overset{\text{def}}{=}tW_{1/t}$ has independent stationary increments with law $N(0,t-s)$. Import $W'_t\xrightarrow[t\to 0]{}0$ from exercise 2.5.21 (ii). Setting $W'_0\overset{\text{def}}{=}0$ we obtain a process W' with almost surely continuous paths, a standard Wiener process (definition 1.2.3).

1.2.15 (i): Take the product of d independent copies of a standard one-dimensional Wiener process. (ii): Every component W^η is a standard one-dimensional Wiener process. (v): Exercise 1.2.10: $\mathbb{E}[W_t^\eta|\mathcal{F}_s^0[W_.]]=W_s^\eta$ and

$$\mathbb{E}\left[W_t^\eta W_t^\theta-W_s^\eta W_s^\theta|\mathcal{F}_s^0[W_.]\right]=(t-s)\cdot\delta^{\eta\theta}\ .$$

Exercise 1.2.11: take $\boldsymbol{z}\in\mathbb{C}^d$ and replace zX by $\langle\boldsymbol{z}|\boldsymbol{X}_t\rangle=\sum_\eta z_\eta X_t^\eta$. Exercise 1.2.12: f is now a Borel function on $\mathbb{R}^d$. The formula reads

$$\begin{aligned}
&\mathbb{E}\left[\phi(\boldsymbol{W}_t)|\mathcal{F}_s^0[W_.]\right]\\
&=\big(2\pi(t-s)\big)^{n/2}\int_{-\infty}^{+\infty}\phi(\boldsymbol{y})\cdot e^{\left(-|\boldsymbol{y}-\boldsymbol{W}_s|^2/2(t-s)\right)}\,dy^1\dots dy^d\ ,
\end{aligned}$$

where $|\ |$ is the euclidean norm on $\mathbb{R}^d$. The proof of exercise 1.2.12 given in this appendix applies literally, if the function $\phi(y)$ is read to mean $\phi(\boldsymbol{y})=\exp(i\langle\boldsymbol{\alpha}|\boldsymbol{y}\rangle)$, etc.

1.2.16 In the proof of theorem 1.2.2 replace $\{\phi_n\}$ by a basis of the Hilbert space of Lebesgue square integrable functions on $\check{\boldsymbol{H}}$. The estimate (1.2.2) that was used in the application on page 14 of Kolmogorov's lemma A.2.37 can be replaced by

$$\mathbb{E}\Big[\big|W_{z_2}-W_{z_1}\big|^6\Big]\le const\cdot n^3\|z_2-z_1\|_\infty^3\quad\text{if}\ \ \|z_i\|_\infty\le n\ .$$

1.3.2 Apply theorem A.5.10.
1.3.6 Let $A = \big[W^\star_\infty < c\big]$. Then A lies in the intersection of the sets $A_n = \big[|W_n - W_{n-1}| < 2c\big]$, each of which has the same probability $q < 1$. Since the A_n form an independent collection, $\mathbb{P}[A] \le q^N$ for all $N \in \mathbb{N}$. Consequently

$$\mathbb{P}\big[W^\star_\infty < \infty\big] \le \sum_{c\in\mathbb{N}} \mathbb{P}\big[W^\star_\infty < c\big] = 0 \,.$$

1.3.10 $Z_{T\vee t} - Z_T = (Z - Z^T)_t$.
1.3.15 (ii): $[\inf_{n\le N} T_n \le t] = \bigcup_{n\le N}[T_n \le t]$. $[\sup_{n\in\mathbb{N}} T_n > t] = \bigcup_{n\in\mathbb{N}}[T_n > t] \in \mathcal{F}_t$.
1.3.16 (i): $A \in \mathcal{F}_S$ implies $A \cap [T \le t] = \big(A \cap [S \le t]\big) \cap [T \le t] \in \mathcal{F}_t$.
(ii): $[S < T] \cap [T \le t] = \bigcup_{q\in\mathbb{Q}}[S \le q] \cap [q < T \le t] \in \mathcal{F}_t$ for all t and thus $[S < T] \in \mathcal{F}_T$. Therefore $[T \le S] = [S < T]^c \in \mathcal{F}_T$ as well. $[S < T] \cap [S \le t]$ $= \big(([S < T] \cap [T \le t]) \cap [S \le t]\big) \cup \big([T > t] \cap [S \le t]\big) \in \mathcal{F}_t$ for all t and thus $[S < T] \in \mathcal{F}_S$ and $[T \le S] \in \mathcal{F}_S$. Finally, $[S < T] = [S\wedge T < T] \in \mathcal{F}_{S\wedge T}$ etc.
1.3.17 $[\![0, T)\!)_t = [T > t] = [T \le t]^c$ belongs to $\mathcal{F}_t$ for all t precisely if T is a stopping time. In that case

$$[\![0, T]\!]_t = [T \ge t] = [T < t]^c = \Big(\bigcup_{\mathbb{Q}\ni q<t} [T \le q]\Big)^c \in \mathcal{F}_t \,.$$

All other stochastic intervals are differences of the two above.
1.3.18 $[T_A \le t] = A \cap [T \le t] \in \mathcal{F}_t \quad \forall\, t$.
1.3.20 $\forall t \ge 0\ \exists k \in \mathbb{N}$ with $k/n \le t < (k+1)/n$. Then $[T^{(n)} \le t]$ $= [T \le k/n] \in \mathcal{F}_t$.
1.3.21 (i) and (ii): Adapt the proof of theorem 2.4.4 on page 69, but note that $\sum_n \Delta X_{T_n} \cdot [\![T_n]\!]$ will generally not converge. (iii): Define inductively $T^{j+1} = \inf\{s > T^j : |h(\Delta X_s)| \ge \epsilon\}$ and set $J^\epsilon \overset{\text{def}}{=} \sum_j h(\Delta X_{T^j})[\![T^j, \infty)\!)$. Clearly $J^\epsilon_{\bullet}$ is adapted, and so is $J_{\bullet} = \lim_{\epsilon\to 0} J^\epsilon_{\bullet}$.
1.3.27 (i): Suppose the stopping times S, T agree almost surely. The set $[S < T] = \bigcup\{[S < q < T] : q \in \mathbb{Q}\}$ belongs to $\mathcal{A}_{\infty\sigma}$ and is negligible, so it is nearly empty.
(ii): $N = \big(N \cap [T = \infty]\big) \cup \bigcup_n N \cap [T \le n]$.
1.3.28 Let X, Y be adapted right-continuous processes. The set where their paths differ: $[X_{\bullet} \ne Y_{\bullet}] = \{\omega \in \Omega : \exists t$ with $X_t(\omega) \ne Y_t(\omega)\} = \bigcup_{q\in\mathbb{Q}}[X_q \ne Y_q]$ then belongs to $\mathcal{A}_{\infty\sigma}$; if X, Y are modifications of each other, it is negligible.
1.3.30 (i): If T is a stopping time, then $[T < t] = \bigcup_n [T \le (t - 1/n) \vee 0]$ $\in \mathcal{F}_t$. Conversely, if $[T < t] \in \mathcal{F}_t \ \ \forall\, t > 0$, then $[T \le t] = \bigcap_n [T < t + 1/n] \in \bigcap_n \mathcal{F}_{t+1/n} = \mathcal{F}_{t+} = \mathcal{F}_t$.

(ii): $[T < t] = \bigcup\{\,[Z_q > \lambda] : \mathbb{Q} \ni q < t\,\}$. The $T^{\lambda+}$ do not change if Z_t is replaced with $Z^+_t \overset{\text{def}}{=} \sup\{Z_s : s \le t\}$. We may thus assume Z is increasing. If $T^{\lambda+} < t$, then $Z_t > \lambda$ and consequently $Z_t > \mu$ for some $\mu > \lambda$; that is to say, $T^{\mu+} \le t$. Thus $\inf_{\mu>\lambda} T^{\mu+} = T^{\lambda+}$, which says that $\lambda \mapsto T^{\lambda+}$ is right-continuous.

(iv): $[T < t] = \bigcup_n [T_n < t]$; and $A \in \bigcap_n \mathcal{F}_{T_n} \implies A \cap [T_n < t] \in \mathcal{F}_t \forall n \forall t$ $\implies A \cap [T < t] \in \mathcal{F}_t \forall t \implies A \cap [T \le t] = A \cap \bigcap_n [T \le t + 1/n] \in \mathcal{F}_{t+} = \mathcal{F}_t$.

(v): If X is left-continuous and adapted to $\mathcal{F}_{.+}$, then $X_{t-1/n} \in \mathcal{F}_{(t-1/n)+} \subset \mathcal{F}_t$ and consequently $X_t = \lim X_{t-1/n} \in \mathcal{F}_t$. Next let X be adapted to $\mathcal{F}_.$ and progressively measurable for $\mathcal{F}_{.+}$, and fix an instant $t > 0$. Evidently

$$\begin{aligned} X^t &= X \cdot [\![0, t)\!) + X_t \cdot [\![t, \infty)\!) \\ &= \lim_{1/t < n \to \infty} X \cdot [\![0, t - 1/n)\!) + X_t \cdot [\![t, \infty)\!) \\ &= \lim_n X^{t-1/n} \cdot [\![0, t - 1/n)\!) + X_t \cdot [\![t, \infty)\!) \ . \end{aligned}$$

The second summand is clearly measurable on $\mathcal{B}^\bullet[0,\infty) \otimes \mathcal{F}_t$. By the above $X^{t-1/n}$ is measurable on $\mathcal{B}^\bullet[0,\infty) \otimes \mathcal{F}_{(t-1/n)+}$, which is contained in $\mathcal{B}^\bullet[0,\infty) \otimes \mathcal{F}_t$. The limit X^t is then also measurable on this product. Since this true for all $t > 0$, X is progressively measurable for $\mathcal{F}_.$.

1.3.31 Let $\mathbb{P} \in \mathfrak{P}$ and $A, A^{(1)}, A^{(2)}, \ldots \in \mathcal{F}_t^{\mathbb{P}}$. Let $\mathcal{N}^{\mathbb{P}}$ denote the $\mathbb{P}$-nearly empty sets. There are sets $A_{\mathbb{P}}, A_{\mathbb{P}}^{(1)}, A_{\mathbb{P}}^{(2)}, \ldots \in \mathcal{F}_t$ with $|A - A_{\mathbb{P}}| \in \mathcal{N}^{\mathbb{P}}$ and $\left|A^{(i)} - A_{\mathbb{P}}^{(i)}\right| \in \mathcal{N}^{\mathbb{P}}$, $i = 1, 2, \ldots$. Evidently $|A^c - A_{\mathbb{P}}^c| \in \mathcal{N}^{\mathbb{P}}$, showing that $\mathcal{F}_t^{\mathbb{P}}$ is closed under taking complements. Similarly, $\left|\bigcup_i A^{(i)} - \bigcup_i A_{\mathbb{P}}^{(i)}\right| \le \sum \left|A^{(i)} - A_{\mathbb{P}}^{(i)}\right| \in \mathcal{N}^{\mathbb{P}}$, showing that $\mathcal{F}_t^{\mathbb{P}}$ is closed under taking countable unions: $\mathcal{F}_t^{\mathbb{P}}$ is a σ-algebra.
Suppose $A \in \mathcal{F}_\infty^*$ is such that there exists an $A_{\mathbb{P}} \in \mathcal{F}_t$ with $|A - A_{\mathbb{P}}| \in \mathcal{N}^{\mathbb{P}}$. Then $A_{\mathbb{P}} \setminus A$ and $A \setminus A_{\mathbb{P}}$ belong to $\mathcal{F}_\infty^*$ and are $\mathbb{P}$-nearly empty. Then $A = (A_{\mathbb{P}} \setminus (A_{\mathbb{P}} \setminus A)) \cup (A \setminus A_{\mathbb{P}})$ belongs to the σ-algebra generated by $\mathcal{F}_t$ and the $\mathbb{P}$-nearly empty sets. This σ-algebra on the other hand is clearly contained in $\mathcal{F}_t^{\mathbb{P}}$, since a $\mathbb{P}$-nearly empty set evidently belongs to it.

1.3.33 Consider the collection $\widetilde{\mathcal{F}}_t^{\mathbb{P}}$ of random variables that satisfy this condition. If $f^{(n)} \in \widetilde{\mathcal{F}}_t^{\mathbb{P}}$ converge pointwise on Ω to f and $f_{\mathbb{P}}^{(n)} \in \mathcal{F}_t$ differ only on a $\mathbb{P}$-nearly empty set from $f^{(n)}$, then set $f_{\mathbb{P}} \stackrel{\text{def}}{=} \limsup f_{\mathbb{P}}^{(n)}$. This random variable is measurable on $\mathcal{F}_t$, and the set $N \stackrel{\text{def}}{=} \left[f \ne f_{\mathbb{P}}\right]$ is contained in the union of the sets $\left[f^{(n)} \ne f_{\mathbb{P}}^{(n)}\right]$, each of which is covered by a countable family of $\mathbb{P}$-negligible sets of $\mathcal{A}_\infty$. Then so is N: $\widetilde{\mathcal{F}}_t^{\mathbb{P}}$ is sequentially closed, and therefore contains the σ-algebra generated by $\mathcal{F}_t$ and the $\mathbb{P}$-nearly empty sets, and every random variable measurable thereon. The converse $\widetilde{\mathcal{F}}_t^{\mathbb{P}} \subset \mathcal{F}_t^{\mathbb{P}}$ is obvious.

1.3.34 (ii): Suppose f satisfies this condition. Given $\mathbb{P} \in \mathfrak{P}$ we can find an $\mathcal{F}_t$-measurable random variable $f_{\mathbb{P}}$ such that $N \stackrel{\text{def}}{=} [f \ne f_{\mathbb{P}}]$ is $\mathbb{P}$-nearly empty. For any $r \in \mathbb{R}$, $\big|[f < r] - [f_{\mathbb{P}} < r]\big|$ is a set of $\mathcal{F}_\infty^*$ contained in N, therefore $[f < r] \in \mathcal{F}_t^{\mathbb{P}}$.
Conversely, assume f is $\mathcal{F}_t^{\mathfrak{P}}$-measurable. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ set

$$f_n = \sum_{k=-\infty}^{\infty} k 2^{-n} \cdot [k 2^{-n} < f \le (k+1) 2^{-n}]_{\mathbb{P}} \ .$$

These are $\mathcal{F}_t$-measurable functions. Clearly $\limsup f_n$ is $\mathcal{F}_t$-measurable and differs only in a $\mathbb{P}$-nearly empty set from f.

1.3.35 (i): For every rational $q > 0$ let $X'_q \in \mathcal{F}_q$ be a random variable nearly equal to X_q (exercise 1.3.33), and set $X'_t \stackrel{\text{def}}{=} \liminf\{X'_q : \mathbb{Q} \ni q \downarrow t\}$. X' is clearly adapted to $\mathcal{F}_{.+} = \mathcal{F}_{.}$. Outside the nearly empty set $N \stackrel{\text{def}}{=} \bigcup_q [X'_q \neq X_q]$, $X'_{.}$ is right-continuous and agrees with $X_{.}$. The set $[X' \neq X]$ is evidently evanescent. If X is a set, choose the X'_q idempotent. If X is increasing, define $X'_t = \lim_{\mathbb{Q} \ni q' \downarrow t} \sup_{q' \geq q \in \mathbb{Q}} X'_q$ instead.

(ii): The sufficiency of the condition is evident. For the necessity consider the $\mathcal{F}^{\mathbb{P}}_{.}$-adapted right-continuous decreasing process $X \stackrel{\text{def}}{=} [\![0, T)\!)$ (convention A.1.5 and exercise 1.3.17). Let X' be a right-continuous $\mathcal{F}_{.}$-adapted set indistinguishable from X, and consider its "right edge"

$$T' \stackrel{\text{def}}{=} \inf\{t \geq 0 : X'_t = 0\} = \inf\{t : (1 - X')_t \geq 1\} .$$

This is an $\mathcal{F}_{.}$-stopping time (proposition 1.3.11), evidently nearly equal to T. If $A \in \mathcal{F}^{\mathbb{P}}_T$, then the reduction T_A is a stopping time on $\mathcal{F}^{\mathbb{P}}_{.}$. Let T' be an $\mathcal{F}_{.}$-stopping time nearly equal to T. Then $A_{\mathbb{P}} \stackrel{\text{def}}{=} [T' < \infty]$ meets the description of the second claim.

1.3.36 (i): Let $A \in \bigcap_n \mathcal{F}^{\mathfrak{P}}_{t+1/n}$, and let $\mathbb{P} \in \mathfrak{P}$. There exist $A^{(n)} \in \mathcal{F}_{t+1/n}$ that is $\mathbb{P}$-nearly equal to A. Then $A_{\mathbb{P}} \stackrel{\text{def}}{=} \liminf_n A^{(n)} \in \mathcal{F}_{t+}$ is $\mathbb{P}$-nearly equal to A. Conversely, assume $A \in \mathcal{F}^{\mathfrak{P}}_{t+}$. Given $\mathbb{P} \in \mathfrak{P}$ we can find an $A_{\mathbb{P}} \in \mathcal{F}_{t+}$ that is $\mathbb{P}$-nearly equal to A. Therefore $A \in \mathcal{F}^{\mathfrak{P}}_u$ for all $u > t$ and $A \in \mathcal{F}^{\mathfrak{P}}_{t+}$.

1.3.42 Fix an instant t and a $\mathbb{P}$-negligible set $A \in \mathcal{F}_t$. Then $A \cap [T > t] \in \mathcal{F}_T$ is $\mathbb{P}'$-negligible for arbitrarily large stopping times T, and then so is A, *provided we understand "arbitrarily large" to mean arbitrarily large with respect to* $\mathbb{P}'$: for every $\epsilon > 0$ and instant t there is a stopping time T with $\mathbb{P}'[T \leq t] < \epsilon$ so that $\mathbb{P}' \ll \mathbb{P}$ on $\mathcal{F}_T$.

1.3.44 Let Ω be the half-line, let $\mathcal{F}_\infty$ be the σ-algebra of Lebesgue measurable subsets of Ω, and let $\mathbb{P}$ be the restriction of the normal law γ_1 to $\mathcal{F}_\infty$. For $\mathcal{F}_t$ take the σ-algebra generated by the sets in $\mathcal{F}_\infty$ that are contained in $[0, t]$ and the interval (t, ∞). The pairs $(\mathcal{F}_t, \mathbb{P})$ are all complete. If $u > t$, then $\{u\} \in \mathcal{A}_\infty$ is $\mathbb{P}$-nearly empty, yet the outer measure that goes with the pair $(\mathcal{F}_t, \mathbb{P}_{|\mathcal{F}_t})$ does not annihilate $\{u\}$.

1.3.47 (i): $\mathcal{F}^0_t$ is the σ-algebra generated by the functions W_s, $0 \leq s \leq t$ (see page 15). Let $t < u$. We have to show that if f is a bounded function measurable on $\mathcal{F}_{t+} = \bigcap_{t < t' < u} \mathcal{F}_{t'}$, then f is already measurable on $\mathcal{F}^{\mathbb{P}}_t$. That is, we have to exhibit a function $f^{\mathbb{P}} \in \mathcal{F}_t$ that $\mathbb{P}$-nearly equals f. We let $f^{\mathbb{P}}$ be the conditional expectation of f on $\mathcal{F}^0_t$ (theorem A.3.24 on page 407). It suffices to show that $f^{\mathbb{P}}$ differs $\mathbb{P}_u$-negligibly from f ($\mathbb{P}_u$ is of course the restriction of $\mathbb{P}$ to $\mathcal{F}_u$). In terms of the $\mathcal{F}_{t+}$-measurable function $g \stackrel{\text{def}}{=} f - f^{\mathbb{P}}$ this means that the measure $\mu \stackrel{\text{def}}{=} g \cdot \mathbb{P}_u$ vanishes on $\mathcal{F}_u$. Suppose we know that $\int \phi(\omega)\, d\mu(d\omega) = 0$ for every function ϕ of the form (1.2.6):

$$\phi(\omega) = \exp\Big(i \sum_{k=1}^{L} r_k \big(W_{t_k}(\omega) - W_{t_{k-1}}(\omega)\big) \Big) , \qquad (*)$$

$r_k \in \mathbb{R}$, $0 = t_0 < t_1 < \ldots < t_L = u$. Then we notice that the family of such functions ϕ forms a multiplicative class and apply exercise A.3.5: μ vanishes on all bounded functions measurable on the σ-algebra generated by the functions $(*)$; this σ-algebra is evidently $\mathcal{F}_u$; and we are done.
In proving that μ vanishes on the function ϕ of $(*)$, we may assume that t is among the t_k, say $t = t_{K-1}$, and by inserting another point strictly between t and the next instant and renaming that new instant t_K we may assume that t_K is as close to t as we please. We get

$$\begin{aligned}
\int \phi\, d\mu &= \mathbb{E}[g \cdot \phi\, d\mathbb{P}] \\
&= \mathbb{E}\Big[g \cdot e^{\left(i \sum_{k \le K} r_k (W_{t_k} - W_{t_{k-1}})\right)} \cdot e^{\left(i \sum_{K<k\le L} r_k (W_{t_k} - W_{t_{k-1}})\right)} \Big] \\
&= \mathbb{E}\Big[g \cdot e^{\left(i \sum_{k \le K} r_k (W_{t_k} - W_{t_{k-1}})\right)} \Big] \cdot \prod_{K<k\le L} \mathbb{E}\Big[e^{(i r_k (W_{t_k} - W_{t_{k-1}}))} \Big] \\
\text{by A.3.45:} \quad &= \mathbb{E}\Big[g \cdot e^{\left(i \sum_{k \le K} r_k (W_{t_k} - W_{t_{k-1}})\right)} \Big] \cdot \prod_{K<k\le L} e^{\left(-r_k^2 (t_k - t_{k-1})/2\right)} .
\end{aligned}$$

We have used the fact that g is measurable on $\mathcal{F}^{\mathbb{P}}_{t_K}$, and that the following increments are independent of this σ-algebra and of each other. Now as $t_K \downarrow t$, the second factor in the last line stays bounded, and the first factor converges to

$$\mathbb{E}\Big[g \cdot \exp\Big(i \sum_{t_k \le t} r_k (W_{t_k} - W_{t_{k-1}}) \Big) \Big] ,$$

which is zero, since the exponential is $\mathcal{F}_t$-measurable.

(ii): The set $[T^{\pm} = 0]$ belongs to $\mathcal{F}^0_{.+}[W] \subseteq \mathcal{F}^{\mathbb{P}}[W]$. The latter σ-algebra contains only sets nearly equal to either $\emptyset$ or Ω. Thus $\mathbb{P}[T^{\pm} = 0]$ equals either zero or one. If $\mathbb{P}[T^{+} = 0] = 0$, then $\mathbb{P}[T^{-} = 0] = 0$ and vice versa – simply consider that T^{-} is "T^{+} for $-W$." In that case $T^{\pm} > 0$ almost surely and $W^2_{T^{\pm} \wedge t} = 0$ for all t. From exercise 1.2.10 in conjunction with the optional stopping theorem 2.5.22 we see that then $E[T^{\pm} \wedge t] = 0$. In other words, we must have $\mathbb{P}[T^{+} = 0] = \mathbb{P}[T^{-} = 0] = 1$.

2.1.10 (i): If S is an elementary stopping time, then

$$\int X\, dZ^S = \int X \cdot [\![0, S]\!]\, dZ \qquad (*)$$

for all $X \in \mathcal{E}$. Let $t > 0$ and let T_n denote the stopping times of exercise 1.3.20. With $S = T_n \wedge t$ $(*)$ gives

$$\int X\, dZ^{T_n \wedge t} = \int X \cdot [\![0, T_n \wedge t]\!]\, dZ , \qquad X \in \mathcal{E} .$$

As X ranges over the unit ball of $\mathcal{E}$, the integrals on the right stay in some bounded subset of L^p, call it $\mathcal{B}$. As $n \to \infty$, the left-hand side converges to $\int X\, dZ^{T\wedge t}$, due to the right-continuity of the paths. As $X \in \mathcal{E}_1$ varies, all these integrals stay in the closure of $\mathcal{B}$, which is again a bounded set. That is to say, $Z^{T\wedge t} = \left(Z^T\right)^t$ is global L^p-integrator for all t, which means that Z^T is an L^p-integrator.

(ii): $\int X\, dZ^{S\vee T} = [S < T] \cdot \int X\, dZ^S + [T \le S] \cdot \int X\, dZ^T$ for $X \in \mathcal{E}$. As X ranges over the unit ball of $\mathcal{E}$, these integrals stay in a bounded set of L^p. This argument covers the "global" case. Replace in it S with $S \wedge t$ and T with $T \wedge t$ to get the "plain" case.

2.1.13 If such an extension is to exist, then Z must satisfy (RC-0). Namely, $Z_u - Z_t$ equals $\int (\!(t, u]\!]\, dZ$, and if $u \downarrow t$ and consequently $(\!(t, u]\!] \downarrow \emptyset$, this must converge to zero in $L^p(\mathbb{P})$, and thus *a fortiori* in measure.

To see the necessity of (B-p), recall first what it means that a subset $\mathcal{I}$ of L^p is bounded: every neighborhood $\mathcal{V}$ of zero in L^p absorbs $\mathcal{V}$; i.e., there is a scalar r such that $\mathcal{I} \subset r \cdot \mathcal{V}$. Now if (B-p) were violated, there would exist a $Y \in \mathcal{E}_+$ such that the collection $\mathcal{I}$ of integrals $\int X\, dZ$ of elementary integrands X in the ***order interval***

$$[-Y, Y] \stackrel{\text{def}}{=} \{X \in \mathcal{E} : \ |X| \le Y\}$$

were unbounded. For $r = n$ there would be $X_n \in [-Y, Y]$ with $\int X_n\, dZ \notin n\mathcal{V}$. The sequence (X_n/n) of elementary integrands would converge pointwise and dominatedly (by Y) to zero, yet their integrals would stay outside $\mathcal{V}$.

2.2.14 Use exercise 1.3.20 and the argument of proposition 2.2.11.

2.3.3 We use again the set $\mathcal{S}$ of 2.3.2 and set $\boldsymbol{Z}^{\mathcal{S}} \stackrel{\text{def}}{=} \sup\{|\boldsymbol{Z}|_s : s \in \mathcal{S}\}$. For $\lambda < \|\boldsymbol{Z}^{\mathcal{S}}\|_{[\alpha]}$, $T \stackrel{\text{def}}{=} \inf\{s \in \mathcal{S} : |\boldsymbol{Z}|_s > \lambda\} \wedge u$ is strictly less than u on $[\boldsymbol{Z}^{\mathcal{S}} > \lambda]$, a set of probability $> \alpha$ where $|\boldsymbol{Z}|_T \ge \lambda$. We get the inequality

$$\lambda = \lambda \Big\| \left[\boldsymbol{Z}^{\mathcal{S}} > \lambda\right] \Big\|_{[\alpha]} \le \big\| |\boldsymbol{Z}|_T \big\|_{[\alpha]} .$$

With the independent Bernoulli random variables $\epsilon_1, \ldots, \epsilon_d$ of theorem A.8.26,

$$|\boldsymbol{Z}|_T \le K_0 \Big\| \sum\nolimits_\eta Z_T^\eta\, \epsilon_\eta \Big\|_{[\kappa_0; \tau]} ,$$

and with exercise A.8.16: $\quad \lambda \le K_0 \Big\| \Big\| \sum\nolimits_\eta Z_T^\eta\, \epsilon_\eta \Big\|_{[\gamma;\mathbb{P}]} \Big\|_{[\alpha\kappa_0 - \gamma; \tau]} , \qquad \gamma < \alpha\kappa_0 .$

Now $\quad \sum\nolimits_\eta Z_T^\eta\, \epsilon_\eta(t) = \int \sum\nolimits_\eta [\![0, T]\!]\, dZ^\eta ,$

and consequently $\quad \lambda \le K_0 \Big\| ⦚\boldsymbol{Z}^u⦚_{[\gamma]} \Big\|_{[\alpha\kappa_0 - \gamma;\tau]} = K_0 ⦚\boldsymbol{Z}^u⦚_{[\gamma]} .$

Now take the supremum over $\gamma < \alpha\kappa_0$, $\lambda < \|\boldsymbol{Z}^{\mathcal{S}}\|_{[\alpha]}$ and $\mathcal{S} \subset [0, u]$ etc.

2.3.7 Let $X = f_0 \cdot \llbracket 0 \rrbracket + \sum_{n=1}^{N} f_n \cdot (\!(t_n, t_{n+1}\rrbracket$ be an elementary integrand that vanishes past t and has $|X| \le 1$, as in equation (2.1.1) on page 46. Then

$$\begin{aligned}\int X \; d|Z| &= f_0 |Z|_0 + \sum_{n=1}^{N} f_n \Big(|Z|_{t_{n+1}} - |Z|_{t_n} \Big) \\ &= f_0 \operatorname{sgn}(Z_0) \cdot Z_0 + \sum_{n=1}^{N} f_n \operatorname{sgn}(Z_{t_n}) \big(Z_{t_{n+1}} - Z_{t_n} \big) \\ &\quad + \sum_{n=1}^{N} f_n \Big(\big|Z_{t_{n+1}}\big| - \big|Z_{t_n}\big| - \operatorname{sgn}(Z_{t_n}) \big(Z_{t_{n+1}} - Z_{t_n} \big) \Big) \\ &\le \int XY \; dZ + A \,, \end{aligned} \tag{$*$}$$

where
$$Y \overset{\text{def}}{=} \operatorname{sgn}(Z_0) \cdot \llbracket 0 \rrbracket + \sum_{n=1}^{N} \operatorname{sgn}(Z_{t_n}) \cdot (\!(t_n, t_{n+1}\rrbracket$$

is an elementary integrand in $\mathcal{E}_1$ and

$$A \overset{\text{def}}{=} \sum_{n=1}^{N} \Big(\big|Z_{t_{n+1}}\big| - \big|Z_{t_n}\big| - \operatorname{sgn}(Z_{t_n}) \big(Z_{t_{n+1}} - Z_{t_n} \big) \Big)$$

is a sum of positive random variables; indeed, since the absolute value function $|\cdot|$ is convex, $|z_2| - |z_1| - \operatorname{sgn}(z_1)\big(z_2 - z_1\big) \ge 0$ for any two reals z_1, z_2. For the choice $X = \llbracket 0, t \rrbracket$ we actually get the equality $|Z|_t = \int Y \; dZ + A$, whence $A = |Z|_t - \int Y \; dZ$. Now $(*)$ stays when X is replaced by $-X$. Therefore

$$\Big| \int X \; d|Z| \Big| \le \Big| \int XY \; dZ \Big| + A \le \Big| \int XY \; dZ \Big| + \Big| \int \llbracket 0, t \rrbracket \; dZ \Big| - \int Y \; dZ \,,$$

sum of three random variables that all have $\lceil\!\lceil \; \rceil\!\rceil_p$-mean less than $\lceil Z^t \rceil_{\mathcal{I}^p}$.

2.3.8 (ii) For an elementary $\overline{\mathcal{F}}_{\bullet}$-stopping time $\overline{T}$, $\llbracket 0, \overline{T} \rrbracket_{\bullet} \circ \underline{R} \in \mathcal{E}$[1]. Conversely, if T is an elementary $\mathcal{F}_{\bullet}$-stopping time, then for every one of its values t_i there is an $\overline{A}_i \in \overline{\mathcal{F}}_{t_i}$ so that $[T = t_i] = \overline{A}_i \circ \underline{R}$. Then $\overline{T} \overset{\text{def}}{=} \sum_i t_i \cdot \overline{A}_i$ is an elementary stopping time on $\overline{\mathcal{F}}_{\bullet}$ with $T = \overline{T} \circ \underline{R}$ and $\llbracket 0, T \rrbracket_{\bullet} = \llbracket 0, \overline{T} \rrbracket_{\bullet} \circ \underline{R}$. Taking differences and linear combinations as in equation (2.1.4) we see that $\mathcal{E}$ consists exactly of the processes $\overline{X} \circ \underline{R}$ with $\overline{X} \in \overline{\mathcal{E}}$. The processes $\overline{X}$ with $\overline{X} \circ \underline{R} \in \mathcal{P}$ are evidently sequentially closed; thus $\overline{\mathcal{P}} \circ \underline{R} \subseteq \mathcal{P}$ – here $\overline{\mathcal{P}}$ denotes the predictables for $\overline{\mathcal{F}}_{\bullet}$, of course. Conversely, the processes X of the form $X = \overline{X} \circ \underline{R}$ are also sequentially closed and contain $\mathcal{E}$, so they exhaust $\mathcal{P}$.

(iv) If $N \in \overline{\mathcal{F}}_t$ is $\overline{\mathbb{P}}$-negligible, then $N \circ \underline{R} = \underline{R}^{-1}(N)$ is $\mathbb{P}$-negligible; therefore the inverse image of a $\overline{\mathbb{P}}$-nearly empty set is $\mathbb{P}$-nearly empty, and $\overline{X} \circ \underline{R}$ is

[1] See convention A.1.5 on page 364

$(\mathcal{F}_{\bm{.}}, \mathbb{P})$-previsible provided $\overline{X}$ is $(\overline{\mathcal{F}}_{\bm{.}}, \overline{\mathbb{P}})$-previsible.
(v) Equation (2.3.9) extends to $\lceil\!\lceil \overline{F} \rceil\!\rceil^*_{\overline{Z}\text{–}p} = \lceil\!\lceil \overline{F} \circ \underline{R} \rceil\!\rceil^*_{Z\text{–}p}$ in the two steps of Daniell's up-and-down procedure. This leads directly to the remaining claims.
2.4.3 Hint: Suppose $\,\vert V\vert_{t+} = \inf\{\vert V\vert_u : u > t\} > \vert V\vert_t + \epsilon$. Set $U \stackrel{\text{def}}{=} \{u > t : |V_u - V_t| < \epsilon/2\}$ and pick $u_1 \in U$. There are $\{t = t_0 < t_1 < \ldots < t_{I+1} = u_1\}$ with $\sum_i |V_{t_{i+1}} - V_{t_i}| > \epsilon$. Repeat this with $u_2 = t_1$ etc. and conclude that $V_{t+} = \infty$.
2.4.9 (i): If $T^\lambda < \tau$, then $I_t \ge \lambda$ for some $t < \tau$ and consequently $I_\tau > \lambda$ and $T^{\lambda+} \le \tau$. Thus $T^{\lambda+} \le T^\lambda$. The reverse inequality is obvious. (ii): $[T^{\Lambda+} < t] = \bigcup [T^{\Lambda^{(n)}+} < t]$, so it suffices to consider the case that Λ takes only countably many values λ_n. As $[T^{\lambda_n+} < t] \cap [\Lambda = \lambda_n] \in \mathcal{F}_t$, $[T^{\Lambda+} < t] = \bigcup [T^{\Lambda+} < t] \cap [\Lambda = \lambda_n] \in \mathcal{F}_t$.
2.5.2 Since by exercise A.3.27 (v)

$$\mathbb{E}[M^g_t | \mathcal{F}_s] = \mathbb{E}\Big[\mathbb{E}[g|\mathcal{F}_t] \, |\mathcal{F}_s\Big] = \mathbb{E}[g|\mathcal{F}_s] = M^g_s \,, \qquad s < t \,,$$

M^g is a martingale. Next, there exists a $K \in \mathbb{R}$ such that the function $g_K = (-K \vee g \wedge K)$ has $\| g - g_K \|_{L^1} < \epsilon$. Let $\mathcal{G}$ be any sub-σ-algebra of $\mathcal{F}_\infty$. Then $\mathbb{E}[g_K|\mathcal{G}]$ lies between $-K$ and K and has L^1-distance less than ϵ from g, since by Jensen's inequality A.3.24

$$|\mathbb{E}[g|\mathcal{G}] - \mathbb{E}[g_K|\mathcal{G}]\,| = |\mathbb{E}[g - g_K|\mathcal{G}]\,| \le \mathbb{E}[|g - g_K| \; |\mathcal{G}] \,.$$

Thus the collection $\{\mathbb{E}[g_K|\mathcal{G}] : \mathcal{G} \subset \mathcal{F}_\infty\}$ is uniformly integrable.
2.5.4 See exercises 1.2.10, 1.2.11, and 1.3.47.
2.5.5 Set $g_\infty \stackrel{\text{def}}{=} \mathbb{E}[g| \bigvee \mathfrak{F}]$. Let $\epsilon > 0$ and consider the algebra $\bigcup \mathfrak{F}$. Its step functions are dense in $L^1(\bigvee \mathfrak{F}, \mathbb{P})$. There is such a step function g^ϵ with $\| g_\infty - g^\epsilon \|_1 \le \epsilon$. It is measurable on some $\mathcal{G} \in \mathfrak{F}$. For $\mathcal{G} \subset \mathcal{G}' \in \mathfrak{F}$

$$\left\| g_\infty - g^{\mathcal{G}'} \right\|_1 \le \| g_\infty - g^\epsilon \|_1 + \left\| g^\epsilon - g^{\mathcal{G}'} \right\|_1 \le 2\epsilon \,.$$

2.5.6 Without loss of generality we may assume that both f and f' are bounded (how?). For every $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the σ-algebra generated by the sets $\big[k2^{-n} < f \le (k+1)2^{-n}\big]$, $k = 0, 1, 2, \ldots$, and the $\mathbb{P}$-negligible subsets of $\mathcal{F}$. Both f and f' are measurable on the σ-algebra $\mathcal{F}_\infty$ generated the $\mathcal{F}_n$. $M_n \stackrel{\text{def}}{=} \mathbb{E}[f|\mathcal{F}_n]$ and $M'_n \stackrel{\text{def}}{=} \mathbb{E}[f'|\mathcal{F}_n]$ are uniformly integrable martingales on the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ (example 2.5.2). The hypothesis translates to $M_n = M'_n$ $\mathbb{P}$-almost surely. Since $M_n \to f$ and $M'_n \to f'$ in $L^1(\mathbb{P})$-mean (exercise 2.5.5), $f = f'$ $\mathbb{P}$-almost surely.
2.5.12 (i): By corollary 2.5.11 this condition is necessary. To see that it is sufficient, let $s < t$ be two instants, and let $A \in \mathcal{F}_s$. Form the elementary stopping time $s_A \wedge t$ which equals s on A and t off A (exercise 1.3.18). The given information $\mathbb{E}[M_t - M_{s_A}] = 0$ can be rewritten as

$$\int_A \mathbb{E}\left[M_t|\mathcal{F}_s\right] \, d\mathbb{P} = \int_A M_s \, d\mathbb{P} \,.$$

(ii): Let M, N be supermartingales, $0 \le t$ and $A \in \mathcal{F}_s$. Then

$$\begin{aligned}\int M_t \wedge N_t \cdot A \, d\mathbb{P} &= \int M_t \wedge N_t \cdot [M_s < N_s] A \, d\mathbb{P} + \int M_t \wedge N_t \cdot [N_s \le M_s] A \, d\mathbb{P} \\ &\le \int M_t \cdot [M_s < N_s] A \, d\mathbb{P} + \int N_t \cdot [N_s \le M_s] A \, d\mathbb{P} \\ &\le \int M_s \cdot [M_s < N_s] A \, d\mathbb{P} + \int N_s \cdot [N_s \le M_s] A \, d\mathbb{P} \\ &= \int M_s \wedge N_s \cdot [M_s < N_s] A \, d\mathbb{P} + \int M_s \wedge N_s \cdot [N_s \le M_s] A \, d\mathbb{P} \\ &= \int M_s \wedge N_s \cdot A \, d\mathbb{P} \, .\end{aligned}$$

2.5.14 For the first statement apply exercise 1.3.35 on page 38. Next, since M is uniformly integrable it is L^1-bounded and $M_\infty \stackrel{\text{def}}{=} \lim_n M_n$ exists pointwise. Again since M is uniformly integrable this limit is taken in L^1-mean (theorem A.8.6).

2.5.16 Set $\mathcal{F}_t = \mathcal{F}_n$ and $M_t = M_n$ for $n \le t < n{+}1$ and apply proposition 2.5.13.

2.5.21 (i): Use exercise 1.2.11 and lemma 2.5.18.

(ii): From (i) $\mathbb{P}[\sup_{s<n} W_s/n{>}\beta/n{+}\alpha/2]{\le}e^{-\alpha\beta}$. With $\beta{=}n^{2/3}$ and $\alpha{=}n^{-1/3}$, the first Borel–Cantelli lemma gives $\mathbb{P}[\sup_{s<n} W_s/n > 2n^{-1/3} \; i.o.] = 0$. This implies $\limsup W_t/t \le 0$, and $\liminf W_t/t = -\limsup -W_t/t \ge 0$.

2.5.23 (ii): Let $0 \le s < t$ and $A \in \mathcal{F}_s$. Then

$$\begin{aligned}\mathbb{E}[M_t^T \cdot A] &= \mathbb{E}\big[M_t^T \cdot A \cdot [T \le s]\big] + \mathbb{E}\big[M_t^T \cdot A \cdot [T > s]\big] \\ &= \mathbb{E}\big[M_{T\wedge t} \cdot A \cdot [T \le s]\big] + \mathbb{E}\big[M_{T\wedge t} \cdot A \cdot [T > s]\big] \\ &= \mathbb{E}\big[M_{T\wedge s} \cdot A \cdot [T \le s]\big] + \mathbb{E}\big[M_{s\vee T\wedge t} \cdot A \cdot [T{\wedge}t > s]\big] \\ \text{by theorem 2.5.22:} \quad &= \mathbb{E}\big[M_{T\wedge s} \cdot A \cdot [T \le s]\big] + \mathbb{E}\big[M_s \cdot A \cdot [T{\wedge}t > s]\big] \\ &= \mathbb{E}\big[M_{T\wedge s} \cdot A \cdot [T \le s]\big] + \mathbb{E}\big[M_{T\wedge s} \cdot A \cdot [T > s]\big] \\ &= \mathbb{E}\big[M_{T\wedge s} \cdot A\big] = \mathbb{E}\big[M_s^T \cdot A\big]\end{aligned}$$

and $$\mathbb{E}[M_t^T|\mathcal{F}_s] = M_s^T \, .$$

(iii): Let M be a local martingale. Given a $t \in \mathbb{R}_+$ and $\epsilon > 0$ one can find a stopping time T with $\mathbb{P}[T < t] < \epsilon$ such that M^T is a martingale. Then $T \wedge t$ has $\mathbb{P}[T \wedge t < t] < \epsilon$ and reduces M to a uniformly integrable martingale.

(iv): Let $0 \le s < t < \infty$ and $A \in \mathcal{F}_s$. There exist stopping times T_n that increase without bound and reduce M to martingales. For $m \in \mathbb{N}$ set $A_m = A \cap [s < T_m]$. Then $A_m \in \mathcal{F}_{s\wedge T_m}$ and by Fatou's lemma A.8.7

$$\mathbb{E}[M_t \, A_m] \le \liminf_{n\to\infty} \mathbb{E}[M_{t\wedge T_n} \, A_m]$$

by part (i): $= \mathbb{E}[M_{s\wedge T_m}\, A_m]$

since $s < T_m$ on A_m: $= \mathbb{E}[M_s\, A_m] \le \mathbb{E}[M_s\, A]$.

Hence $\mathbb{E}[M_t\, A] \le \mathbb{E}[M_s\, A]$.

For the last statement of (iv) write

$$\begin{aligned}
\mathbb{E}[M_{S\vee T}] &= \mathbb{E}[M_{S\vee T}[S \le T]] + \mathbb{E}[M_{S\vee T}[S > T]] \\
&= \mathbb{E}[M_T[S \le T]] + \mathbb{E}[M_S[S > T]]
\end{aligned}$$

by 1.3.16 (iv): $= \mathbb{E}\big[\mathbb{E}[M_T|\mathcal{F}_{S\wedge T}][S \le T]\big] + \mathbb{E}\big[\mathbb{E}[M_S|\mathcal{F}_{S\wedge T}][S > T]\big]$

by part (i): $= \mathbb{E}[M_{S\wedge T}[S \le T]] + \mathbb{E}[M_{S\wedge T}[S > T]] = \mathbb{E}[M_{S\wedge T}] = \mathbb{E}[M_0]$

2.5.25 For $X = f_0 \cdot [\![0]\!] + \sum_{n=1}^N f_n \cdot (\!(t_n, t_{n+1}]\!]$ an elementary integrand as in the proof of theorem 2.5.24 we take the square root in

$$\begin{aligned}
\mathbb{E}\Big[\Big(\int X\, dW\Big)^2\Big] &= \mathbb{E}\Big[\Big(\sum_{n=1}^N f_n \cdot (W_{t_{n+1}} - W_{t_n})\Big)^2\Big] \\
&= \mathbb{E}\Big[\sum_{n=1}^N f_n^2 \cdot \big(W_{t_{n+1}} - W_{t_n}\big)^2\Big] \\
&= \mathbb{E}\Big[\sum_{n=1}^N f_n^2 \cdot \big(W_{t_{n+1}}^2 - 2W_{t_{n+1}}W_{t_n} + W_{t_n}^2\big)\Big]
\end{aligned}$$

by 2.5.4 and 2.5.3: $= \mathbb{E}\Big[\sum_{n=1}^N f_n^2 \cdot \big(W_{t_{n+1}}^2 - W_{t_n}^2\big)\Big]$

by 2.5.4: $= \mathbb{E}\Big[\sum_{n=1}^N f_n^2 \cdot \big(t_{n+1} - t_n\big)\Big] = \int\int X_s^2\, ds\, d\mathbb{P}$.

For the second equality chose $X = [\![0, t]\!]$.

2.5.31 From proposition A.8.24

$$A_p^{(2.5.6)} \le \begin{cases} \Big(\dfrac{4p}{(p-1)(2-p)}\Big)^{1/p} & \text{for } 1 < p < 2, \\ 1 & \text{for } p = 2 \\ \Big(\dfrac{3 \cdot 8^{3/2} \cdot p}{p - 3/2} + \dfrac{8^3 \cdot 3 \cdot p}{3 - p}\Big)^{1/p} & \text{for } 2/3 < p < 3 \\ \Big(\dfrac{4p}{p-2}\Big)^{1 - 1/p} & \text{for } 2 < p < \infty. \end{cases}$$

(Since $\lim_{2 \ne p \to 2} A_p \approx 57.8$, this is an unsatisfactory formula; the problem arises to fashion a better one. Using complex interpolation one can show that $A_p \le \big(\frac{16}{p-1}\big)^{2(2-p)}$ for $1 < p < 2$, which has at least the right limit behavior at $p = 2$)

2.5.32 There is a nearly empty set N_1 off which the path $\mathbb{Q} \ni q \mapsto S_q$ has no oscillatory discontinuities (lemma 2.5.27 and lemma 2.3.1). Use this to define $S'_t \stackrel{\text{def}}{=} \lim_{q\downarrow t} S_q$, a supermartingale with right-continuous paths and adapted to $\mathcal{F}^{\mathfrak{P}}_{.+}$. For $\delta > 0$ set $T_\delta \stackrel{\text{def}}{=} \inf\{t : S_t < \delta\}$. This is a $\mathcal{F}^{\mathfrak{P}}_{.+}$-stopping time at which $S'_{T_\delta} \le \delta$. Let $T^{(n)}_\delta$ be the stopping times of exercise 1.3.20 and fix an instant t. By exercise 2.5.12, $\mathbb{E}\big[(S_t - S_{t\wedge T^{(n)}_\delta}) \cdot [T^{(n)}_\delta < t]\big] \le 0$, which in the limit produces $\mathbb{E}\big[S_t \cdot [T_\delta < t]\big] = \mathbb{E}\big[S'_t \cdot [T_\delta < t]\big] \le \mathbb{E}\big[S_{t\wedge T_\delta} \cdot [T_\delta < t]\big] \le \delta \cdot \mathbb{P}[T_\delta < t] \le \delta$ and exhibits $N_2 \stackrel{\text{def}}{=} \bigcap_\delta [T_\delta < t]$ as $\mathbb{P}$-nearly empty, inasmuch as S_t is almost surely strictly positive. Now if the restriction to $\mathbb{Q}$ of $S_{\bullet}(\omega)$ is not bounded away from zero on $[0,t]$, then neither is $S'_{\bullet}(\omega)$ and ω must lie in N_2: $N \stackrel{\text{def}}{=} N_1 \cup N_2$ meets the description.

2.5.33 Let $s < t$ and $A \in \mathcal{F}_s$. There are stopping times U_n that converge almost surely to ∞ and reduce M to martingales, so that $\mathbb{E}[A \cdot (M^{U_n}_t - M^{U_n}_s)] = 0$. Now $M^{U_n}_t \xrightarrow[n\to\infty]{} M_t$ with $|M^{U_n}_t| \le M^\star_t \in L^1$ when M is an L^1-integrator (see theorem 2.3.6 on page 63). Then $\mathbb{E}[A \cdot (M_t - M_s)] = 0$, and we see M is a martingale. The second claim is established similarly.

3.1.2 Let $S^{(n)}, T^{(n)}$ be the stopping times of exercise 1.3.20. Then clearly $X^{(n)} \stackrel{\text{def}}{=} n \cdot ((S^{(n)} \wedge n, T^{(n)} \wedge n]]$ are elementary integrands whose supremum $H = \infty \cdot ((S,T]] \in \mathcal{E}^{\uparrow}_{+}$ majorizes $|F|$. It suffices to show that $\lceil\!\lceil H \rceil\!\rceil^*_{Z\text{–}0} \le \epsilon$. But this is evident: the stochastic integral $f \stackrel{\text{def}}{=} \int X\,dZ$ of any $X \in \mathcal{E}$ with $|X| \le H$ vanishes off $[S < T]$ and therefore has $\lceil\!\lceil f \rceil\!\rceil_0 \le \epsilon$; the supremum of such $\lceil\!\lceil f \rceil\!\rceil_0$ is $\lceil\!\lceil H \rceil\!\rceil^*_{Z\text{–}0}$.

3.1.3 This is immediate from exercise 2.5.25 and Daniell's construction.

3.2.2 For every n there is a countable collection $\{X^{(n,k)}\}$ in $\mathcal{E}$ whose pointwise supremum is $H^{(n)}$. The functions $X^{(n)} \stackrel{\text{def}}{=} \sup_{\nu,k\le n} X^{(\nu,k)} \le H^{(n)}$ belong to $\mathcal{E}$ and increase pointwise to $\sup_n H^{(n)}$. Hence

$$\Big\lceil\!\!\Big\lceil \sup_n H^{(n)} \Big\rceil\!\!\Big\rceil^* = \sup_n \Big\lceil\!\!\Big\lceil X^{(n)} \Big\rceil\!\!\Big\rceil^* \le \sup_n \Big\lceil\!\!\Big\lceil H^{(n)} \Big\rceil\!\!\Big\rceil^* \le \Big\lceil\!\!\Big\lceil \sup_n H^{(n)} \Big\rceil\!\!\Big\rceil^* .$$

3.2.5 Suppose F is evanescent. Then the projection N of $[F \ne 0]$ on Ω is nearly empty. By the regularity of the filtration, $X \stackrel{\text{def}}{=} N \times [0,\infty)$ is an elementary integrand, and evidently $\lceil\!\lceil X \rceil\!\rceil_{Z\text{–}p} = 0$. The countable subadditivity of the mean implies that $\lceil\!\lceil \infty \cdot X \rceil\!\rceil^*_{Z\text{–}p} = 0$, the solidity then that $\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}p} = 0$.

3.2.11 The first statement was done in 3.2.7 if F is everywhere defined.

3.2.12 (ii): Given an $\epsilon > 0$ find $N \in \mathbb{N}$ so that $\sum_{n\ge N} \lceil\!\lceil F_n \rceil\!\rceil^* < \epsilon/2$. Then find $0 < r < 1$ so that $\sum_{n<N} \lceil\!\lceil r \cdot |F_n| \rceil\!\rceil^* < \epsilon/2$. Use the countable subadditivity and solidity to conclude that $\lceil\!\lceil r \cdot \sum |F_n| \rceil\!\rceil^* < \epsilon$.

3.2.15 (i): If $F \in \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]$, then there are elementary integrands X_n with $\lceil\!\lceil F - X_n \rceil\!\rceil^* < 2^{-n}$. The proof of theorem 3.2.10, suitably adapted, shows that the sequence $F_1 = X_1, F_2 = X_2 - X_1, \ldots, F_n = X_n - X_{n-1}, \ldots$ will meet the description. For the converse note that for $X_1, \ldots, X_N \in \mathcal{E}$

$$\left\lceil\!\left\lceil F - \sum_{n=1}^{N} X_n \right\rceil\!\right\rceil^* \le \sum_{n=1}^{N} \lceil\!\lceil F_n - X_n \rceil\!\rceil^* + \sum_{n=N+1}^{\infty} \lceil\!\lceil F_n \rceil\!\rceil^* .$$

Given $\epsilon > 0$, find first N so large that the second sum is less than $\epsilon/2$, then find $X_n \in \mathcal{E}$ such that the (finite) first sum is also less than $\epsilon/2$.

(ii) is plain from (i) and the countable subadditivity of the mean.

(iii): Let $F_n \in \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]_+$ with $\lceil\!\lceil \sum_n F_n \rceil\!\rceil^* < \infty$. Find $X_n \in \mathcal{E}_+$ with $\lceil\!\lceil F_n - X_n \rceil\!\rceil^* \le 2^{-n}$ (why can the X_n be chosen to be positive?). Then

$$\begin{aligned}\left\lceil\!\left\lceil \sum_n X_n \right\rceil\!\right\rceil^* &\le \left\lceil\!\left\lceil \sum_n F_n \right\rceil\!\right\rceil^* + \left\lceil\!\left\lceil \sum_n (X_n - F_n) \right\rceil\!\right\rceil^* \\ &\le \left\lceil\!\left\lceil \sum_n F_n \right\rceil\!\right\rceil^* + 2 < \infty ,\end{aligned}$$

and therefore $\lceil\!\lceil X_n \rceil\!\rceil^* \xrightarrow[n\to\infty]{} 0$.

3.2.16 (i): Let $(X_n), (Y_n)$ be sequences in $\mathcal{E}$ that converge in mean to $F, G \in \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]$, respectively, and let $r, s \in \mathbb{R}$. Then $(rX_n + sY_n)$ converges to $rF + sG$ in mean, since

$$\lceil\!\lceil (rF + sG) - (rX_n + sY_n) \rceil\!\rceil^* \le |r| \lceil\!\lceil F - X_n \rceil\!\rceil^* + |s| \lceil\!\lceil G - Y_n \rceil\!\rceil^* ,$$

which converges to zero as $n \to \infty$. Thus

$$\begin{aligned}\int (rF + sG) = \lim_n \int (rX_n + sY_n) &= \lim_n \left(r \int X_n + s \int Y_n \right) \\ &= r \int F + s \int G .\end{aligned}$$

This shows that the integral is linear. Next let (X_n) be a sequence of elementary integrands with $\lceil\!\lceil F - X_n \rceil\!\rceil^* \xrightarrow[n\to\infty]{} 0$. Then $\lceil\!\lceil X_n \rceil\!\rceil^* \xrightarrow[n\to\infty]{} \lceil\!\lceil F \rceil\!\rceil^*$ by exercise 3.2.9, and so $|\int F\, dZ| = \lim_n |\int X_n\, dZ| \le \lim_n \lceil\!\lceil X_n \rceil\!\rceil^* = \lceil\!\lceil F \rceil\!\rceil^*$.

3.2.30 If not, one of the collections $\mathcal{M}_k = \{A \cap (\!(k-1, k]\!] : A \in \mathcal{M}\}$ would contain uncountably many non-negligible sets. For some $r \in \mathbb{N}$ we would have $\lceil\!\lceil A \rceil\!\rceil^* > 1/r$ for uncountably many $A \in \mathcal{M}_k$, which is impossible since the measure of $(\!(k-1, k]\!]$ is finite.

3.3.3 The General Stone–Weierstraß theorem A.2.2 permits the extension of the bounded linear map $\int \cdot\, dZ : \mathcal{E}^0 \to L^p$ to $\overline{\mathcal{E}}^0$, so that the $\mathcal{A}_t$ and $\mathcal{E}^0$ may be assumed to be both algebras AND vector lattices closed under chopping. The argument of lemma 3.3.1, which does not refer to the nature of the elementary integrands, shows that $\int \cdot\, Z^t$ is σ-additive in probability. The argument of proposition 3.3.2 also carries through, showing that $\int \cdot\, Z$ is a σ-additive vector measure to L^p. Daniell's mean furnishes an extension satisfying the Dominated Convergence Theorem, and therefore integrating every elementary integrand in $\mathcal{E}$.

3.4.1 There is a sequence of elementary integrands X^k with $\lceil\!\lceil X^k \rceil\!\rceil^* < 2^{-k}$ for $k \geq 1$ and $F = \sum_{k\geq 0} X^k$ a.e. and in mean. Then $Y^K \stackrel{\text{def}}{=} \sum_{k\leq K} k|X^k|$ converges to an integrable function G. Set $U \stackrel{\text{def}}{=} [G > M] \leq G/M$. Then $U = \sup_{n,K} 1 \wedge \big(n{\cdot}(Y^K - (Y^K \wedge M))\big) \in \mathcal{E}_+^{\uparrow}$ and $\lceil\!\lceil U \rceil\!\rceil^* < \epsilon$ for a suitable choice of M. On U^c, $\sum_k X^k$ converges uniformly to F.

3.4.7 The class of functions ϕ such that $\phi(F_1, \ldots, F_N)$ is measurable is closed under pointwise limits, by Egoroff's theorem, and contains the continuous functions, by theorem 3.4.6. Thus it contains the smallest class of functions closed under pointwise limits and containing the continuous functions, *viz.* the Borel functions.

3.4.8 Needed

3.5.4 The first statement is clearly true if Z is continuous and adapted. The collection of adapted processes Z such that Z^T is predictable is closed under pointwise limits. If the predictable process V has right-continuous paths of finite variation, then $\left|V\right| = \sup\{\sum_i \left|V^{q_{i+1}} - V^{q_i}\right|\}$, where the supremum is extended over all finite rational partitions $\{0 = q_0 < q_1 < q_2 < \ldots\}$ of $[0, \infty)$.

3.5.5 Let $\epsilon > 0$. There is a K such that $\mathbb{P}[|f| \geq K] \leq \epsilon$. Let us write $f{\cdot}(\!(S,T]\!] \cdot G = G^{(K)} + G'$, with $G^{(K)} \stackrel{\text{def}}{=} f{\cdot}[|f| < K] \cdot (\!(S,T]\!] \cdot G$ and $G' \stackrel{\text{def}}{=} f{\cdot}[|f| \geq K] \cdot (\!(S,T]\!] \cdot G = f \cdot (\!(S_{[|f|\geq K]}, T]\!] \cdot G$. $G^{(K)}$, being Z–0-measurable and majorized by KG, is Z–0-integrable. G' vanishes off the stochastic interval $(\!(S_{[|f|\geq K]}, T]\!]$, whose projection on Ω has measure less than ϵ, and thus has $\lceil\!\lceil G' \rceil\!\rceil^*_{Z\text{–}0} \leq \epsilon$ (exercise 3.1.2). That is to say, $f{\cdot}(\!(S,T]\!] \cdot G$ differs arbitrarily little (by less than ϵ) from a Z–0-integrable process ($G^{(K)}$), so it is Z–0-integrable itself. Furthermore, $\int f{\cdot}(\!(S,T]\!] \cdot G\, dZ \doteq \lim_{K\to\infty} \int G^{(K)}\, dZ$. It suffices to show that $\int G^{(K)}\, dZ \doteq f[|f| < K] \int (\!(S,T]\!] \cdot G\, dZ$; in other words, that equation (3.5.2) holds when f is bounded. If it holds when S and T are elementary, we apply it to the stopping times $S^{(n)} \wedge n, T^{(n)} \wedge n$ and take the limit as $n \to \infty$: we may assume that S, T are elementary. If it holds when f is a set in $\mathcal{F}_S$, then it holds for linear combinations of them and their uniform limits: we may assume in addition that f is a set $A \in \mathcal{F}_S$. In that case the equality in question reads $\int (\!(S_A, T_A]\!] \cdot G\, dZ = \dot{A} \cdot \int (\!(S,T]\!] \cdot G\, dZ$. If $G \in \mathcal{E}$ is an elementary stochastic interval or a linear combination thereof, it is true by inspection; it follows in general by approximation with elementary integrands.

3.5.7 (i): Let $\big(X^{(n)}\big)$ be a sequence in $\mathcal{P}^{\mathbb{P}}$ with pointwise limit X. There are predictable processes $X^{(n)}_{\mathbb{P}}$ such that $N_n \stackrel{\text{def}}{=} \pi_\Omega[X^{(n)} \neq X^{(n)}_{\mathbb{P}}]$ is nearly empty. Set $X_{\mathbb{P}} = \limsup X^{(n)}_{\mathbb{P}}$. This is a predictable process. The projection of $[X \neq X_{\mathbb{P}}]$ on Ω is contained in the union of the N_n and therefore in a negligible set of $\mathcal{A}_{\infty\sigma}$: it is nearly empty itself. In other words, the paths of X and $X_{\mathbb{P}}$ agree outside a nearly empty set.

(ii): Let us start by showing that a measurable (see page 23) evanescent process X is predictable. There is a nearly empty set N such that X vanishes off $\mathcal{N} \stackrel{\text{def}}{=} [0,\infty) \times N$. Now the collection $\mathcal{M}_{\mathcal{N}}$ of processes Y such that $Y \cdot \mathcal{N} \in \mathcal{P}$ is a monotone class and contains every generator of the measurable processes of the form $(s,t] \times A$, $A \in \mathcal{F}^*_\infty$. Indeed, $\big((s,t] \times A\big) \cdot \mathcal{N}$ $= (s,t] \times (A \cap N)$ is in $\mathcal{E}$ on the grounds that the nearly empty set $A \cap N$ belongs to $\mathcal{F}_0 = \mathcal{F}^{\mathbb{P}}_{0+}$. Therefore $\mathcal{M}_{\mathcal{N}}$ contains all measurable processes, in particular X. That is to say, $X \in \mathcal{P}$. Now if X is a measurable previsible process and $X_{\mathbb{P}} \in \mathcal{P}$ cannot be distinguished from X with $\mathbb{P}$, then $X = X_{\mathbb{P}} + \big(X - X_{\mathbb{P}}\big)$ is the sum of two processes in $\mathcal{P}$.

3.5.10 (i): $(t-1/n) \vee 0$ predicts t. $(T+(\epsilon-1/n) :\ n > 1/\epsilon)$ predicts $T+\epsilon$. If $T^1, T^2, \ldots$ are announced by $T^1_n, T^2_n, \ldots$, then $T'_N = \bigvee_{k,n\le N} T^k_n$ predicts $\bigvee_k T^k$, and $T^1_n \wedge \ldots \wedge T^k_n$ predicts $T^1 \wedge \ldots \wedge T^k$. (ii): $0_A \wedge n$ predicts 0_A. $S_A = S \vee 0_A$. If $\big(S^{(n)}\big)$ announces S, then $\big(S^n_{[S^n \le T]} \wedge n\big)$ announces $S_{[S\le T]}$.

3.5.11 It suffices to show that $[\![0,T)\!)$ is predictable; all other intervals in question can be gotten from this one by taking intersections and relative complements with intervals then known to be predictable. Let $\big(T_n\big)$ be a sequence of stopping times announcing T and simply observe that $[\![0,T)\!)$ is a simple combination of predictable sets:

$$[\![0,T)\!) = \bigcup_n [\![0,T_n]\!] \setminus [\![T]\!] \,.$$

3.5.19 (i) The set $\{t :\ \Delta V_t \ge \lambda\}$ is left closed, on the grounds that the non-oscillatory nature of the path of $V \in \mathfrak{D}$ prevents it from having an accumulation point. The graph of $T^\lambda_{\Delta V}$ is the intersection of the previsible sets $[\![0,T^\lambda_{\Delta V}]\!]$ and $[\Delta V \ge \lambda]$. The claim follows from theorem 3.5.13. (ii) (See the proof of theorem 2.4.4). For every $i,j \in \mathbb{N}$ define inductively $T^{i,0} = 0$ and

$$T^{i,j+1} = \inf\big\{t > T^{i,j} :\ \Delta I_t \ge 1/i\big\} \ .$$

From exercise 3.5.19 we know that the $T^{i,j}$ are predictable stopping times. They are countable in number, so we count them: $\big\{T^{i,j}\big\} = \big\{T'_1, T'_2, \ldots\big\}$. The T'_n do not have disjoint graphs, of course, so we force the issue: since $P_n \stackrel{\text{def}}{=} [\![T_n]\!] \setminus \bigcup_{\nu<n} [\![T'_\nu]\!]$ is previsible, the reduction T_n of T'_n to $\bigcap_{\nu<n} [T_\nu \ne T_n]$, which has P_n for its graph, is a predictable stopping time (see theorem 3.5.13). The T_n meet the claim.

3.5.20 Let $\big(S_n\big)$ be a sequence of stopping times announcing T. By Doob's optional stopping theorem 2.5.22,

$$M_{T-} = \lim M_{S_n} = \lim \mathbb{E}\big[M_\infty | \mathcal{F}_{S_n}\big]$$

both almost surely and in L^1-mean. For every n and $A \in \mathcal{F}_{S_n}$ the integrals of M_∞ and M_{T-} over A agree. Since $\mathcal{F}_{T-} = \bigvee \mathcal{F}_{S_n}$, these integrals agree also if taken over any set of $\mathcal{F}_{T-}$.

3.6.8 $X\widetilde{F} \geq XF$ is predictable, so $X\widetilde{F} \geq \widetilde{XF}$ a.e. By the same argument, $X^{-1}\widetilde{XF} \geq \widetilde{\widetilde{F}}$. Division by zero does not really occur (why?).

3.6.13 Let $T^{(n)}$ be the stopping times of exercise 1.3.20. The equality

$$\int X \; dZ^{T^{(n)}\wedge k} = \int X \cdot [\![0, T^{(n)} \wedge k]\!] \; dZ$$

holds trivially for all $X \in \mathcal{E}$ and all $k, n \in \mathbb{N}$. By right-continuity we have

$$\int X \; dZ^{T\wedge k} \doteq \int X \cdot [\![0, T \wedge k]\!] \; dZ$$

for all $X \in \mathcal{E}$. Consider the collection of all predictable X for which the equality above obtains. By the Dominated Convergence Theorem this collection is closed under limits of bounded sequences and therefore contains all bounded predictable processes. Letting $k \to \infty$ shows that for all $X \in \mathcal{P}_{00}$

$$\int X \; dZ^T = \int X \cdot [\![0, T]\!] \; dZ \; . \qquad (*)$$

Now let $\widetilde{G}$ be a Z-envelope of $|G|$. Choose it so that it is also a Z^T-envelope of $|G|$. Then by exercise 3.6.8

$$\lceil\!\lceil G \cdot [\![0, T]\!] \rceil\!\rceil^*_{Z\text{–}p} = \left\lceil\!\left\lceil \widetilde{G} \cdot [\![0, T]\!] \right\rceil\!\right\rceil^*_{Z\text{–}p}$$

by corollary 3.6.10: $$= \sup\left\{ \left\lceil\!\left\lceil \int Y \; dZ \right\rceil\!\right\rceil_p \; : \; Y \in \mathcal{P}_{00}, \; |Y| \leq \widetilde{G} \cdot [\![0, T]\!] \right\}$$

by $(*)$: $$= \sup\left\{ \left\lceil\!\left\lceil \int Y \; dZ^T \right\rceil\!\right\rceil_p \; : \; Y \in \mathcal{P}_{00}, \; |Y| \leq \widetilde{G} \right\}$$

$$= \lceil\!\lceil G \rceil\!\rceil^*_{Z^T\text{–}p} \; .$$

The rest is even simpler. Suppose, say, $G \cdot [\![0, T]\!]$ is Z–p-integrable. There is a sequence of elementary integrands $X^{(n)}$ with $\lceil\!\lceil G \cdot [\![0, T]\!] - X^{(n)} \rceil\!\rceil^*_{Z\text{–}p} \to 0$. Then $\lceil\!\lceil G - X^{(n)} \rceil\!\rceil^*_{Z^T\text{–}p} = \lceil\!\lceil (G - X^{(n)}) \cdot [\![0, T]\!] \rceil\!\rceil^*_{Z\text{–}p} \to 0$ and $\int G \; dZ^T = \lim \int X^{(n)} \; dZ^T = \lim \int X^{(n)} \cdot [\![0, T]\!] \; dZ$.

3.6.14 The mean $\lceil\!\lceil \;\; \rceil\!\rceil^* \stackrel{\text{def}}{=} \lceil\!\lceil \;\; \rceil\!\rceil^*_{Z\text{–}0} + \lceil\!\lceil \;\; \rceil\!\rceil^*_{Z'\text{–}0}$ majorizes $d(Z + Z')$ on $\mathcal{E}$ and therefore $\lceil\!\lceil \;\; \rceil\!\rceil^*_{(Z+Z')\text{–}0}$ on predictable processes (exercise 3.6.16). Let $\underline{F}, \overline{F}$ be predictable with $\underline{F} \leq F \leq \overline{F}$ and $\lceil\!\lceil \overline{F} - \underline{F} \rceil\!\rceil^*_{Z\text{–}0} = 0$, and let $\underline{F}', \overline{F}'$ be similar, constructed with Z' instead (proposition 3.6.6 (ii)). Set $\overline{\Phi} \stackrel{\text{def}}{=} \overline{F} \wedge \overline{F}'$ and $\underline{\Phi} \stackrel{\text{def}}{=} \underline{F} \wedge \underline{F}'$. Then $\lceil\!\lceil \overline{\Phi} - \underline{\Phi} \rceil\!\rceil^* = 0$ and therefore $\lceil\!\lceil \overline{\Phi} - F \rceil\!\rceil^*_{(Z+Z')\text{–}0} = 0$. If F is integrable for both Z and Z', then $\overline{\Phi}$ is finite for $\lceil\!\lceil \;\; \rceil\!\rceil^*_{Z\text{–}0}$ and $\lceil\!\lceil \;\; \rceil\!\rceil^*_{Z'\text{–}0}$, therefore for $\lceil\!\lceil \;\; \rceil\!\rceil^*$ and for $\lceil\!\lceil \;\; \rceil\!\rceil^*_{(Z+Z')\text{–}0}$: it and with it F is $(Z + Z')$–0-integrable; etc.

3.6.16 If $\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}p} > a$, then $\lceil\!\lceil \int Y \; dZ \rceil\!\rceil_{L^p} > a$ for some Y as in corollary 3.6.10. Such Y is integrable for any mean; there is a sequence $(X^{(n)})$ of elementary integrands converging in the mean $\lceil\!\lceil\ \rceil\!\rceil^* + \lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$ to Y. Then $\lceil\!\lceil F \rceil\!\rceil^* \ge \lceil\!\lceil Y \rceil\!\rceil^* = \lim_n \lceil\!\lceil X^{(n)} \rceil\!\rceil^* \ge \lim_n \lceil\!\lceil X^{(n)} \rceil\!\rceil^*_{Z\text{–}p} > a$.

3.6.18 Start with $p \ge 1$. Consider pairs (A, μ_A) consisting of a Z-non-negligible predictable Z–p-integrable set A and a positive σ-additive measure μ_A that satisfies $|\mu(X)| \le \lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}p}$ and

$$\mu_A(P) = 0 \Longleftrightarrow \lceil\!\lceil P \rceil\!\rceil^*_{Z\text{–}p} = 0 \qquad \forall P \in \mathcal{P}.$$

A maximal collection of such pairs with mutually disjoint first entries is at most countable, so we write it $\{(A^{(1)}, \mu_{A^{(1)}}), \ldots\}$. The complement B of $\bigcup_k A_k$ is Z-negligible; if it were not, then the Hahn–Banach theorem A.2.25 would provide a linear functional μ on $L^{(1)}[Z\text{–}p]$ with $\mu(B) > 0$. Such μ would be a measure on $\mathcal{P}$, and it could be chosen positive. It is not hard to restrict μ to a subset $A^{(0)}$ of B such that $(A^{(0)}, \mu_{|A^{(0)}})$ could be adjoined to the supposedly maximal collection. $\mu := \sum_k 2^{-k}\mu_{A_k}$ meets the description. If $0 \le p < 1$, then theorem 4.1.2 provides a probability $\mathbb{P}' \approx \mathbb{P}$ for which Z is an L^2-integrator. The measure μ produced above in this situation is a control measure for Z.

3.7.2 $T^{(1)} \vee T^{(2)}$ reduces Z to an $L^p(\mathbb{P})$-integrator. (exercise 2.1.10). We may assume without loss of generality that G is positive. By exercise 3.6.13 the $G \cdot (\!(S^{(i)}, T^{(i)}]\!]$ are $Z^{T^{(1)}\vee T^{(2)}}$–p-integrable and then so is their supremum $G \cdot (\!(S^{(1)} \wedge S^{(2)}, T^{(1)} \vee T^{(2)}]\!]$.

3.7.8 For every $\mathbb{P} \in \mathfrak{P}$ let $\mathbb{P}\text{–}\int G\,dZ$ denote the stochastic integral computed in $L^0(\mathbb{P})$. Let Ξ denote the collection of bounded predictable processes X such that there exists a right-continuous process $X{*}Z$ with left limits and with $(X{*}Z)_t \in \mathbb{P}\text{–}\int X \cdot [\![0, t]\!]\,dZ$ for all $\mathbb{P} \in \mathfrak{P}$ and all $t > 0$. Clearly $\mathcal{E} \subset \Xi$. Let $\big(X^{(n)}\big)$ be a bounded sequence in Ξ that converges pointwise to some $X \in \Xi$. By lemma 2.3.2 and exercise 3.6.13

$$\left\lceil\!\!\left\lceil \Big(X^{(n)}{*}Z - X^{(m)}{*}Z\Big)^\star_t \right\rceil\!\!\right\rceil_{L^0(\mathbb{P})} \xrightarrow[m,n\to\infty]{} 0$$

for every $\mathbb{P} \in \mathfrak{P}$ and $t > 0$.[2] That is to say, the set of points in Ω where the path of $X^{(n)}{*}Z$ does not converge uniformly on every finite interval belongs to $\mathcal{A}_{\infty\sigma}$ by its description and is $\mathbb{P}$-negligible for every $\mathbb{P} \in \mathfrak{P}$. We set $X{*}Z = 0$ on this set and $X{*}Z = \lim X^{(n)}{*}Z$ elsewhere. Using the regularity of $\mathcal{F}_{\bf .}$ we conclude that $X \in \Xi$: Ξ contains all bounded predictable processes. If X is predictable and locally Z–0;$\mathbb{P}$-integrable for every $\mathbb{P} \in \mathfrak{P}$, we apply the result to $\big((-n) \vee X \wedge n\big) \cdot [\![0, n]\!]$ and take the limit.

3.7.9 There are stopping times T_n increasing to ∞ that reduce M to global L^1-integrators for which G is M^{T_n}–1-integrable (corollary 2.5.29). The situation is reduced to the case that M is a martingale and global L^1-integrator and G is M–1-integrable. We have to show that then $G{*}M$ is

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[2] Previous faulty formula corrected by **Roger Sewell**, `<rfs@cambridgeconsultants.com>`

a martingale, i.e., that $\mathbb{E}\big[\int X\ d(G{*}M)\big] = 0$ for $X \in \mathcal{E}$ with $X_0 = 0$ (see proposition 2.5.10). This is evident if G is elementary, and follows in general by approximating G with elementary integrands in $\lceil\!\lceil\ \rceil\!\rceil^*_{1\text{–}M}$-mean.

3.7.15 Apply exercise 3.6.13 and lemma 3.7.5: for any t

$$(G{*}Z^T)_t \in \int_0^t G\ dZ^T = \int [\![0, T\wedge t]\!] \cdot G\ dZ = \int [\![0, T\wedge t]\!]\ d(G{*}Z)$$
$$\ni (G{*}Z)_{T\wedge t} = (G{*}Z)^T_t\ .$$

Now use the right-continuity of the ultimate right and left-hand sides.

3.7.16 We start with the case that the graph $[\![T]\!]$ is included. Setting $B \stackrel{\text{def}}{=} \big\{(s,\omega) :\ \omega \in \Omega_0,\ s \le T(\omega)\big\}$ write

$$X{*}Z - X'{*}Z' = (X - X'){*}Z + X'{*}(Z - Z')\ . \tag{$*$}$$

It is clear upon inspection that $X{*}(Z - Z') = 0$ on B if X is an elementary integrand. We know from exercise 3.6.14 that X' is $(Z - Z')$–0-integrable. There is a sequence $\big(X^{(n)}\big)$ of elementary integrands such that $X^{(n)}{*}(Z - Z')$ converges to $X'{*}(Z - Z')$ uniformly on bounded intervals, except in a nearly empty set (corollary 3.7.11). We conclude that, on B, $X'{*}(Z - Z')$ is indistinguishable from 0.

To show that the first term of $(*)$ is evanescent on B let $\epsilon > 0$ and set $S = \inf\big\{t :\ \big((X - X'){*}Z\big)^\star_t \ge \epsilon\big\}$. This is a stopping time (proposition 1.3.11). On $\Omega_0 \cap [S \le T]$ the entire path of $(X - X') \cdot [\![0, S]\!]$ vanishes. From corollary 3.7.13 in conjunction with lemma 3.7.5 and exercise 3.6.13 we know that for all t

$$\big((X - X'){*}Z\big)_{S\wedge t} \in \int_0^{S\wedge t} (X - X')\ dZ = \int (X - X') \cdot [\![0, S]\!]\ dZ^t$$

almost surely vanishes on this set. The right-continuity of $(X - X'){*}Z$ shows that the whole path of $(X - X'){*}Z$ vanishes almost surely up to and including time S on $\Omega_0 \cap [S \le T]$. Since $\big((X - X'){*}Z\big)^\star_S \ge \epsilon$ on $[S < \infty]$, this implies that $S > T$ almost surely on Ω_0. We take $\epsilon \to 0$ and conclude that $(X - X'){*}Z$ vanishes almost surely on Ω_0 up to and including time T.

If $[\![T]\!]$ is excluded, we apply the above to $(T - \epsilon) \vee 0$ and let $\epsilon \to 0$.

3.7.18 needed

3.7.19 The first statement was done in exercise 3.5.8. For the second statement note that $(\!(S, T]\!]{*}Z = Z^T - Z^S$ is previsible if S, T are stopping times. Thus $X{*}Z$ is previsible for $X \in \mathcal{E}$. Now approximate the general integrand as in corollary 3.7.11.

3.7.20 (i) To say that $f \cdot G$ is Z–0-integrable on $(\!(S, T]\!]$ means that $f \cdot G \cdot (\!(S, T]\!]$ is Z–0-integrable (exercise 3.6.13). In view of definition 3.7.1 neither hypothesis nor conclusion are changed if we replace G by $G \cdot (\!(S, T]\!]$: we may assume that G vanishes off $(\!(S, T]\!]$ and is Z–0-integrable. Then $f \cdot G$

is Z–0-integrable on $(\!(S,T]\!]$ if and only if $f \cdot (\!(S,T]\!]$ is $G{*}Z$–0-integrable (theorem 3.7.10), which is true because $G{*}Z$ is a global L^0-integrator and $f \cdot (\!(S,T]\!]$ has almost surely finite maximal function at ∞ (theorem 3.7.17). Thus

$$\int_{S+}^{T} f \cdot G \, dZ = \int f \cdot (\!(S,T]\!] \, d(G{*}Z)$$

by exercise 3.5.5: $$= \dot{f} \cdot \big((G{*}Z)_T - (G{*}Z)_S\big)^{\cdot}$$

by exercise 3.6.13: $$= \dot{f} \cdot \int (\!(S,T]\!] \, d(G{*}Z)$$

by theorem 3.7.10: $$= \dot{f} \cdot \int G \cdot (\!(S,T]\!] \, dZ$$

by exercise 3.6.13: $$= \dot{f} \cdot \int_{S+}^{T} G \, dZ \, .$$

(ii) Again we may assume that G vanishes off the predictable interval $[\![S,T]\!]$ and is Z–0-integrable. Again we know a priori that $f \cdot G$ is Z–0-integrable on $[\![S,T]\!]$: it amounts to saying that $f \cdot [\![S,T]\!]$ is $G{*}Z$–0-integrable, which is true for the same reason as above. This allows us to reduce the problem to the case that f is bounded. For if we have the equality (3.7.6) in this case, we apply it to $(-k) \vee f \wedge k$ and let $k \to \infty$. Say $|f| \le k$.
Let $(S^{(n)})$ be an increasing sequence of stopping times announcing S. There is a sequence of $\mathcal{F}_{S^{(n)}}$-measurable functions $f^{(n)}$ with $|f^{(n)}| \le k$ that converges almost surely to f; we can take for $f^{(n)}$ the conditional expectation of f given $\mathcal{F}_{S^{(n)}}$, or by density first find the $f^{(n)}$ so as to converge in $\lceil\!\lceil \ \rceil\!\rceil_0$-mean to f and then go to an almost surely convergent subsequence. Now from (i)

$$\int_{S^{(n)}+}^{T} f^{(m)} \cdot G \, dZ = \dot{f}^{(m)} \cdot \int_{S^{(n)}+}^{T} G \, dZ \, , \qquad m \le n \in \mathbb{N} \, .$$

We let first $n \to \infty$ and then $m \to \infty$ and get the claim.

(iii) G is predictable (proposition 3.5.2) and has almost surely finite maximal function at any finite instant t. It is thus Z–0-integrable on every interval $[\![0,t]\!]$ (theorem 3.7.17) with integral $\int G \cdot [\![0,t]\!] \, dZ \doteq \sum f_k \cdot (Z^t_{S_{k+1}} - Z^t_{S_k})$ (proposition 3.5.2 and theorem 3.2.24).

3.7.21 By corollary A.5.13, $|X^{(n)} - X|^\star_T$ is $\mathcal{F}_T$-measurable. Thus every subsequence of $X^{(n)}$ has a further subsequence $X^{(n_k)}$ with $|X^{(n_k)} - X|^\star_T \xrightarrow[k\to\infty]{} 0$ almost surely. Then $X^{(n_k)} \cdot [\![0,T]\!] \to X \cdot [\![0,T]\!]$ nearly uniformly and thus Z-almost everywhere. By theorem 3.7.17, $X^{(n_k)} \cdot [\![0,T]\!] \to X \cdot [\![0,T]\!]$ in Z–p-mean, by equation (3.7.5) $\lceil X^{(n_k)}{*}Z^T - X{*}Z^T \rceil_{Z\text{–}p} \xrightarrow[k\to\infty]{} 0$, and the claim follows from theorem 2.3.6.

3.7.24 Make $\| (X_{\bullet-} - X^{n_k}) \cdot [\![0,k]\!] \|^*_{Z\text{–}p}$ summable and use Borel–Cantelli.

3.7.25 Let $\mathcal{S}_1, \mathcal{S}_2$ be stochastic partitions. Then $S \overset{\text{def}}{=} \bigcup\{[\![S]\!] : S \in \mathcal{S}_1 \cup \mathcal{S}_2\}$ is progressively measurable. Define by induction $T_1 = \inf\{t : t \in S\}$ and $T_{k+1} \overset{\text{def}}{=} \inf\{t > T_k : t \in S\}$. T_1, being the infimum of the first stopping time of $\mathcal{S}_1$ and the first stopping time of $\mathcal{S}_2$ is a stopping time (exercise 1.3.15). If T_k is a stopping time, then T_{k+1} is the debut of the progressively measurable set $(\!(T_1, \infty)\!) \cap S$ and therefore is a stopping time, by corollary A.5.12. This big result uses the natural conditions, so it is better to argue as follows: $[T_{k+1} \le t] = \bigcup\{[T_k < S \le t] : S \in \mathcal{S}_1 \cup \mathcal{S}_2\}$. Now $[T_k < S] \in \mathcal{F}_S$ (exercise 1.3.16) and so $[T_k < S \le t] = [T_k < S] \cap [S \le t] \in \mathcal{F}_t$. The T_k and $T_\infty \overset{\text{def}}{=} \sup_{k\in\mathbb{N}} T_k$ make up a partition that refines both $\mathcal{S}_1$ and $\mathcal{S}_2$.

3.7.27 Let $R \overset{\text{def}}{=} (X, Z)$, $\underline{R} : \Omega \to \mathscr{D}^2$ the associated representation so that $X = \overline{X} \circ \underline{R}$, $Z = \overline{Z} \circ \underline{R}$. Now define stopping times $\overline{S}_k : \mathscr{D}^2 \to \overline{\mathbb{R}}$ and processes $\overline{Y}^{(\delta)}_t$ on $\mathscr{D}^2$ by (3.7.10) and (3.7.11). Choose δ_n so that $\sum_n f(n\delta_n) < \infty$, let and set $(x_{\bullet-} \circledast z)_{\bullet} \overset{\text{def}}{=} \lim \overline{Y}^{(\delta_n)}_{\bullet}(x_{\bullet}, z_{\bullet})$ where this limit exists uniformly on bounded intervals, $x_{\bullet-} \circledast z \overset{\text{def}}{=} 0$ elsewhere. This produces a map: $\mathscr{D}^2 \to \mathscr{D}$ which is easily seen to be adapted to the canonical (i.e., right-continuous) filtrations on these path spaces. Clearly the version of $X_{\bullet-}{*}Z$ produced by the algorithm (3.7.11) is nothing but the composition $X_{\bullet-} \circledast Z$ of (X, Z) with this map, as long as Z satisfies (3.7.13).

3.7.29 Apply the Borel–Cantelli lemma to this consequence of inequality (3.7.14):

$$\mathbb{P}\Big[\big|X_{\bullet-}{*}Z - Y^{(\delta_n)}\big|^\star_\infty > 1/n\Big] \le \lceil n\delta_n \cdot Z \rceil_{\mathcal{I}^0} \,.$$

3.7.31 (ii) The set $[N(U) > K]$ is the same as $[S_K < U]$, and on it

$$K\delta^2 \le \sum_{k<K} \big(X^U_{S_{k+1}} - X^U_{S_k}\big)^2 \le L^2 \sum_{k<K} \big(X'^U_{S_{k+1}} - X'^U_{S_k}\big)^2 \,.$$

We take the square root and apply corollary 3.1.7, getting

$$\sqrt{K}\delta\big(\mathbb{P}[N > K]\big)^{1/q} \le LK_q^{(\text{A.8.5})} \lceil X'^U \rceil_{\mathcal{I}^q} \,.$$

Raising this to the power q and using the estimate of $K_q^{(\text{A.8.5})}$ from exercise A.8.29 gives the claim.

3.7.32 (i) The approximands of corollary 3.7.11 have continuous paths.

(ii) Let $\big(X^{(n)}\big)$ be a sequence of elementary integrands approximating X in mean, fast enough so that $X^{(n)}{*}Z \to X{*}Z$ uniformly on compacta a.s. (corollary 3.7.11). Since the equation in question holds by inspection for elementary integrands,

$$\Delta(X{*}Z) = \lim \Delta(X^{(n)}{*}Z) = \lim X^{(n)} \cdot \Delta Z = X \cdot \Delta Z \,.$$

(iii) The same argument works if $X^{(n)}$ is chosen as in theorem 3.7.26.

3.7.35 Use theorem A.3.18 on page 403.

3.8.3 Convolving the absolute value function $|\cdot|$ with a suitable approximate identity δ that is of class C^∞, has compact support, and is symmetric about the origin produces C^∞-functions $|\cdot|^{(\delta)}$ that converge uniformly on every compact set in $\mathbb{R}^d$ to $|\cdot|$; also, the gradients of the $|\cdot|^{(\delta)}$ will converge pointwise and dominatedly on every compact to $\nabla_\eta|\cdot|$. Applying theorem 3.8.1 to the $|\cdot|^{(\delta)}$ and taking the limit will result in the claim.

3.8.6 Apply exercise 3.7.19.

3.8.8 For $\eta = 1, \ldots, d$ let $\mathcal{S} = \{0 = S_0 \le S_1 \le S_2 \le \ldots \le \infty\}$ be a random partition[8], and set $f_0^\eta = Z_0^\eta$, $f_1^\eta = Z_{S_1}^\eta - Z_0^\eta$, $f_k^\eta = Z_{S_k}^\eta - Z_{S_{k-1}}^\eta$, $k = 2, 3 \ldots, K$. Enumerate the f's: $f_1, \ldots f_{(K+1)h}$. Let ϵ_ν be the Bernoulli random variables of theorem A.8.26. We have

$$\Big(\sum_{\nu=1}^{(K+1)d} f_\nu^2\Big)^{1/2} \le K_p^{(\mathrm{A.8.5})} \cdot \Big\| \sum_{\nu=1}^{(K+1)d} f_\nu \epsilon_\nu \Big\|_{L^p(\tau)}$$

and so
$$\Big\|\Big(\sum_{\nu=1}^{(K+1)d} f_\nu^2\Big)^{1/2}\Big\|_{L^p(\mathbb{P})} \le K_p \cdot \Big\|\Big\| \sum_{\nu=1}^{(K+1)d} f_\nu \epsilon_\nu \Big\|_{L^p(\tau)}\Big\|_{L^p(\mathbb{P})}$$
$$= K_p \cdot \Big\|\Big\| \sum_{\nu=1}^{(K+1)d} f_\nu \epsilon_\nu \Big\|_{L^p(\mathbb{P})}\Big\|_{L^p(\tau)}$$
$$\le K_p \cdot \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil_{\mathcal{I}^p[\mathbb{P}]} .$$

This is because the sum in the penultimate line is of the form $\int_0^T \langle \boldsymbol{X} | d\boldsymbol{Z} \rangle$ with $\boldsymbol{X} \in \mathcal{E}_1^d$. As the partition $\mathcal{S}$ runs through a sequence whose mesh tends to zero, the left-hand side of this inequality converges to $\big(\sum_{\eta=1}^d [Z^\eta, Z^\eta]_T\big)^{1/2}$. The case $p = 0$ is handled similarly (see the proof of corollary 3.1.7).

3.8.11 Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>`, who saw that the claim needed additional hypotheses and the right constant to become true.

There are arbitrarily large stopping times T_1 such that $M_{.-}^{T_1}$ and $N_{.-}^{T_1}$ are bounded. For instance, the infimum of

$$\inf\{t : |M_t| > k\} \quad \text{and} \quad \inf\{t : |N_t| > k\}$$

tends to ∞ as $k \to \infty$. There are arbitrarily large stopping times T_2 such that both M^{T_2} and N^{T_2} are L^2-integrators. $T \stackrel{\text{def}}{=} T_1 \wedge T_2$ can be made arbitrarily large. The first two terms on the right in

$$M_T \cdot N_T = \int_{0+}^T M_{.-}\, dN + \int_{0+}^T N_{.-}\, dM + [M, N]_T$$

are the values at T of martingales that vanish at $t = 0$. Their expectation is zero. The last claim follows from theorem 2.5.19.

Suppose M is merely a càdlàg local martingale. There are arbitrarily large stopping times T such that both $M^{T}_{.-}$ is bounded and M^T an L^1-integrator (corollary 2.5.29). In the formula

$$M_T^2 = 2\int_{0+}^{T} M_{.-}\, dM + [M,M]_T$$

either side is integrable iff $(\Delta_T M)^2$ is, and if it is not then both sides have expectation ∞. Thus

$$\mathbb{E}\big[M_T^2\big] = \mathbb{E}\big[[M,M]_T\big] \quad \text{and} \quad \mathbb{E}\big[M_T^{\star 2}\big] \le 4\cdot \mathbb{E}\big[[M,M]_T\big]$$

for arbitrarily large stopping times T.

3.8.12 Given $\epsilon > 0$ set $S_0 \stackrel{\text{def}}{=} 0$ and $S_{k+1} \stackrel{\text{def}}{=} T \wedge \inf\{t > S_k : |V_t^c - V_{S_k}^c| \ge \epsilon\}$. For this partition[8]

$$A_T^{\mathcal{T}}[\,\cdot^2; V^c] \le \epsilon \sum_n |V_{S_{k+1}} - V_{S_k}| \le \epsilon \cdot \lvert V \rvert_T\ .$$

Hence $S[{}^{c}V] = 0$ and $S[V] = S[{}^{j}V + {}^{j}V] \le S[{}^{j}V] = S[V - {}^{c}V] \le S[V]$. The second claim follows from the inequality of Kunita–Watanabe.

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` for the following emphasis: The last equality makes sense only if $\int Z\, dV$ is understood as an ordinary pathwise Lebesgue-Stieltjes integral (see page 144). Then $\int_0^t Z\, dV = \int_0^t Z_{.-}\, dV + \int_0^t \Delta Z\, dV = \int_0^t Z_{.-}\, dV + \sum_{0\le s<t} \Delta Z_s \Delta V_s = \int_0^t Z_{.-}\, dV + [Z,V]_t$, and the stated equality follows from the definition (3.8.8) of the square bracket. [By proposition 3.7.33 on page 144, $\int_0^t Z_{.-}\, dV$ can be understood as a Lebesgue Stieltjes integral or a stochastic integral; either way its value is the same.]

3.8.13 Without loss of generality $M_0 = 0$. Since M is continuous and has finite variation, $[M,M] = 0$ (exercise 3.8.12). There are thus arbitrarily large times T with $\mathbb{E}[M_T^2] = 0$ (exercise 3.8.11). By theorem 2.5.19 there are arbitrarily large bounded stopping times T with $\mathbb{E}[M_T^{\star 2}] = 0$: the set $[M^{\star}_{.-} \neq 0]$ is evanescent.

3.8.14

$$\begin{aligned} Y_s Z_s &= \int_{0+}^{s} Y_{.-}\, dZ + \int_{0+}^{s} Z_{.-}\, dY + [Y,Z]_s \\ &= Y_0 Z_0 + \sum_{0\le k<\infty} \big(Y^{S_{k+1}} Z^{S_{k+1}} - Y^{S_k} Z^{S_k}\big)_s \\ &= Y_0 Z_0 + \sum_{0\le k<\infty} \big(Y_s^{S_{k+1}} - Y_s^{S_k}\big)\big(Z_s^{S_{k+1}} - Z_s^{S_k}\big) \\ &\quad + \sum_{0\le k<\infty} Y_{S_k}\big(Z_s^{S_{k+1}} - Z_s^{S_k}\big) + \sum_{0\le k<\infty} Z_{S_k}\big(Y_s^{S_{k+1}} - Y_s^{S_k}\big) \end{aligned}$$

$$= Y_0 Z_0 + \sum_{0 \le k < \infty} \big(Y_s^{S_{k+1}} - Y_s^{S_k}\big)\big(Z_s^{S_{k+1}} - Z_s^{S_k}\big)$$

$$+ \int_{0+}^{s} Y_{.-}^{\mathcal{S}} \, dZ + \int_{0+}^{s} Z_{.-}^{\mathcal{S}} \, dY$$

Subtract the first line from the last and rearrange to obtain

$$[Y, Z]_s - \Big(Y_0 Z_0 + \sum_{0 \le k < \infty} (Y_s^{S_{k+1}} - Y_s^{S_k})(Z_s^{S_{k+1}} - Z_s^{S_k})\Big)$$

$$= \int_0^s (Y^{\mathcal{S}} - Y)_{.-} \, dZ + \int_0^s (Z^{\mathcal{S}} - Z)_{.-} \, dY \, .$$

Now apply theorem 2.3.6 and theorem 3.7.10.

3.8.24 **Roger Sewell**, `<rfs@cambridgeconsultants.com>`, points out that this is put way too cavalierly: "is constant" should be read "has a nearly constant modification," and even this statement needs the natural conditions. Namely, only when the filtration is right continous do proposition 2.5.13 on page 75 or the finite variation furnish an adapted càdlàg modification to which the following arguments apply.

(i) Let Z be a previsible local martingale of finite variation, which in vieew of the natural conditions may be assumed càdlàg. Set $M \stackrel{\text{def}}{=} Z - Z_0$. 2.5.29 and 3.5.16 provide arbitrarily large stopping times T such that M^T is a bounded global L^1-integrator. Since M has finite variation, the continuous bracket $\{M, M\}$ vanishes (exercise 3.8.12) and proposition 3.8.22 results in

$$M_T^2 = \int \big(M + M_{.-}\big) \, dM^T \, .$$

Since M is a martingale, the right-hand side has expectation zero (exercise 3.2.17). By ***Doob***'s maximal theorem 2.5.19, $\mathbb{E}[M_T^{\star 2}] = 0$ for arbitrarily large stopping times T: M is indeed evanescent, and $Z = M_0$ is nearly constant.

(ii) Again 2.5.29 and 3.5.16 provide arbitrarily large stopping times T such that M^T is a global L^1-integrator and $\left\lvert V \right\rvert^T$ is bounded. By exercise 3.8.12

$$[M, V]_S^T - [M, V]_0^T = \sum_{0 < s \le T \wedge S} \Delta M_s \cdot \Delta V_s = \int_{0+}^{S} \Delta V \, dM^T$$

for any stopping time S; since this quantity has expectation zero, $[M, V]^T$ is by proposition 2.5.10 a martingale, and $[M, V]$ is a local martingale.

3.8.25 (i) Linearity and the Dominated Convergence Theorem reduce the situation to the case that f is positive and bounded. By proposition 3.6.6 (ii) there is a positive previsible process $P \le f \cdot [\![T]\!]$ that differs Z–0-negligibly from $f \cdot [\![T]\!]$. By theorem 3.5.13, $[P > 0]$ is the graph of a predictable stopping time S, announced by the sequence (S_n), say. By lemma 3.5.15, P_S is measurable on the strict past $\mathcal{F}_{S-}$ and can be approximated in measure by random variables $f_n \in \mathcal{F}_{S_n}$. Taking a subsequence, we can arrange things so that $f_n \to P_S$ almost surely. Then $f_n \cdot (\!(S_n, T]\!] \to P = P_S \cdot [\![S]\!]$ except on an evanescent set. Taking the limit in

$$\int f_n \cdot (\!(S_n, S]\!] \, dZ = f_n \cdot \big(Z_S - Z_{S_n}\big)$$

results in

$$\int f \cdot [\![T]\!] \, dZ = \int P \, dZ = P_S \cdot \Delta Z_S \ .$$

Now since the L^0-integrators $f \cdot [\![T]\!] * Z$ and $P * Z$ are the same and have the same jumps, and since the jump $P_S \cdot \Delta Z_S$ of the latter vanishes almost surely on $[T < S] = [S = \infty]$, the jump $f \cdot \Delta Z_T$ of the former also vanishes almost surely on this set, and the consequent equality $P_S \cdot \Delta Z_S = f \cdot \Delta Z_T$ results in the claim.

(ii) In the proof of proposition 3.8.22 apply (i) instead of theorem 3.5.13 and continue as in the proof of proposition 3.8.22.

3.9.1 For any $F \in C^2(D)$ set $\overline{F} \stackrel{\text{def}}{=} F \circ \boldsymbol{Z}$, so that $\overline{\Phi} = \Phi(\boldsymbol{Z})$ and $\overline{\Phi_{;\eta}} = \Phi_{;\eta}(\boldsymbol{Z})$, etc. Let $\Phi, \Psi \in \mathcal{I}$. That is to say,

$$d\overline{\Phi}_t = \overline{\Phi_{;\eta}}_{t-} dZ^\eta_t + \frac{1}{2} \overline{\Phi_{;\eta\theta}}_t d[Z^\eta, Z^\theta]^c_t + (\Delta\overline{\Phi}_t - \overline{\Phi_{;\eta}}_{t-} \Delta Z^\eta_t)$$

and
$$d\overline{\Psi}_t = \overline{\Psi_{;\eta}}_{t-} dZ^\eta_t + \frac{1}{2} \overline{\Psi_{;\eta\theta}}_t d[Z^\eta, Z^\theta]^c_t + (\Delta\overline{\Psi}_t - \overline{\Psi_{;\eta}}_{t-} \Delta Z^\eta_t) \ .$$

Hence
$$\begin{aligned} d[\overline{\Phi}, \overline{\Psi}]_t &= d[\overline{\Phi}, \overline{\Psi}]^c_t + \Delta\overline{\Phi}_t \Delta\overline{\Psi}_t \\ &= \overline{\Phi_{;\eta}}_t \overline{\Psi_{;\theta}}_t d[Z^\eta, Z^\theta]^c_t + \Delta\overline{\Phi}_t \Delta\overline{\Psi}_t \end{aligned}$$

and
$$\begin{aligned} d(\overline{\Phi} \cdot \overline{\Psi})_t &= \overline{\Phi}_{t-} d\overline{\Psi}_t + \overline{\Psi}_{t-} d\overline{\Phi}_t + d[\overline{\Phi}, \overline{\Psi}]_t \\ &= \big(\overline{\Psi} \cdot \overline{\Phi_{;\eta}} + \overline{\Phi} \cdot \overline{\Psi_{;\eta}}\big)_{t-} dZ^\eta_t && (*) \\ &+ \frac{1}{2} \big(\overline{\Psi} \cdot \overline{\Phi_{;\eta\theta}} + \overline{\Phi} \cdot \overline{\Psi_{;\eta\theta}} + 2\overline{\Phi_{;\eta}} \cdot \overline{\Psi_{;\theta}}\big)_t d[Z^\eta, Z^\theta]^c_t && (**) \\ &+ \overline{\Psi}_{t-}(\Delta\overline{\Phi}_t - \overline{\Phi_{;\eta}}_{t-} \Delta Z^\eta_t) + \overline{\Phi}_{t-}(\Delta\overline{\Psi}_t - \overline{\Psi_{;\eta}}_{t-} \Delta Z^\eta_t) && (***) \\ &+ \Delta\overline{\Phi}_t \Delta\overline{\Psi}_t \ . && (****) \end{aligned}$$

Now
$$\Delta\overline{\Phi\Psi}_t = \overline{\Psi}_{t-} \Delta\overline{\Phi}_t + \overline{\Phi}_{t-} \Delta\overline{\Psi}_t + \Delta\overline{\Phi}_t \Delta\overline{\Psi}_t$$

and $\overline{(\Phi\Psi)_{;\eta}}_{t-} \Delta Z^\eta_t = (\overline{\Phi} \cdot \overline{\Psi_{;\eta}} + \overline{\Psi} \cdot \overline{\Phi_{;\eta}})_{t-} \Delta Z^\eta_t$.

We see that the items in lines $(***)$ and $(****)$ add up to $\Delta\overline{\Phi\Psi}_t - \overline{\Phi\Psi_{;\eta}}_{t-} \Delta Z^\eta_t$. Identifying the products in lines $(*)$ and $(**)$ by means of Leibniz' product rule gives

$$d\overline{\Phi\Psi}_t = \overline{(\Phi\Psi)_{;\eta}}_{t-} dZ^\eta + \frac{1}{2} \overline{(\Phi\Psi)_{;\eta\theta}}_t d[Z^\eta, Z^\theta]^c_t + \Delta\overline{\Phi\Psi}_t - \overline{(\Phi\Psi)_{;\eta}}_{t-} \Delta Z^\eta_t \ .$$

This says that $\Phi\Psi \in \mathcal{I}$ as claimed.

3.9.3 Both existence and uniqueness are immediate from theorem 5.2.15 on page 291. It is also easy to see that a solution $\mathscr{E}$ must equal the right-hand side $'\mathscr{E}$ of (3.9.4): by exercise 3.7.16, $\mathscr{E}$ satisfies $\mathscr{E} = 1 + \mathscr{E}_{.-} {*} Z^{T_1-}$ on $[\![0, T_1)\!)$, where $T_1 = \inf\{t : \mathscr{E}_t \le 0\}$. Applying Itô's Formula with $\Phi = \ln$ shows that $\mathscr{E} = {}'\mathscr{E}$ on $[\![0, T_1)\!)$. By proposition 3.8.21, $\mathscr{E} = {}'\mathscr{E}$ on $[\![0, T_1]\!]$. Then continue as in the existence proof. For the second claim, compute: $d(\mathscr{E}[Z]\mathscr{E}[Z']) = \mathscr{E}[Z]_{.-} d\mathscr{E}[Z'] + \mathscr{E}[Z']_{.-} d\mathscr{E}[Z] + d\big[\mathscr{E}[Z], \mathscr{E}[Z']\big] = (\mathscr{E}[Z]\mathscr{E}[Z'])_{.-} d(Z + Z' + d[Z, Z'])$.

3.9.4 (ii): $t \mapsto e^{Z_t - Z_0 - {}^c[Z,Z]_t/2}$ is clearly bounded away from zero for $0 \le t < S \wedge u$ so the question is whether the product

$$\prod_{0<s<S\wedge t} (1+\Delta Z_s) e^{-\Delta Z_s} = \prod_{0<s<S\wedge t, |\Delta Z_s|<1/2} (1+\Delta Z_s) e^{-\Delta Z_s} \cdot \prod_{0<s<S\wedge t, |\Delta Z_s|\ge 1/2} (1+\Delta Z_s) e^{-\Delta Z_s}$$

is bounded away from zero for such t. Now between 0 and $S \wedge u$ there are only finitely many jumps of size $|\Delta Z_s(\omega)| \ge 1/2$, by assumption none of them equal to -1, so the contribution of the second product on the right is bounded away from zero. Since $\ln(1+u) - u \ge -2u^2$ for $|u| < 1/2$, the contribution of the first product exceeds $e^{-2{}^j[Z,Z]_u(\omega)}$.

3.9.8 Since $[M, M]_\infty = \infty$, the $T^{\lambda+}$ are almost surely finite. The time transformation $\lambda \mapsto T^\lambda$ is left-continuous, $\lambda \mapsto T^{\lambda+}$ is right-continuous (theorem 2.4.7) and so is $\mathcal{G}_{.}$ (exercise 1.3.30 (iii)). Due to theorem 2.5.22, $(W_\lambda, \mathcal{G}_\lambda)$ is a right-continuous local martingale. Let $0 \le \lambda < \mu < \infty$ and $A \in \mathcal{G}_\lambda$, and set $X \stackrel{\text{def}}{=} A \cdot (\lambda, \mu] \in \mathcal{P}[\mathcal{G}]$.

Since $$\lambda < [M, M]_\tau \le \mu \Longleftrightarrow T^{\lambda+} < \tau \le T^{\mu+} ,$$

we have $X_{[M,M]} = A \cdot (\!(T^{\lambda+}, T^{\mu+}]\!] \in \mathcal{P}[\mathcal{F}]$

and $$\int X \, dW = A \cdot (W_\mu - W_\lambda) = A \cdot \big(M_{T^{\mu+}} - M_{T^{\lambda+}}\big) = \int X_{[M,M]} \, dM .$$

That is to say, the vector space and bounded monotone class of processes $X \in \mathcal{P}[\mathcal{G}]$ such that $X_{[M,M]} \in \mathcal{P}[\mathcal{F}]$ and $\int X \, dW = \int X_{[M,M]} \, dM$ contains a linear generator of $\mathcal{P}[\mathcal{G}]$ and thus contains all bounded $\mathcal{G}$-predictable processes (theorem A.3.4). The usual truncation argument does the rest (use corollary 3.6.10).

To prove that $W_{.}$ is a standard Wiener process we will show that $[W, W]_\mu = \mu$ and invoke corollary 3.9.5. First, for $\mu > 0$,

$$[W, W]_\mu = W_\mu^2 - 2\int_0^\mu W_{\lambda-} \, dW_\lambda$$

$$= M_{T^{\mu+}}^2 - 2\int_0^{T^{\mu+}} W_{[M,M]_s-} \, dM_s$$

$$= [M,M]_{T^{\mu+}} + 2\int_0^{T^{\mu+}} \big(M_s - W_{[M,M]_s}\big)\, dM_s$$

$$= \mu + 2\int_0^{T^{\mu+}} \big(M_s - W_{[M,M]_s-}\big)\, dM_s\ . \qquad (*)$$

The last integral is the value at $T^{\mu+}$ of a continuous local martingale whose square at $T^{\mu+}$ has expectation

$$\mathbb{E}\Big[\int_0^{T^{\mu+}} \big(M_s - W_{[M,M]_s-}\big)^2\, d[M,M]_s\Big]$$

by 2.4.7:

$$=\mathbb{E}\Big[\int_0^{\mu} \big(M_{T^{\lambda+}} - W_{[M,M]_{T^{\lambda+}}-}\big)^2\, d\lambda\Big]$$

$$=\mathbb{E}\Big[\int_0^{\mu} \big(M_{T^{\lambda+}} - M_{T^{\lambda}}\big)^2\, d\lambda\Big]$$

$$=\int_0^{\mu} \mathbb{E}\Big[\big(M_{T^{\lambda+}} - M_{T^{\lambda}}\big)^2\Big]\, d\lambda\ = 0\ ,$$

because M is almost surely constant on $[\![T^{\lambda}, T^{\lambda+}]\!]$. Thus $[W,W]_\mu = \mu$ for all μ, and W is a standard Wiener process.

Next let X be left-continuous and adapted to $\mathcal{F}_{\bullet}$. Then $X_{T^{\cdot}}$ is left-continuous and adapted to $\mathcal{G}_{\bullet}$. The monotone class theorem shows that $X_{T^{\cdot}}$ is predictable on $\mathcal{G}_{\bullet}$ for all $X \in \mathcal{P}[\mathcal{F}_{\bullet}]$. Let X be the elementary integrand $A\cdot(s,t]$, $A \in \mathcal{F}_s$. To see that $\int X\, dM = \int X_{T^{\lambda}}\, dW_\lambda$ it suffices by the above to show that $\int X_{T^{[M,M]}}\, dM = \int X\, dM$. Now

$$X_{T^{[M,M]}} = A \cap \big[s < T^{[M,M]} \le t\big] = A \cap \big[[M,M]_s < [M,M] \le [M,M]_t\big]$$
$$= A \cap \big\{\sigma : T^{[M,M]_s+} < \sigma \le T^{[M,M]_t+}\big\}$$

so that $\Delta \stackrel{\text{def}}{=} \big|X - X_{T^{[M,M]}}\big| = A \cap \big((\!(s, T^{[M,M]_s+}]\!] \cup (\!(t, T^{[M,M]_t+}]\!]\big)\ .$

$T^{[M,M]_s+}$, being the first time $[M,M]$ is strictly greater than $[M,M]_s$, is a stopping time. As $[M,M]$ is constant on Δ, $[\Delta{*}M, \Delta{*}M] = \Delta^2{*}[M,M] = 0$, and we see as above that, indeed, $\int X\, dM = \int X_{T^{[M,M]}}\, dM$ and therefore

$$\int X_t\, dM_t = \int X_{T^{\lambda}}\, dW_\lambda\ . \qquad (*)$$

The monotone class of processes for which this equality holds contains thus a generator of $\mathcal{P}[\mathcal{F}_{\bullet}]$, and then every bounded predictable process.

Let now $0 \le p < \infty$ and let $X_{\bullet}$ be M–p-integrable. In showing that $X_{T^{\cdot}}$ is W–p-integrable and satisfies $(*)$, we may without loss of generality assume that $X \ge 0$. Proposition 3.6.6 on page 125 (ii) provides upper and lower envelopes $\underset{\sim}{X} \in \mathcal{P}[\mathcal{F}_{\bullet}]$ and $\widetilde{X} \in \mathcal{P}[\mathcal{F}_{\bullet}]$ that differ M-negligibly, are finite for $[\![\ \ ]\!]^*_{M\text{–}p}$, and sandwich X: $0 \le \underset{\sim}{X} \le X \le \widetilde{X}$. Clearly

$\underset{\sim}{X}_{T^\cdot} \le X_{T^\cdot} \le \widetilde{X}_{T^\cdot}$. Several applications of regularity (corollary 3.6.10 on page 128) now show that the lower and upper $\mathcal{P}[\mathcal{G}_\bullet]$-envelopes $\underset{\sim}{X}_{T^\cdot}$ and $\widetilde{X}_{T^\cdot}$ of $X_{T^\cdot}$ differ W-negligibly, that $[\![X_\bullet]\!]^*_{M\text{–}p} = [\![X_{T^\cdot}]\!]^*_{W\text{–}p}$, and that $(*)$ obtains.

Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>`, who pointed out that the solution previously given here was garbled and beset with typos.

3.9.12 (i) Let T be a stopping time at which G'^T and M^T are bounded and therefore are martingale L^2-integrators. Then

$$\begin{aligned}\left(G'^T_t\right)^2 &= 1 + 2\,(G'^T{*}G'^T)_t + \int_0^t \left(G'^T_s\right)^2 \, d[M,M]^T_s \\ &\le 1 + 2\,(G'^T{*}G'^T)_t + \int_0^t \left(G'^T_s\right)^2 \eta_s^2 \, ds\end{aligned}$$

and so $$\mathbb{E}\Big[\left(G'^T_t\right)^2\Big] \le 1 + \mathbb{E}\Big[\int_0^t (G'^T_s)^2 \eta_s^2 \, ds\Big] = 1 + \int_0^t \mathbb{E}\Big[(G'^T_s)^2\Big]\eta_s^2 \, ds\ .$$

Gronwall's lemma A.2.35 gives $\mathbb{E}[\left(G'_{t\wedge T}\right)^2] \le \exp(\int_0^t \eta_s^2 \ ds)$. Now take $T \uparrow \infty$ and apply Fatou's lemma to arrive at $\mathbb{E}[G'^2_t] \le \exp(\int_0^t \eta_s^2 \ ds)$. For (ii) see [82] and [55, page 199]. For the last claim consult [66].

3.9.15 There is an increasing sequence of stopping times $T_n \in \mathfrak{T}$ whose pointwise limit is ∞ $\mathbb{P}$-almost surely. Let $0 < t < \infty$ and set $N \overset{\text{def}}{=} \bigcap_n [T_n \le t]$. This is a $\mathbb{P}$-nearly empty set and belongs to $\mathcal{F}_{T_n} \cap \mathcal{F}_t$ since $\mathcal{F}_\bullet$ is regular. The σ-additivity of $\mathbb{P}'$ gives

$$\mathbb{E}\big[G'_t\big] \ge \mathbb{E}\big[G'_t[T_n > t]\big] = \mathbb{P}'[[T_n > t] \cup N] \to \mathbb{P}'[\Omega] = 1\ .$$

3.9.18 Let $\Omega' \overset{\text{def}}{=} \Omega \backslash N$, and $\mathcal{F}'_t = \{A \cap \Omega' : A \in \mathcal{F}_t\}$ the filtration induced on Ω'. It is naturally measured by the collection $\mathfrak{P}'$ of probabilities $\mathbb{P}' : A \cap \Omega' \mapsto \int_A d\mathbb{P}$, $\mathbb{P} \in \mathfrak{P}$. Let then $(\mathcal{F}'_t, \overline{\mathbb{P}}'_t)$ be a consistent system of σ-additive probabilities with $\overline{\mathbb{P}}'_t \ll \mathbb{P}'t$ on $\mathcal{F}'_t$, $\mathbb{P}' \in \mathfrak{P}', t \ge 0$. Let $\overline{\mathbb{P}}' : \mathcal{A}'_\infty \overset{\text{def}}{=} \bigcup_t \mathcal{F}'_t \to \mathbb{R}$ be its projective limit. Again it is to be shown that if $\mathcal{A}'_\infty \ni A'_n \downarrow \emptyset$, then $\overline{\mathbb{P}}'[A'_n] \to 0$. We write $A'_n = A_n \cap \Omega'$, where $A_n \in \mathcal{A}_\infty$ can be chosen to decrease as n increases. Now $\mathcal{F}_t \ni A \mapsto \overline{\mathbb{P}}_t[A] \overset{\text{def}}{=} \overline{\mathbb{P}}'_t[A \cap \Omega']$, $t \ge 0$, clearly defines a consistent system $(\mathcal{F}_\bullet, \overline{\mathbb{P}}_\bullet)$ such that $\overline{\mathbb{P}}_t \ll \mathbb{P}_t$ on $\mathcal{F}_t$, $\mathbb{P} \in \mathfrak{P}, t \ge 0$. For if $\mathbb{P} \in \mathfrak{P}$ vanishes on $A \in \mathcal{F}_t$, then $\mathbb{P}'[A \cap \Omega'] = 0$ and thus $\overline{\mathbb{P}}_t[A] = \overline{\mathbb{P}}'[A \cap \Omega'] = 0$. Let $\overline{\mathbb{P}}$ denote its σ-additive projective limit on $\mathcal{F}_\infty$. Then $\lim \overline{\mathbb{P}}'[A'_n] = \lim \overline{\mathbb{P}}[A_n] = \overline{\mathbb{P}}[\bigcap A_n] \le \overline{\mathbb{P}}[N] = 0$ (see equation (3.9.8) on page 167).

3.9.23 $X \cdot Z = X{*}Z + Z{*}X + {}^c[X,Z] = \big(X{*}Z + {}^c[X,Z]/2\big) + \big(Z{*}X + {}^c[X,Z]/2\big) = X{\circ}Z + Z{\circ}X$.

3.10.4 This is evident if $\check{F}$ is a linear combination of functions in $\mathcal{H}$. An arbitrary function in $\mathcal{E}[\boldsymbol{H}]$ is the confined uniform limit of such and its indefinite integral is therefore uniformly on bounded intervals the limit of the indefinite integrals of such, at least nearly (see inequality (3.10.3)). Then clearly $\check{X}*\zeta$ is $\mathcal{F}_{\bullet}[\zeta]$-adapted, and another application of inequality (3.10.3) gives the claim.

4.1.3 An application of Hölder's inequality with conjugate exponents q/p and $q/(q-p)$ yields

$$\int |f|^r \, d\mu = \int g \cdot |f|^r \, d\mu/g \le \Big(\int g^{q/(q-p)} \, d\mu/g\Big)^{(q-p)/p} \cdot \Big(\int |f|^{rq/p} d\mu/g\Big)^{p/q}$$
$$\le c \cdot \|f\|^r_{L^{rq/p}(\mu/g)} .$$

(ii) is the first half of exercise A.8.17. (iii) Let $X \in \mathfrak{L}^1[Z\text{–}q;\mathbb{P}']$. There are elementary integrands $X^{(n)}$ converging in $\mathfrak{L}^1[Z\text{–}q;\mathbb{P}']$ to X. Then $\Delta^{(n)} \stackrel{\text{def}}{=} X*Z - X^{(n)}*Z$ has $\|\,|Y*\Delta^{(n)}|^\star_\infty\,\|_{L^q(\mathbb{P}')} \to 0$ uniformly for $Y \in \mathcal{P}$ with $|Y| \le 1$. Then, with $r = p$ in (i), $\sup_Y \|\,|Y*\Delta^{(n)}|^\star_\infty\,\|_{L^p(\mathbb{P})} \to 0$, which implies $\lceil \Delta^{(n)} \rceil_{\mathcal{I}^p[\mathbb{P}]} \to 0$ and $X^{(n)} \to X$ in $\mathfrak{L}^1[Z\text{–}p;\mathbb{P}]$.

4.1.5 According to criterion 4.1.4 there is a strictly positive function g_0 with $\|g_0\|_{L^{p/(q-p)}(\mathbb{P})} \le 1$ such that for every $x \in \mathcal{E}$

$$\Big(\int |\mathcal{I}(x)|^q \cdot \frac{d\mathbb{P}}{g_0}\Big)^{1/q} \le \eta_{p,q}(\mathcal{I}) \cdot \|x\|_{\mathcal{E}} , \qquad x \in \mathcal{E} .$$

What is left to do is to adjust g_0 so that $\dfrac{d\mathbb{P}}{g_0}$ becomes a probability.

Now since $$\int \frac{d\mathbb{P}}{g_0 + 0} = \int \Big(g_0^{p/(q-p)}\Big)^{(p-q)/p} \, d\mathbb{P}$$

by theorem A.3.24: $$> \Big(\int g_0^{p/(q-p)} \, d\mathbb{P}\Big)^{(p-q)/p} \ge 1 \qquad (*)$$

and $$\int \frac{d\mathbb{P}}{g_0 + 1} < 1 ,$$

there is a number $\sigma \in (0,1)$ such that the function $g \stackrel{\text{def}}{=} g_0 + \sigma$ satisfies $\int g^{-1} d\mathbb{P} = 1$. Then $g' \stackrel{\text{def}}{=} g^{-1}$ is bounded and $\mathbb{P}' = g'\mathbb{P}$ is a probability. (If the inequality on the left in $(*)$ is not strict, we face by exercise A.3.27 (iii) the case of a constant g_0; it is silly to contemplate this case, and anyway then $g' = 1/g_0 = 1$ is a priori bounded.) The remaining inequalities are easy to establish: by exercise A.8.2,

$$\|g\|_{L^{p/(q-p)}(\mathbb{P})} \le 2^{\left(\left(1-\frac{p}{q-p}\right)/\frac{p}{q-p}\right) \vee 0} \cdot (1+\sigma) \le 2^{(p\vee(q-p))/p} ,$$

which is inequality (4.1.14). The estimate (4.1.15) follows from exercise 4.1.3. [Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>`, who spotted a couple of typos in the first version (12/21/2006).]

4.1.6 (i): Let $0 < p < q < \overline{q}$ and suppose $\overline{C} \stackrel{\text{def}}{=} \eta_{p,\overline{q}}(\mathcal{I})$ is finite. Then there is a $\overline{g} \geq 0$ with $\int \overline{g}^{p/(\overline{q}-p)}\, d\mu \leq 1$ and $\int \left|\mathcal{I}(x)\right|^{\overline{q}} / \overline{g}\, d\mu \leq \overline{C}^{\overline{q}} \| x \|^{\overline{q}}$. Set $g \stackrel{\text{def}}{=} \overline{g}^{(q-p)/(\overline{q}-p)} = \overline{g}^{q/\overline{q}} \cdot \overline{g}^{p(q-\overline{q})/\overline{q}(\overline{q}-p)}$. Then $\int g^{p/(q-p)}\, d\mu \leq 1$, and Hölder's inequality gives $\int \left|\mathcal{I}(x)\right|^{q} / g\, d\mu \leq \overline{C}^{q} \| x \|^{q}$.
(ii): Let $x_1, \ldots, x_n \in E$, let $C > \eta_{p,q}(\mathcal{I})$ and $C' > \eta_{p,q}(\mathcal{I}')$, and set $\boldsymbol{\xi} \stackrel{\text{def}}{=} (\mathcal{I}x_1, \ldots, \mathcal{I}x_n)$ and $\boldsymbol{\xi}' \stackrel{\text{def}}{=} (\mathcal{I}'x_1, \ldots, \mathcal{I}'x_n)$. Then by exercise A.8.2

$$\begin{aligned}\Big\| \|\boldsymbol{\xi} + \boldsymbol{\xi}'\|_{\ell^q} \Big\|_{L^p(\mu)} &\leq 2^{0 \vee [(1-q)/q]} \times \Big\| \|\boldsymbol{\xi}\|_{\ell^q} + \|\boldsymbol{\xi}'\|_{\ell^q} \Big\|_{L^p(\mu)} \\ &\leq 2^{0 \vee [(1-q)/q]} \cdot 2^{0 \vee [(1-p)/p]} \times \Big[\Big\| \|\boldsymbol{\xi}\|_{\ell^q} \Big\|_{L^p(\mu)} + \Big\| \|\boldsymbol{\xi}'\|_{\ell^q} \Big\|_{L^p(\mu)} \Big] \\ &\leq 2^{0 \vee [(1/q-1)]} \cdot 2^{0 \vee [(1/p-1)]} \times [C + C'] \Big(\sum_{\nu=1}^{n} \|x_\nu\|_E^q \Big)^{1/q} .\end{aligned}$$

That is to say,

$$\eta_{p,q}(\mathcal{I} + \mathcal{I}') \leq 2^{0 \vee (1-q)/q} \cdot 2^{0 \vee (1-p)/p} \times \big[\eta_{p,q}(\mathcal{I}) + \eta_{p,q}(\mathcal{I}') \big] .$$

Page 195.. It suffices to prove inequality (4.1.13) for a step function

$$f = \sum_i x_i\, A_i\,, \qquad\qquad x_i \in E\,,\ A_i \in \mathcal{T}\,.$$

Then

$$\begin{aligned}\Big\| \|\mathcal{I}f\|_{L^q(\tau)} \Big\|_{L^p(\mu)} &= \Big\| \Big\| \sum_i \mathcal{I}x_i\, A_i \Big\|_{L^q(\tau)} \Big\|_{L^p(\mu)} \\ &= \Big\| \Big(\sum_i \left|\mathcal{I}x_i\right|^q \tau(A_i) \Big)^{1/q} \Big\|_{L^p(\mu)} \\ &\leq C \Big(\sum_i \|x_i\|_E^q\ \tau(A_i) \Big)^{1/q} = C\, \|f\|_{L^q(\tau,E)}\,.\end{aligned}$$

4.1.8 Such $\mathcal{I}$ can be extended via the methods of chapter 3 to a continuous linear map of the bounded Baire functions on K to $L^p(\mu)$, preserving the modulus of continuity. For the second claim use the first one on the Gelfand transform $\widehat{\mathcal{I}}$ (see page 370).
(4.1.18) Indeed, let $\{x_\nu\}_{\nu=1}^n$ be a finite collection of vectors in $\ell^\infty(k)$. x_ν has an expansion $x_\nu = \sum_\kappa e_\kappa \cdot x_\nu^\kappa$ in terms of the standard basis $\{e_\kappa\}_{\kappa=1}^k$ of $\ell^\infty(k)$, and

$$|\mathcal{I}x_\nu|^2 = \Big(\sum_{\kappa=1}^{k} \mathcal{I}e_\kappa \cdot x_\nu^\kappa \Big)^2 \leq \sum_\kappa (\mathcal{I}e_\kappa)^2 \sum_\kappa (x_\nu^\kappa)^2 \leq k \sum_\kappa (\mathcal{I}e_\kappa)^2 \| x_\nu \|_{\ell^\infty}^2 .$$

Therefore

$$\Big\|\Big(\sum_\nu |\mathcal{I}x_\nu|^2\Big)^{\frac12}\Big\|_{L^p(\mu)} \le \sqrt{k}\cdot\Big\|\Big(\sum_\kappa (\mathcal{I}e_\kappa)^2\Big)^{\frac12}\Big\|_{L^p(\mu)}\cdot\Big(\sum_\nu \|x_\nu\|^2_{\ell^\infty(k)}\Big)^{\frac12}$$
$$\le \sqrt{k}\cdot\Big\|\Big(\sum_\kappa (\mathcal{I}e_\kappa)^p\Big)^{\frac1p}\Big\|_{L^p(\mu)}\cdot\Big(\sum_\nu \|x_\nu\|^2_{\ell^\infty(k)}\Big)^{\frac12}$$
$$= \sqrt{k}\cdot\Big(\sum_\kappa \|\mathcal{I}e_\kappa\|^p_{L^p(\mu)}\Big)^{\frac1p}\cdot\Big(\sum_\nu \|x_\nu\|^2_{\ell^\infty(k)}\Big)^{\frac12}$$
$$\le k^{1/p+1/2}\|\mathcal{I}\|_p\cdot\Big(\sum_\nu \|x_\nu\|^2_{\ell^\infty(k)}\Big)^{\frac12}\,.$$

4.1.13 Choose $p = 0.5$, $p_1 = .253$, $p_2 = .559$, $q = 0.8$, $\beta = 1/4$, and $\delta = \alpha/8$ and evaluate on the computer $T_{p,q}(E) \le 4.327\ldots$, $B^{(A.8.13)}_{[\beta],q} \le 17.76\ldots$ (cf. inequality (A.8.16)), and get

$$\frac{B_{[\beta],q}\cdot T_{p,q}(E)\cdot\|\mathcal{I}\|_{[\delta]}}{(\alpha\beta-\delta)^{1/p}} \le \frac{76.87\ldots\cdot\|\mathcal{I}\|_{[\alpha/8]}}{(\alpha/8)^2} \le \frac{5000\,\|\mathcal{I}\|_{[\alpha/8]}}{\alpha^2}\,.$$

Page 208.. Exercise A.8.17 gives, with $f = g'$ and $r = p/(q-p)$,

$$\big\|g'g\big\|_{[\alpha;\mathbb{P}]} \le \big\|g'\big\|_{L^{p/(q-p)}(\mathbb{P}/g)}\cdot\Big(\frac{2}{\alpha}\Big)^{q/p}\cdot\|g\|^{q/(q-p)}_{[\alpha/2;\mathbb{P}]}$$
$$\le E^{(4.1.6)}_{p,q}\cdot\Big(\frac{2}{\alpha}\Big)^{(q-p)/p}\cdot\Big(E^{(4.1.9)}_{[\alpha/2],\,p}[\,\lceil\!\lceil \boldsymbol{Z}\rceil\!\rceil_{[\cdot]}]\Big)^{q/p}\,.$$

Thus $$E^{(4.1.9)}_{[\alpha],q} \le E_{p,q}\cdot\big(2/\alpha\big)^{(q-p)/p}\cdot\Big(E^{(4.1.9)}_{[\alpha/2],p}[\,\lceil\!\lceil \boldsymbol{Z}\rceil\!\rceil_{[\cdot]}]\Big)^{q/p}$$

for any $p \in (0,1)$, $q \ge 1$, and $\alpha \in (0,1)$.

4.1.14 (i) Apply theorem 4.1.7. (ii) Reduce to (i) by proposition 4.1.12.

4.2.1 $\mathcal{K}[Z] \subset \mathcal{K}[Z']$ and $S_\infty[Z] \in \mathcal{K}[Z]$.

4.2.9 Let $\mu(dt) = f_t\,dy_t$ and $\nu(dt) = dz_t$. Clearly $\mu(\{0\}) \le \nu(\{0\})$. Consider next an interval of the form $[s,t)$. Given $\epsilon > 0$ set $t_0 = s$ and $t_{n+1} = \inf\{t > t_n : |f_t - f_{t_n}| \ge \epsilon\}$. Since f does not oscillate at any point, $\sup_n t_n > t$. There will be an N such that $t_N < t \le t_{N+1}$. Redefine $t_{N+1} = t$ and consider the partition $\{t_0 < t_1 \ldots < t_{N+1} = t\}$ of $(s,t]$ into half-open intervals $[t_n, t_{n+1})$. Since f does not oscillate by more than ϵ on any of these intervals, we have

$$\begin{aligned}\mu\big([s,t)\big) &= \sum_n \mu\big([t_n,t_{n+1})\big)\\ &\le \sum_n \sup_{t_n\le t<t_{n+1}} f(t)\cdot\big(y_{t_{n+1}}-y_{t_n}\big)\\ &\le \epsilon(y_t-y_s)+\sum_n \inf_{t_n\le t<t_{n+1}} f(t)\cdot\big(y_{t_{n+1}}-y_{t_n}\big)\\ &\le \epsilon(y_t-y_s)+\sum_n\big(z_{t_{n+1}}-z_{t_n}\big)\\ &= \epsilon(y_t-y_s)+\nu\big([s,t)\big)\ .\end{aligned}$$

Since this is true for all $\epsilon>0$, we have $\mu\le\nu$ on all intervals of the form $[s,t)$ and their finite unions, then on the sequential closure of these, which is the Borel σ-algebra.

4.2.11 There are arbitrarily large stopping times T such that M^T is a global L^1-integrator and $M^T_{.-}$ is bounded (corollary 2.5.29). Let N be a bounded martingale and $X\in\mathcal{E}_1$. The first two terms in

$$(X{*}M)_T\cdot N_T=\int_0^T (X{*}M)_{.-}\,dN+\int_0^T X\cdot N_{.-}\,dM+\big[(X{*}M),N\big]_T$$

have expectation zero. Thus

$$\mathbb{E}\big[(X{*}M)_T\cdot N_T\big]=\mathbb{E}\big[\,[(X{*}M),N]_T\,\big]=\mathbb{E}\Big[\big(X{*}[M,N]\big)_T\Big]\le\mathbb{E}\big[\,\big|[M,N]\big|_\infty\,\big]$$

by corollary 4.2.8: $\quad\le 2\sqrt{2p}\cdot\|S_\infty[M]\|_{L^p}\cdot\|N_\infty\|_{L^{p'}}\ .$

Taking the supremum over N with $\|N_\infty\|_{L^{p'}}\le 1$ and all $X\in\mathcal{E}_1$, and letting $T\to\infty$ yields the claim. The last statement is better done this way: for $X\in\mathcal{E}_1$

$$\mathbb{E}\big[(X{*}M)^2_\infty\big]=\mathbb{E}\big[[X{*}M,X{*}M]_\infty\big]=\mathbb{E}\big[(X^2{*}[M,M])_\infty\big]\le\mathbb{E}\big[[M,M]_\infty\big]\ .$$

4.2.13 Using lemma 4.2.2 on page 210, continue at inequality (4.2.7) on page 214 instead with

$$\begin{aligned}\mathbb{E}\big[|M_T|^{\star q}\big] &\le \frac{eq^2}{2}\cdot\left(\mathbb{E}\Big[\Big(M_T^{\star(q-2)}\Big)^{q'/(2-q')}\Big]^{(2-q')/q'}\right)\cdot\|M\|^2_{\mathcal{K}^q}\\ &= \frac{eq^2}{2}\cdot\Big(\mathbb{E}[M_T^{\star q}]\Big)^{(q-2)/q}\cdot\|M\|^2_{\mathcal{K}^q}\ ,\end{aligned}$$

divide, and take the square root. The first inequality is corollary 4.2.4.

4.2.14 In view of exercise 2.5.17 only the case $1<q<2$ is new. We continue the notations from its answer on page 470. The crucial inequality $\|S_n\|_{L^2}\le\sigma\sqrt{n}$ can be replaced by

$$\begin{aligned}\|S_n\|_{L^q} &\le C_q^{(4.2.4)}\Big\|[S,S]_n^{1/2}\Big\|_{L^q}=C_q\Big\|\Big(\sum_{\nu=1}^n F_\nu^2\Big)^{1/2}\Big\|_{L^q}\\ &\le C_q\Big\|\Big(\sum_{\nu=1}^n |F_\nu|^q\Big)^{1/q}\Big\|_{L^q}\le C_q\sigma_q n^{1/q}\ .\end{aligned}$$

A similar argument gives

$$\Big\| |\widetilde{Z}_n|^\star \Big\|_{L^q} \le C_q \Big\| \Big(\sum_{\nu=1}^{n} \frac{F_\nu^2}{\nu^2} \Big)^{1/2} \Big\|_{L^q} \le C_q \Big\| \Big(\sum_{\nu=1}^{n} \frac{|F_\nu|^q}{\nu^q} \Big)^{1/q} \Big\|_{L^q}$$
$$\le C_q \sigma_q \big(\sum \nu^{-q} \big)^{1/q} < \infty \,,$$

showing that $\widetilde{Z}$ is a global L^q-integrator and thus has almost surely a limit at infinity. The expectation of the total variation of $\widehat{Z}$ can be estimated by

$$\sum_{2\le\nu<\infty} \frac{\mathbb{E}[|S_{\nu-1}|]}{\nu(\nu-1)} \le \sum_{2\le\nu<\infty} \frac{\| S_{\nu-1} \|_{L^q}}{\nu(\nu-1)} \le C_q \sigma_q \sum_{2\le\nu<\infty} \frac{(\nu-1)^{1/q}}{\nu(\nu-1)} < \infty \,.$$

4.2.17 **Roger Sewell**, `<rfs@cambridgeconsultants.com>`, noticed that these inequalities hold for arbitrarily L^0-integrators Z rather than only for martingales M, as the original formulation of the exercise would suggest.
The left-hand inequality becomes obvious upon the choice $T = 0$ in the definition of $\mathcal{K}[Z]$, given our convention that $[Z,Z]_{0-} = 0$. For the right-hand inequality replace the estimate in the proof of inequality (4.2.5) for $S \stackrel{\text{def}}{=} S[Z]$ and $2 \le q < \infty$ by

$$\begin{aligned} \mathbb{E}[S_\infty^q] &\le \frac{q}{2} \cdot \mathbb{E}\Big[\int_0^\infty S^{q-2}\, d(S^2) \Big] \\ &\le \frac{q}{2} \cdot \inf \Big\{ \mathbb{E}\big[S^{q-2} \cdot g^2 \big] : g \in \mathcal{K}^q[Z] \Big\} \\ &\le \frac{q}{2} \cdot \big(\mathbb{E}[S_\infty^q] \big)^{q/(q-2)} \cdot \inf \Big\{ \big(\mathbb{E}[g^q] \big)^{2/q} : g \in \mathcal{K}[Z] \Big\} \,. \end{aligned}$$

Thus $\|S_\infty[Z]\|_{L^q} \le \sqrt{q/2} \cdot \|Z\|_{\mathcal{K}^q}$.

4.2.19 As $r < p$, M is an L^r-integrator. Since X is M-measurable (page 118) it suffices to show that $\| X \|^*_{M\text{–}r}$ is finite (theorem 3.4.10). Now

$$\begin{aligned} \|X\|^*_{M\text{–}r} &\le C_r^{(4.2.8)} \cdot \Big\| \Big(\int X^2 \, d[M,M] \Big)^{1/2} \Big\|_{L^r} \\ &\le C_r \cdot \|X^\star_\infty \cdot S_\infty[M]\|_{L^r} \\ &\le C_r \cdot \|X^\star_\infty\|_{L^q} \cdot \|S_\infty[M]\|_{L^p} < \infty \,. \end{aligned}$$

4.2.21 By exercise 1.3.6 on page 26, $T^c < \infty$ so that $T^c \wedge n \uparrow T^c$. Taking the expectation in

$$W^2_{T^c\wedge n} = 2 \int_0^{T^c\wedge n} W \, dW + T^c \wedge n$$

yields

$$\mathbb{E}\big[W^2_{T^c\wedge n}\big] = \mathbb{E}\big[T^c \wedge n\big] \,,$$

and Fatou's lemma A.8.7 gives

$$\mathbb{E}[c^2] = \mathbb{E}\big[\liminf W^2_{T^c\wedge n}\big] \le \liminf \mathbb{E}\big[T^c \wedge n\big] = \mathbb{E}\big[T^c\big] \le c^2 \ .$$

T^{c+} is handled the same way.

4.2.22 Set $V \stackrel{\text{def}}{=} \sum_i [M^i, M^i]$. In view of the inequalities of Kunita–Watanabe (theorem 3.8.9), $d[M^i, M^j]$ is absolutely continuous with respect to $d[M^i, M^i]$ for $1 \le i, j \le n$ and then with respect to dV. Therefore there exists a symmetric positive semidefinite matrix G^{ij} of well-measurable processes, the Radon–Nikodym derivatives $d[M^i, M^j]/dV$, so that $d[M^i, M^j] = G^{ij}\, dV$. It has trace 1. The usual diagonalization process for a symmetric matrix furnishes further an orthonormal matrix U^i_ν and scalars $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge 0$ so that $G^{ij} = \sum_\nu U^i_\nu U^j_\nu \lambda_\nu$. The similarity-invariance of the trace gives $\sum \lambda_\nu = 1$. Repeated applications of lemma A.2.21 on page 378 during the diagonalization process show that the U^i_ν and the λ_ν can be chosen to depend Borel measurably on G^{ij}, so that they are again well-measurable processes. Next observe that for $\boldsymbol{X} = (X_i) \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]$

$$\begin{aligned} ⦙\boldsymbol{X}*\boldsymbol{M}⦙_{\mathcal{I}^p} &\sim \|S_\infty[\boldsymbol{X}*\boldsymbol{M}]\|_{L^p} \\ &= \Big\|\Big(\int \sum_{i,j=1}^n X_i X_j \, d[M^i, M^j]\Big)^{1/2}\Big\|_{L^p} \\ &= \Big\|\Big(\int \sum_{i,j} X_i X_j \, G^{ij}\, dV\Big)^{1/2}\Big\|_{L^p} \\ &= \Big\|\Big(\int^* \sum_\nu \langle \boldsymbol{X}|\boldsymbol{U}_\nu\rangle^2 \lambda_\nu \, dV\Big)^{1/2}\Big\|^*_{L^p} \\ &= \|\langle \boldsymbol{X}|\boldsymbol{U}_{\bullet}\rangle\|^* \ , \end{aligned}$$

where $$\|\boldsymbol{F}\|^* \stackrel{\text{def}}{=} \Big\|\Big(\int^* \sum_{\nu=1}^n |F_\nu|^2 \lambda_\nu \, dV\Big)^{1/2}\Big\|^*_{L^p}$$

on functions $\boldsymbol{F} = (F_\nu)$ that live on $\check{\boldsymbol{B}} \stackrel{\text{def}}{=} \{1, \ldots, n\} \times \boldsymbol{B}$. After these preliminaries, let $({}^1\boldsymbol{X}, {}^2\boldsymbol{X}, {}^3\boldsymbol{X} \ldots)$ be a sequence in $\mathfrak{L}^1[\boldsymbol{M}\text{–}p]$ such that ${}^n\boldsymbol{X}*\boldsymbol{M}$ converges in $\mathcal{H}^p_0$ to some martingale L. Then ${}^n\boldsymbol{F}(\nu, \varpi) \stackrel{\text{def}}{=} \langle {}^n\boldsymbol{X}(\varpi)|(\boldsymbol{U}_\nu)(\varpi)\rangle$ defines a Cauchy sequence $({}^n\boldsymbol{F})$ for the mean $\| \ \|^*$, and replacing $({}^n\boldsymbol{X})$ by a subsequence we may in view of theorem 3.2.22 assume that $\langle {}^n\boldsymbol{X}|\boldsymbol{U}_{\bullet}\rangle$ converges both $\| \ \|^*$-a.e. and in $\| \ \|^*$-mean to some function $\boldsymbol{F}$ on $\check{\boldsymbol{B}}$. Since $\sum_\nu U^i_\nu U^j_\nu = \delta^{ij}$, ${}^nX_i = \sum_\nu \langle {}^n\boldsymbol{X}|\boldsymbol{U}_\nu\rangle U^i_\nu \to X_i \stackrel{\text{def}}{=} \sum_\nu F_\nu U^i_\nu$ both $\| \ \|^*$-a.e. and in $\| \ \|^*$-mean, for $i = 1, \ldots, n$. For ease of thinking and without loss of generality (see page 116) we assume the ${}^nX_i$ and X_i are predictable. Now in case a) G^{ij} is diagonal, $U^i_\nu = \delta^i_\nu$, $\boldsymbol{X} \mapsto \|\langle \boldsymbol{X}|\boldsymbol{U}_{\bullet}\rangle\|^*$ is solid and thus is a mean; in fact, it is equivalent to the Daniell mean $\llbracket\ \rrbracket^*_{\boldsymbol{M}\text{–}p}$ on previsibles (proposition 3.6.1 and exercise 3.6.16). Then $\boldsymbol{X} \in \mathfrak{L}^1[\boldsymbol{M}\text{–}p]$, and $L = \boldsymbol{X}*\boldsymbol{M}$. In case b) the brackets $[M^i, M^j]$ are previsible, so are the G^{ij}, λ_ν, and the U^i_ν; we set $N_\nu \stackrel{\text{def}}{=} \boldsymbol{U}_\nu*\boldsymbol{M}$, obtaining martingales satisfying a) and having $\{N^1, \ldots, N^n\}^\| = \mathcal{A}^\|$. Now an $L \in \mathcal{A}^\|$ can be written $L = \sum_\nu X^\nu*N^\nu = (\sum_\nu X^\nu U^i_\nu)*M^i$. In case c) we replace the $[M^i, M^j]$ by the previsible brackets $\langle M^i, M^j\rangle$ (see page 228), obtain previsible G^{ij}, λ_ν, and U^i_ν, and continue as in b).

4.2.23 We prove the very last statement: $(\mathcal{A}^{\perp\!\!\perp})^{\perp\!\!\perp} = \mathcal{A}^\|$. The inclusion $\mathcal{A}^\| \subseteq (\mathcal{A}^{\perp\!\!\perp})^{\perp\!\!\perp}$ is obvious. For the converse suppose $N \in \mathcal{H}^p_0$ lies outside $\mathcal{A}^\|$. The general theorem of Hahn–Banach (A.2.25 (i)) provides a martingale N^* in the dual $\mathcal{H}^{p*}_0$ with $\langle M|N^*\rangle \le 1$ for all $M \in A^\|$ and $\langle N|N^*\rangle > 1$. Here $\langle\,|\,\rangle$ denotes the duality. Since $\mathcal{A}^\|$ is a subspace, $\langle M|N^*\rangle = 0$ for all $M \in \mathcal{A}^\|$ so that $N^* \in (\mathcal{A}^\|)^{\perp\!\!\perp} \subseteq \mathcal{A}^{\perp\!\!\perp}$. Since $\langle N|N^*\rangle > 1$, N also lies outside $(\mathcal{A}^{\perp\!\!\perp})^{\perp\!\!\perp}$.

4.3.5 A process Z such that $\{Z_T : T \in \mathfrak{T}[\mathcal{F}_.], T < \infty\}$ is uniformly integrable is called a ***process of class (D)*** in the literature ([73, page 101]); it is called a ***process of class (DL)*** if the stopped process Z^u is of class (D) for all instants $u < \infty$ (*ibidem*).

Suppose then that Z is a positive supermartingale that is continuous in probability. By lemma 2.5.27 (ii) on page 80, $Z \wedge c$ is a global L^2-integrator with a càdlàg modification, for any $c > 0$. We may thus assume that Z itself has càdlàg paths and a limit $Z_\infty \in \overline{\mathbb{R}}$ at ∞.

The necessity of (D) for the conclusion is obvious: The global integrator $\widehat{Z}$ is of class (D) since $|\widehat{Z}_T| \le \widehat{Z}^\star_\infty \in L^1$ for all $T \in \mathfrak{T}[\mathcal{F}_.]$ (see theorem 2.3.6 on page 63); and $\widetilde{Z}_t = \mathbb{E}[\widetilde{Z}_\infty|\mathcal{F}_t]$ is of class (D) by example 2.5.2 on page 72. Thus their sum Z is of class (D) as well.

To show the sufficiency, assume that Z is of class (D). Then $Z_\infty = \lim_{t\to\infty} Z_t$ also in L^1-mean.

An aside: The martingale $M_t \stackrel{\text{def}}{=} \mathbb{E}[Z_\infty|\mathcal{F}_t]$ is of class (D) as well (*ibidem*), and $Z' \stackrel{\text{def}}{=} Z - M$ is a positive supermartingale of class (D) with $\mathbb{E}[Z'_t] \xrightarrow[t\to\infty]{} 0$.

A positive supermartingale Z' with $\mathbb{E}[Z'_t] \xrightarrow[t\to\infty]{} 0$ is called a ***potential***. Thus a positive supermartingale Z of class (D) can be written as the sum of a potential Z' of class (D) and a uniformly integrable martingale M. This decomposition $Z = Z' + M$ is known as the ***Riesz decomposition*** of Z.

For any $c > 0$ set $T_c \stackrel{\text{def}}{=} \inf\{t : Z_t \ge c\}$. Then $Z^{T_c} \wedge c$ is a global L^2-integrator (see lemma 2.5.27 (ii)) that differs by the global L^1-integrator $(Z_{T_c} - c)_+ \cdot [\![T_c, \infty)\!)$ from Z^{T_c}. The latter is therefore a global L^1-integrator. Now $B \stackrel{\text{def}}{=} [\sup_c T_c < \infty]$ is negligible: if it were not, then the random variables $Z_{T_c} \ge c \cdot B$, $c < \infty$, would not be uniformly integrable. This means that $T_c \xrightarrow[c\to\infty]{} \infty$ and implies that Z is a local L^1-integrator and has a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$ (with $\widehat{Z}_0 = Z_0$ and $\widetilde{Z}_0 = 0$ by our convention). Since $\mu_Z \le 0$, $\widehat{Z}$ is a decreasing process; since $\mathbb{E}[\widehat{Z}_t - \widehat{Z}_0] = \mathbb{E}[Z_t - Z_0] \ge \mathbb{E}[Z_\infty - Z_0]$, we have $\mathbb{E}[\widehat{Z}_t] \ge \mathbb{E}[Z_\infty]$ for all t, so $\widehat{Z}_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} \widehat{Z}_t$ exists almost surely and in L^1-mean. Therefore $\widehat{Z}$ is a global L^1-integrator of size $\lceil \widehat{Z} \rceil_{\mathcal{I}^1} \le \mathbb{E}[Z_0] + \mathbb{E}[Z_0 - Z_\infty]$ and is of class (D). Then $\widetilde{Z} = Z - \widehat{Z}$ is a martingale of class (D) as well. The sufficiency of the condition (D) is established.

An aside: Assume that Z is a potential and set $M \stackrel{\text{def}}{=} \widetilde{Z} + Z_0$ and $A \stackrel{\text{def}}{=} Z_0 - \widehat{Z}$. Then M is a uniformly integrable martingale with $M_0 = Z_0$ and A is an increasing predictable process with $A_0 = 0$ and $\mathbb{E}[A_\infty] < \infty$; and $Z = M - A$. Since $Z_\infty = 0$, we have $M_\infty = A_\infty$ and $M_t = \mathbb{E}[A_\infty | \mathcal{F}_t]$. That is to say, $Z_t = \mathbb{E}[A_\infty|\mathcal{F}_t] - A_t$ is determined entirely by the predictable increasing process A; it is called the ***potential generated by A***.

By applying the result above to the stopped processes Z^u, $u < \infty$, we get the following: For a positive supermartingale Z to have a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$ with $\widehat{Z}$ of integrable finite variation ($\lceil Z \rceil_t \in L^1 \quad \forall\, t < \infty$) and $\widetilde{Z}$ a martingale (rather than merely a local one) it is necessary and sufficient that Z be of class (DL).

By applying it to the stopped processes Z^U, U a finite stopping time, we get the following: For a positive supermartingale Z to have a Doob–Meyer decomposition it is necessary and sufficient that Z be locally of class (D), i.e., Z^U to be of class (D) for arbitrarily large stopping times U.

4.3.6 Let $(S^{(n)})$ be a sequence of stopping times that announces T. Then $\mathbb{E}[\widetilde{Z}_T | \mathcal{F}_{S^{(n)}}] = \widetilde{Z}_{S^{(n)}} \xrightarrow[n\to\infty]{} \widetilde{Z}_{\cdot-T}$ and thus $\mathbb{E}[\widetilde{Z}_T - \widetilde{Z}_{\cdot-T} | \mathcal{F}_{T-}] = 0$. Since $\widehat{Z}_T \in \mathcal{F}_{T-}$, $\widehat{Z}_T - \widehat{Z}_{\cdot-T} = \mathbb{E}[\widehat{Z}_T - \widehat{Z}_{\cdot-T}|\mathcal{F}_{T-}] = \mathbb{E}[Z_T - Z_{T-}|\mathcal{F}_{T-}]$.
Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` (12/21/2006), who noticed that the result extends to more general circumstances: for the various conditional expectations above to make sense it is not necessary that Z be a global L^1-integrator, That T reduce it to one will do. This happens, for instance, if Z is a plain L^1-integrator and T is bounded. In fact, the equality $\Delta\widehat{Z}_T = \mathbb{E}\left[\Delta Z_T \middle| \mathcal{F}_{T-}\right]$ holds whenever Z is a local L^1-integrator and ΔZ_T is integrable. To see this, let U_n be a sequence of stopping times that reduce Z to a global L^1-integrator and increase to ∞. Let $B \stackrel{\text{def}}{=} A \cap [t < T]$, $A \in \mathcal{F}_t$, be a typical generator of $\mathcal{F}_{T-}$ (as $\Delta Z_0 = \Delta\widehat{Z}_0$ we need not worry about $t = 0$). Then $[T < U_n] = \bigcup_{q\in\mathbb{Q}}[T \wedge U_n \le q] \cap [q < U_n]$ belongs to $\mathcal{F}_{T\wedge U_n -}$ and is contained in $[T \wedge U_n = T]$, $B_n \stackrel{\text{def}}{=} A \cap [t < T \wedge U_n] \cap [T < U_n] \in \mathcal{F}_{T\wedge U_n -}$ is contained in $[T = T \wedge U_n]$ as well, and therefore

$$\mathbb{E}[\Delta Z_T \cdot B_n] = \mathbb{E}\Big[\Delta Z^{U_n}_{T\wedge U_n} \cdot B_n\Big] = \mathbb{E}\Big[\Delta \widehat{Z}^{U_n}_{T\wedge U_n} \cdot B_n\Big] = \mathbb{E}\Big[\Delta \widehat{Z}_T \cdot B_n\Big] .$$

As $n \to \infty$, the Dominated Convergence Theorem gives $\mathbb{E}[\Delta Z_T \cdot B] = \mathbb{E}[\Delta\widehat{Z}_T \cdot B]$. As $\Delta\widehat{Z}_T \in \mathcal{F}_{T-}$, we get $\Delta\widehat{Z}_T = \mathbb{E}\left[\Delta Z_T \middle| \mathcal{F}_{T-}\right]$.

For the second claim reduce to global L^1-integrators; at the predictable time $T^\epsilon = \inf\{t : \Delta\widehat{Z}_t \ge \epsilon\}$ the jump is zero. Thus $T^\epsilon = \infty$.

To prove the inequality, let M be a martingale with $\|M_\infty\|_{(q/2)'} \le 1$. There is a countable family $\{T_n\}$ of stopping times, all of them necessarily predictable, at which the jumps of $\widehat{Z}$ occur (theorem 2.4.4). Then

$$\mathbb{E}\Big[[\widehat{Z},\widehat{Z}]_\infty \cdot M_\infty\Big] = \mathbb{E}\Big[\sum_n \big|\mathbb{E}\big[\Delta Z_{T_n}|\mathcal{F}_{T_n-}\big]\big|^2 \cdot M_\infty\Big]$$

by inequality (A.3.10):
$$\le \mathbb{E}\Big[\sum_n \mathbb{E}\big[(\Delta Z_{T_n})^2|\mathcal{F}_{T_n-}\big] \cdot M_{T_n-}\Big]$$

$$= \mathbb{E}\Big[\sum_n (\Delta Z_{T_n})^2 \cdot M_{T_n-}\Big] \le \mathbb{E}\big[{}^j[Z,Z]_\infty \cdot M^\star_\infty\big]$$

by theorem 2.5.19:
$$\le \Big\|{}^j[Z,Z]_\infty\Big\|_{q/2} \cdot (q/2) = (q/2)\,\Big\|{}^jS_\infty[Z]\Big\|_q^2 .$$

Taking the supremum over M and then the square root gives the claim.

4.3.8 Reduce to the case that Z is a global L^1-integrator and let M be a bound for the jumps of Z, let $Z = \widehat{Z} + \widetilde{Z}$ be the Doob–Meyer decomposition of Z and S a stopping time, necessarily predictable (exercise 3.5.19), at which $\widehat{Z}$ jumps. In view of exercise 4.3.6 we have $|\Delta\widehat{Z}_S| \le M$, and therefore $|\Delta\widetilde{Z}_S| \le 2M$. Thus $|\Delta\widetilde{Z}| \le 2M$. Let $K > 0$ and $U = \inf\{t : ¦\widehat{Z}¦_t \vee S_t[\widetilde{Z}] \ge K\}$. U can be made arbitrarily large by the choice of K. Then $\widehat{Z}^U$ is a global L^q-integrator for all q since $\|\,¦\widehat{Z}^U¦_\infty\|_{L^\infty} \le K + M$, and $\widetilde{Z}^U$ is a global L^q-integrator for all q since $\|\,S_\infty[\widetilde{Z}^U]\,\|_{L^\infty} \le K + 2M$ (exercise 4.2.18).

4.3.9 We may without loss of generality assume that Z is a global L^1-integrator. Let $U = \inf\{t : |\Delta Z_t| \ge M\}$. This stopping time can be made arbitrarily large by the choice of $M > 0$. Let Z^{U-} be the process Z stopped just before U:

$$Z^{U-} = Z \cdot [\![0,U)\!) + Z_{U-} \cdot [\![U,\infty)\!) = (Z - \Delta Z_U)^U .$$

Since Z^{U-} differs by the L^1-bounded finite variation process $\Delta Z_U \cdot [\![U,\infty)\!)$ from the L^1-integrator Z^U, it is a global L^1-integrator itself, with jumps uniformly bounded by M. Now apply exercise 4.3.8.

4.3.12 (i)⇒(ii) Let N be a previsible set with $\mu_{V'}(N) = 0$. By Fubini's theorem A.3.18 we have $\int_N dV'_t(\omega) = 0$ and then $\int_N dV_t(\omega) = 0$ for almost all $\omega \in \Omega$ and, taking the expectation, $\mu_V(N) = 0$.

(ii)⇒(iii) is immediate from the theorem of Radon–Nikodym.

It is left to prove the last statement under the assumption (iii); it will entrain the implication (iii)⇒(i). Let $\overline{V} = G{*}V'$. This is a previsible (exercise 3.7.19) positive increasing process. Both V and $\overline{V}$ are Doléans–Dade processes for the measure μ_V, so they are indistinguishable.

4.3.13 Let D be a Radon–Nikodym derivative of μ with respect to the variation measure $¦\mu¦$. This is a previsible process of absolute value one. Since $¦\mu¦ = D \cdot \mu$ and $\mu \ge ¦\mu¦$ we have $¦V¦ \ge V = D{*}V$. On the other hand, $d(D{*}V)$ is a positive measure on the line majorizing $|dV|$, so $D{*}V \ge ¦V¦$. To prove the remaining inequality $¦V¦_{\mathcal{I}^p} \ge \|\,¦V¦_\infty\|_{L^p}$, note that $¦V¦_{\mathcal{I}^p} \ge \|\int D\,dV\,\|_{L^p} = \|\,¦V¦_\infty\|_{L^p}$.

4.3.14 There is an increasing sequence of stopping times T_n so that $M^{(n)} \stackrel{\text{def}}{=} M^{T_{n+1}} - M^{T_n}$ is the sum of a finite variation process $V^{(n)}$ and a global L^2-integrator $Z^{(n)}$ (corollary 2.5.29). Let $Z^{(n)} = \widehat{Z}^{(n)} + \widetilde{Z}^{(n)}$ be the Doob–Meyer decomposition and write

$$M = \sum_n \Big(V^{(n)} + \widehat{Z}^{(n)}\Big) + \sum_n \widetilde{Z}^{(n)} .$$

4.3.15 By corollary A.5.13 on page 438 the maximal process $X^\star$ is progressively measurable and therefore adapted. Since it is increasing it has a càdlàg version $X^\star_{.+}$, which is a global L^p-integrator iff $\| X^\star_\infty \|_{L^p}$ is finite. (Thanks to **Roger Sewell**, who added this remark and further improvements to my previous slapdash version of this answer.)
First the case $r \ge 1$, which implies that $q > 1$. Then Z has a Doob–Meyer decomposition $Z = \widehat{Z} + \widetilde{Z}$,

$$\int X\, dZ = \int X\, d\widehat{Z} + \int X\, d\widetilde{Z} \le X^\star_\infty \cdot \lceil \widehat{Z} \rceil_\infty + \int X\, d\widetilde{Z} ,$$

and
$$\Big\|\int X\, dZ\Big\|_{L^r} \le \Big\|X^\star_\infty \cdot \lceil \widehat{Z} \rceil_\infty\Big\|_{L^r} + \Big\|\int X\, d\widetilde{Z}\Big\|_{L^r}$$

by A.8.4:
$$\begin{aligned}
&\le \|X^\star_\infty\|_{L^p} \cdot \Big\|\lceil \widehat{Z} \rceil_\infty\Big\|_{L^q} + C_r^{(4.2.4)} \cdot \Big\|S_\infty[X{*}\widetilde{Z}]\Big\|_{L^r} \\
&= \|X^\star_\infty\|_{L^p} \cdot \Big\|\lceil \widehat{Z} \rceil_\infty\Big\|_{L^q} + C_r \cdot \Big\|\Big(\int_0^\infty X^2\, d[\widetilde{Z},\widetilde{Z}]\Big)^{1/2}\Big\|_{L^r} \\
&\le \|X^\star_\infty\|_{L^p} \cdot \lceil \widehat{Z} \rceil_{\mathcal{I}^q} + C_r \cdot \Big\|X^\star_\infty \cdot S_\infty[\widetilde{Z}]\Big\|_{L^r} \\
&\le \|X^\star_\infty\|_{L^p} \cdot \lceil \widehat{Z} \rceil_{\mathcal{I}^q} + C_r \cdot \|X^\star_\infty\|_{L^p} \cdot \Big\|S_\infty[\widetilde{Z}]\Big\|_{L^q} \\
&\le \|X^\star_\infty\|_{L^p} \cdot \widehat{C}_q^{(4.3.1)} \lceil Z \rceil_{\mathcal{I}^q} \\
&\qquad + C_r \cdot \|X^\star_\infty\|_{L^p} \cdot K_q^{(3.8.6)} \lceil \widetilde{Z} \rceil_{\mathcal{I}^q} \\
&\le C_{q,r} \cdot \|X^\star_\infty\|_{L^p} \cdot \lceil Z \rceil_{\mathcal{I}^q} \\
&\qquad + C_{q,r} \cdot \|X^\star_\infty\|_{L^p} \cdot \widetilde{C}_q^{(4.3.1)} \lceil Z \rceil_{\mathcal{I}^q} \\
&= C_{q,r} \cdot \|X^\star_\infty\|_{L^p} \cdot \lceil Z \rceil_{\mathcal{I}^q} \qquad (*)
\end{aligned}$$

— the constants C_r and $C_{q,r}$ above are understood to vary from one occurrence to the next.
Now if $r < 1$ pick a $q' \ge q/r > 1$ and let $\mathbb{P}'$ be a probability equivalent to $\mathbb{P}$ under which both global L^q-integrators $X^{\star p/q}_{.+}$ and Z are global $L^{q'}$-integrators. By theorem 4.1.2 on page 191, such $\mathbb{P}'$ exists if $\| X^\star_\infty \|_{L^p} < \infty$, the only case where there is anything to prove.

Now
$$\frac{q}{rq'} = \frac{q}{pq'} + \frac{1}{q'} \quad \text{, i. e.,} \quad \frac{1}{rq'/q} = \frac{1}{pq'/q} + \frac{1}{q'} ,$$

and as $1 \le rq'/q < pq'/q$ we may apply the case treated above:

$$\begin{aligned}\Big\|\int X\,dZ\Big\|_{L^r(\mathbb{P})} &\le \big(E^{(4.1.7)}_{q,q'}\big)^{q/q'r} \cdot \Big\|\int X\,dZ\Big\|_{L^{rq'/q}(\mathbb{P}')} \\ &\le E_{\dots} \cdot C^{(*)}_{q',r} \cdot \|X^\star_\infty\|_{L^{pq'/q}(\mathbb{P}')} \cdot ⦚Z⦚_{\mathcal{I}^{q'}(\mathbb{P}')} \\ &= C_{\dots} \cdot \Big(\int \big(X^{\star p/q}_\infty\big)^{q'}\,d\mathbb{P}'\Big)^{q/pq'} \cdot D^{(4.1.5)}_{q,q',2}\,⦚Z⦚_{\mathcal{I}^q(\mathbb{P})} \\ &= C_{\dots} \cdot ⦚X^{\star p/q}_{\bullet+}⦚^{q/p}_{\mathcal{I}^{q'}(\mathbb{P}')} \cdot ⦚Z⦚_{\mathcal{I}^q(\mathbb{P})} \\ &\le C_{\dots} \cdot D^{(4.1.5)}_{q,q',2}\,⦚X^{\star p/q}_{\bullet+}⦚^{q/p}_{\mathcal{I}^q(\mathbb{P})} \cdot ⦚Z⦚_{\mathcal{I}^q(\mathbb{P})} \\ &= C_{\dots} \cdot \Big\|X^{\star p/q}_\infty\Big\|^{q/p}_{L^q(\mathbb{P})} \cdot ⦚Z⦚_{\mathcal{I}^q(\mathbb{P})} \\ &= C_{\dots} \cdot \|X^\star_\infty\|_{L^p(\mathbb{P})} \cdot ⦚Z⦚_{\mathcal{I}^q(\mathbb{P})}\,.\end{aligned}$$

Applying this to YX with $Y \in \mathcal{E}_1$ gives

$$\Big\|\int YX\,dZ\Big\|_{L^r} = \Big\|\int Y\,d(X{*}Z)\Big\|_{L^r} \le C_{p,q} \cdot \|X^\star_\infty\|_{L^p} \cdot ⦚Z⦚_{\mathcal{I}^q}\,,$$

whence $\quad ⦚X{*}Z⦚_{\mathcal{I}^r} \le C_{p,q} \cdot \|X^\star_\infty\|_{L^p} \cdot ⦚Z⦚_{\mathcal{I}^q}\,, \qquad 0 < \frac{1}{r} = \frac{1}{p} + \frac{1}{q} < \infty\,.$

4.3.17 Copy the proofs of the corresponding statements for the square bracket.

4.3.18 Use exercise 3.8.24 on page 157 (ii).

4.3.20 If $p \le 2$, argue that, for $X \in \mathcal{E}_1$, $\|X{*}M^t\|_{L^p} \le \|(X{*}M^t)^\star\|_{L^p} \le C^{(4.3.6)}_p \cdot \|s_\infty[(X * M^t)]\|_{L^p} = C_p \cdot \|(\int_0^t X^2\,d\langle M,M\rangle)^{1/2}\|_{L^p} \le C_p \cdot \|s_t[M]\|_{L^p} = C_p \cdot \|s_\infty[M^t]\|_{L^p}$ implies $⦚M^t⦚_{\mathcal{I}^p} \le C_p \cdot \|s_t[M]\|_{L^p}$.

If $p \ge 2$, use the fact that $s[M] = S[M]$ as $[M,M]$ is previsible, and employ inequality (4.2.4) on page 213 instead.

4.3.21 Taking the expectation in

$$\int X\,dZ \le \int |X|\,d⦚\widehat{Z}⦚ + \int X\,d\widetilde{Z} \le X^\star_\infty \cdot ⦚\widehat{Z}⦚_\infty + \int X\,d\widetilde{Z}$$

gives $\|Z\|^\wedge_p \le \|⦚\widehat{Z}⦚_\infty\|_{L^p}$. The measurability of $X^\star_\infty$ is not required for this argument. The remaining inequality $\|⦚\widehat{Z}⦚_\infty\|_{L^p} \le p{\cdot}\|Z\|^\wedge_p$ is nontrivial only if $\|Z\|^\wedge_p < \infty$. In this case $|\widehat{Z}_0| = \int \operatorname{sgn}(\widehat{Z}_0){\cdot}[\![0]\!]\,dZ$ is p-integrable. There are then arbitrarily large stopping times S such that $⦚\widehat{Z}^S⦚ = ⦚\widehat{Z}⦚^S$ is bounded (see corollary 3.5.16). Let $D = D^{-1}$ be a previsible derivative of the Doléans–Dade measure $\nu_{\widehat{Z}}$ with respect to its variation and set $X \stackrel{\text{def}}{=} p{\cdot}D{\cdot}⦚\widehat{Z}^S⦚^{p-1}$. This previsible process has its càdlàg maximal process in $L^{p'}$:

$$\|X^\star_\infty\|_{L^{p'}} = \|X_\infty\|_{L^{p'}} = p{\cdot}\Big(\mathbb{E}\Big[\,⦚\widehat{Z}⦚_S^{(p-1)\frac{p}{p-1}}\Big]\Big)^{\frac{p-1}{p}} = p \cdot \Big\|\,⦚\widehat{Z}⦚_S\Big\|^{p-1}_{L^p}\,.$$

Now $d\,\big\lvert\widehat{Z}\big\rvert^{p} \le p\,\big\lvert\widehat{Z}\big\rvert^{p-1} d\,\big\lvert\widehat{Z}\big\rvert = pD\,\big\lvert\widehat{Z}\big\rvert^{p-1} d\widehat{Z} = X\,d\widehat{Z}$ on $[\![0,S]\!]$ and consequently

$$\Big\|\,\big\lvert\widehat{Z}\big\rvert_S\Big\|_{L^p}^p \le \mathbb{E}\Big[\int X\;d\widehat{Z}^S\Big] \le p\cdot\Big\|\,\big\lvert\widehat{Z}\big\rvert_S\Big\|_{L^p}^{p-1}\cdot\|Z\|_p^{\wedge}\;.$$

Division by the middle factor yields

$$\Big\|\,\big\lvert\widehat{Z}\big\rvert_S\Big\|_{L^p} \le p\cdot\|Z\|_p^{\wedge}\;.$$

Now take $S\uparrow\infty$ to arrive at the claim. Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>` for pointing out various typos in the first version (01/16/2007).

4.3.22 Take the supremum over $g\in L_1^{p'}$ over

$$\begin{aligned}\mathbb{E}\Big[\widehat{I}_\infty\cdot g\Big] &= \mathbb{E}\Big[\int_0^\infty M^g_{\bullet-}\;d\widehat{I}\Big] = \mathbb{E}\Big[\int_0^\infty M^g_{\bullet-}\;dI\Big]\\ &\le \mathbb{E}[M^{g\star}_\infty\cdot I_\infty] \le \|M^{g\star}_\infty\|_{L^{p'}}\cdot\|I_\infty\|_{L^p}\\ \text{by theorem 2.5.19:}\quad &\le p\cdot\|I_\infty\|_{L^p} = p\cdot\big\lvert I\big\rvert_{\mathcal{I}^p}\;.\end{aligned}$$

4.3.27 Applying the expectation in equation (3.10.6) on page 182 gives (using the Einstein convention)

$$\begin{aligned}\mathbb{E}[\Phi(\boldsymbol{Z}_t)] &= \mathbb{E}[\Phi(\boldsymbol{Z}_0)]\\ &\quad+\mathbb{E}\Big[\int_{0+}^t \Phi_{;\eta}(\boldsymbol{Z}_{\bullet-})\;d\widehat{Z}^\eta\Big]\\ &\quad+\frac{1}{2}\,\mathbb{E}\Big[\int_{0+}^t \Phi_{;\eta\theta}(\boldsymbol{Z}_{\bullet-})\;d^c[Z^\eta,Z^\theta]\Big]\\ &\quad+\mathbb{E}\Big[\int_0^t\Big(\Phi(\boldsymbol{Z}_{s-}+\boldsymbol{y})-\Phi(\boldsymbol{Z}_{s-})-\Phi_{;\eta}(\boldsymbol{Z}_{s-})\cdot y^\eta\Big)\,\widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y},ds)\Big]\;.\end{aligned}$$

To see that this clumsy looking formula might have some interest, consider the case that $\boldsymbol{Z}$ is a Lévy process. Then $\widehat{Z}^\eta_t = A^\eta\times t$, ${}^c[Z^\eta,Z^\theta]_t = B^{\eta\theta}\times t$, and $\widehat{\jmath_{\boldsymbol{Z}}}(d\boldsymbol{y},ds)=\nu(d\boldsymbol{y})\times ds$, with constant vector A^η, constant matrix $B^{\eta\theta}$, and constant measure $\nu(d\boldsymbol{y})$ on $\mathbb{R}^d$ (lemma 4.6.7 and lemma 4.6.8 on page 258). Then we get the more informative formula

$$\begin{aligned}\mathbb{E}[\Phi(\boldsymbol{Z}_t)] &= \mathbb{E}[\Phi(\boldsymbol{Z}_0)]\\ &\quad+\int_0^t \mathbb{E}\big[A^\eta\Phi_{;\eta}(\boldsymbol{Z}_{s-})\big]\;ds\\ &\quad+\frac{1}{2}\int_0^t \mathbb{E}\Big[B^{\eta\theta}\Phi_{;\eta\theta}(\boldsymbol{Z}_{s-})\Big]\;ds\end{aligned}$$

$$+\int_0^t \int_{\mathbb{R}^d} \mathbb{E}\Big[\Phi(\boldsymbol{Z}_{s-}+\boldsymbol{y})-\Phi(\boldsymbol{Z}_{s-})-\Phi_{;\eta}(\boldsymbol{Z}_{s-})\cdot y^\eta\Big]\,\nu(d\boldsymbol{y})\,ds\ ,$$

or, as $\boldsymbol{Z}_{s-}=\boldsymbol{Z}_s$ almost surely at all instants s,

$$\frac{d\mathbb{E}[\Phi(\boldsymbol{Z}_t)]}{dt} = A^\eta\,\mathbb{E}\big[\Phi_{;\eta}(\boldsymbol{Z}_t)\big] + \big(B^{\eta\theta}/2\big)\,\mathbb{E}\big[\Phi_{;\eta\theta}(\boldsymbol{Z}_t)\big]$$
$$+\int_{\mathbb{R}^d}\nu(d\boldsymbol{y})\,\mathbb{E}\Big[\Phi(\boldsymbol{Z}_t+\boldsymbol{y})-\Phi(\boldsymbol{Z}_t)-\Phi_{;\eta}(\boldsymbol{Z}_t)\cdot y^\eta\Big]\ .$$

Let us assume that $\boldsymbol{Z}$ starts with $\boldsymbol{Z}_0=0$, and consider the function $(t,\boldsymbol{x})\mapsto u(t,\boldsymbol{x})\overset{\text{def}}{=}\mathbb{E}[\Phi(\boldsymbol{x}+\boldsymbol{Z}_t)]$. Then, applying the previous equality to the function $\boldsymbol{z}\mapsto\Phi(\boldsymbol{x}+\boldsymbol{z})$, we see that u solves the initial value problem

$$u(0,x)=\Phi(x)\ ,$$
$$\frac{du(t,\boldsymbol{x})}{dt} = A^\eta\,u_{;\eta}(t,\boldsymbol{x}) + \big(B^{\eta\theta}/2\big)\,u_{;\eta\theta}(t,\boldsymbol{x})$$
$$+\int_{\mathbb{R}^d}\Big(u(t,\boldsymbol{x}+\boldsymbol{y})-u(t,\boldsymbol{x})-u_{;\eta}(t,\boldsymbol{x})\cdot y^\eta\Big)\,\nu(d\boldsymbol{y})\ .$$

(Thanks to **Roger Sewell**, `<rfs@cambridgeconsultants.com>`, for spotting a typo here in the first answer given.) Since to a given *suitable* triple $(\boldsymbol{A},B,\nu)$ a process with characteristic triple $(t\boldsymbol{A},tB/2,\nu(d\boldsymbol{y})dt)$ exists (theorem 4.6.17 on page 267), we know now that this initial value problem has a solution for any $\Phi\in C_b^2$. Or, in other words, the clumsy formula above leads to a description of the generator of a Lévy process.

4.4.1 We have shown that for every $n\in\mathbb{N}$ there are a stopping time T_n with $\mathbb{P}[T_n<n]<2^{-n}$, a process ${}^nV$ of finite variation, and a L^2-bounded martingale ${}^nM$ with $|\Delta^nM|\le 1$, both stopping at time T_n, such that $Z^{T_n}={}^nV+{}^nM$. Replacing if necessary T_n by $T'_n=\inf\{T_\nu:\ \nu\ge n\}$ and ${}^nV,{}^nM$ by the stopped processes ${}^nV^{T'_n},{}^nM^{T'_n}$, we may assume that the T_n *increase* to ∞. It looks as though the right-continuity of the filtration is used here (see exercise 1.3.30); it is left to the reader to show that its regularity suffices. The decomposition

$$Z^{T_{n+1}}-Z^{T_n}=\Big({}^{n+1}V-{}^{n+1}V^{T_n}\Big)+\Big({}^{n+1}M-{}^{n+1}M^{T_n}\Big)$$
$$\overset{\text{def}}{=}{}^{n\prime}V+{}^{n\prime}M$$

has the property that its finite variation and martingale parts stop at T_{n+1} and vanish on $[\![0,T_n]\!]$. We get the required decomposition $Z=V+M$ by setting

$$V={}^1V+\sum_{n=1}^\infty {}^{n\prime}V\quad\text{and}\quad M={}^1M+\sum_{n=1}^\infty {}^{n\prime}M\ ,$$

sums which at every point $\varpi\in\boldsymbol{B}$ have only finitely many non-zero terms.

4.4.4 By proposition 3.7.33 F is V–0-integrable and by exercise 3.6.16 and definition (4.2.9) F is M–2-integrable; now apply exercise 3.6.14.

4.4.6 There is a countable $\mathbb{Q}$-algebra $\mathcal{C} \subset C_{00}(\boldsymbol{H})$ whose sequential closure is $\mathcal{B}^\bullet(\boldsymbol{H})$. For every $\phi \in \mathcal{C}$ let P^ϕ be the sparse predictable support of $\phi{*}\zeta$ and set $P = \bigcup_{\phi\in\mathcal{C}} P^\phi$. Let S be a predictable stopping time. Then so is the time S' whose graph is $[\![S]\!] \setminus P$ (theorem 3.5.13). The $\check{H} \in \check{\mathcal{P}}$ with the property that $\Delta\big(\check{H}{*}\zeta\big)_{S'}$ is nearly zero is sequentially closed and contains ϕX for all $\phi \in \mathcal{C}$ and all $X \in \mathcal{E}$ (equation (3.10.2)). Then it contains $\check{\mathcal{P}}$.

4.5.3 Let us write $\|\,\boldsymbol{X}\,\|_T^\diamond$ for the right-hand side of (4.5.1). To start with assume T reduces $\boldsymbol{Z}$ to a global L^p-integrator. For any bounded $\boldsymbol{Y} \in \mathcal{P}^d$ with $|Y_\eta| \le |X_\eta|$ clearly $\|\int \boldsymbol{Y}\, d\boldsymbol{Z}^T\|_{L^p} \le \|\,\boldsymbol{X}\,\|_T^\diamond$. Corollary 3.6.10 implies that X_η is Z_η–p-integrable for $1 \le \eta \le d$. Setting $\boldsymbol{X}^{(n)} \stackrel{\text{def}}{=} \boldsymbol{X} \cdot [|\boldsymbol{X}| \le n]$, we have $\lceil\!\lceil (\boldsymbol{X} - \boldsymbol{X}^{(n)}){*}\boldsymbol{Z}^T \rceil\!\rceil_{\mathcal{I}^p} \le \sum_\eta \|\,X_\eta - X_\eta^{(n)}\,\|^*_{Z^T\!-p} \xrightarrow[n\to\infty]{} 0$ and by theorem 2.3.6 $\|\,(\boldsymbol{X} - \boldsymbol{X}^{(n)}){*}\boldsymbol{Z}_T^\star\,\| \xrightarrow[n\to\infty]{} 0$. Since $\|\,|\boldsymbol{X}^{(n)}{*}\boldsymbol{Z}|_T^\star\,\|_{L^p} \le \|\,\boldsymbol{X}\,\|_T^\diamond$, we have $\|\,|\boldsymbol{X}{*}\boldsymbol{Z}|_T^\star\,\|_{L^p} \le \|\,\boldsymbol{X}\,\|_T^\diamond$ in the limit. In general let T_n be stopping times reducing $\boldsymbol{Z}$ to global L^p-integrators. Then the previous argument gives $\|\,|\boldsymbol{X}{*}\boldsymbol{Z}|_{T\wedge T_n}^\star\,\|_{L^p} \le \|\,\boldsymbol{X}\,\|_T^\diamond$, which produces (4.5.1) in the limit as $n \to \infty$.

4.5.9 From $(1+c)^x - 1 \ge x\ln(1+c)$ we get, with $x = 1/p$,

$$\Big(\big(1 + (1/p')^p\big)^{1/p} - 1\Big) \ge \frac{1}{p}\ln\big(1 + (1-\ 1/p)^p\big)\ .$$

Since $p \mapsto (1-\ 1/p)^p$ increases this exceeds $p^{-1}\ln\big(1 + (1/2)^2\big) \ge 1/(5p)$.

4.5.11 Since $d[Z,Z] \le d\Lambda$, Itô's formula gives

$$|X{*}f(Z)|_T^\star \le |Xf'(Z){*}Z|_T^\star + |Xf''(Z){*}[Z,Z]|_T^\star/2 \qquad \text{, which implies}$$

$$\||X{*}f(Z)|_T^\star\|_{L^p} \le C_p^\diamond L \max_{\rho=1^\diamond,2}\Big\|\Big(\int_0^T |X|_s^\rho\, d\Lambda_s\Big)^{1/\rho}\Big\|_{L^p} + \frac{L}{2}\Big\|\Big(\int_0^T |X|_s\, d\Lambda_s\Big)\Big\|_{L^p}\ .$$

4.5.12 The Doléans–Dade measure of $\boldsymbol{Z}^{\langle 1\rangle}$ is the Doléans–Dade measure of ${}^1\boldsymbol{X}{*}\boldsymbol{Z}$ and by inequality (4.5.21) majorizes the Doléans–Dade measure of any other $\boldsymbol{X}'{*}\boldsymbol{Z}$ with $\boldsymbol{X}' \in \mathcal{E}_1^d$.

4.5.13 The differential form of equation (4.5.23) exhibits the Doléans–Dade measure of $\boldsymbol{Z}^{\langle 2\rangle}$ as the Doléans–Dade measure of $[{}^2\boldsymbol{X}{*}\boldsymbol{Z}, {}^2\boldsymbol{X}{*}\boldsymbol{Z}]$.

4.5.14 Set ${}^q\check{H} \stackrel{\text{def}}{=} \langle {}^q\boldsymbol{X}_s|\boldsymbol{y}\rangle$. The Doléans–Dade measure of $|{}^q\check{H}|^q{*}\jmath_{\boldsymbol{Z}}$ coincides by equation (4.5.25) with that of $\boldsymbol{Z}^{\langle q\rangle}$ and majorizes by definition the Doléans–Dade measures of all other $|\check{H}|^q{*}\jmath_{\boldsymbol{Z}}$, $\check{H}$ as indicated.

4.5.20 (i) Inequality (4.5.27) is obvious with $C_0 = 1$ when $\ell = 0$. Since $|d[Z^\eta, Z^\theta]| \le d\Lambda$, (4.5.28) follows from

$$\Big\|\int_{T^\kappa}^{T^\mu} g_\iota\cdot|Z^\iota - Z^{\iota T^\kappa}|^{\star\ell}\,|d[Z^\eta,Z^\theta]|\Big\|_{L^p} \le \Big\|\int_{T^\kappa}^{T^\mu} g_\iota\cdot\big|Z^\iota - Z^{\iota T^\kappa}\big|_s^{\star\ell}\, d\Lambda_s\Big\|_{L^p}$$

by theorem 2.4.7:

$$= \Big\|\int_\kappa^\mu g_\iota\cdot\big|Z^\iota - Z^{\iota T^\kappa}\big|_{T^\lambda}^{\star\ell}\, d\lambda\Big\|_{L^p}$$

by exercise A.3.29: $$\leq \int_\kappa^\mu \Big\| g_\iota \cdot |Z^\iota - Z^{\iota T^\kappa}|^{\star\ell}_{T^\lambda} \Big\|_{L^p} \, d\lambda$$

by inequality (4.5.27): $$\leq C_\ell \cdot \|\mathbf{g}\|_{L^p} \int_\kappa^\mu (\lambda{-}\kappa)^{\ell/2} \, d\lambda$$

$$\leq \frac{1}{\ell/2+1} C_\ell \, (\mu{-}\kappa)^{\ell/2+1} \cdot \|\mathbf{g}\|_{L^p} \,. \tag{C.1}$$

For $\ell = 1$, $g_\eta \cdot |Z^\eta - Z^{\eta T^\kappa}|^\star$ equals $\big|g_\eta \cdot ((T^\kappa, T^\mu]\!]{*}Z^\eta\big|^\star$, and theorem 4.5.1 gives

$$\Big\| g_\eta \cdot |Z^\eta - Z^{\eta T^\kappa}|^\star_{T^\mu} \Big\|_{L^p} \leq C_p^\diamond \cdot \max_{\rho=1,2} \Big\| \Big(\int_{T^\kappa}^{T^\mu} |g|^\rho \, d\Lambda \Big)^{1/\rho} \Big\|_{L^p}$$

by theorem 2.4.7: $$\leq C_p^\diamond \cdot \|\mathbf{g}\|_{L^p} \cdot \max_{\rho=1,2} (\mu{-}\kappa)^{1/\rho}$$

since $\mu{-}\kappa \leq 1$: $$\leq C_p^\diamond \, (\mu{-}\kappa)^{1/2} \cdot \|\mathbf{g}\|_{L^p} \,.$$

Let then $\ell > 1$ and assume inequality (4.5.27) has been established for indices up to ℓ. Now

$$\begin{aligned} (Z^\eta{-}Z^{\eta T^\kappa})^{\ell+1}_t &= (\ell{+}1)\Big(((T^\kappa, T^\mu]\!](Z^\eta{-}Z^{\eta T^\kappa})^\ell {*} Z^\eta \Big)_t \\ &\quad + \frac{(\ell{+}1)\ell}{2} \int_{T^\kappa}^t (Z^\eta{-}Z^{\eta T^\kappa})^{\ell-1} \, d[Z^\eta, Z^\eta] \\ &\leq (\ell{+}1) \Big| ((T^\kappa, T^\mu]\!](Z^\eta{-}Z^{\eta T^\kappa})^\ell {*} Z^\eta \Big|^\star_t \\ &\quad + \frac{(\ell{+}1)\ell}{2} \int_{T^\kappa}^t \big|Z^\eta{-}Z^{\eta T^\kappa}\big|^{\ell-1} \, \big|d[Z^\eta, Z^\eta]\big| \end{aligned}$$

implies

$$\begin{aligned} |Z^\eta{-}Z^\eta_{T^\kappa}|^{\star\ell+1}_{T^\mu} &\leq (\ell{+}1) \Big| ((T^\kappa, T^\mu]\!](Z^\eta{-}Z^{\eta T^\kappa})^\ell {*} Z^\eta \Big|^\star_{T^\mu} \\ &\quad + \frac{(\ell{+}1)\ell}{2} \int_{T^\kappa}^{T^\mu} \big|Z^\eta{-}Z^{\eta T^\kappa}\big|^{\ell-1} \, \big|d[Z^\eta, Z^\eta]\big| \,, \end{aligned} \tag{C.2}$$

the first term of which contributes not more than

$$\begin{aligned} &(\ell{+}1) C_p^\diamond \cdot \max_{\rho=1,2} \Big\| \Big(\int_{T^\kappa}^{T^\mu} \big|g_\eta |Z^\eta{-}Z^{\eta T^\kappa}|^{\star\ell}_s\big|^\rho \, d\Lambda_s \Big)^{1/\rho} \Big\|_{L^p} \\ &\quad = (\ell{+}1) C_p^\diamond \cdot \max_{\rho=1,2} \Big\| \Big(\int_\kappa^\mu \big|g_\eta |Z^\eta{-}Z^{\eta T^\kappa}|^{\star\ell}_{T^\lambda}\big|^\rho \, d\lambda \Big)^{1/\rho} \Big\|_{L^p} \end{aligned}$$

by A.3.29: $$\leq (\ell{+}1) C_p^\diamond \cdot \max_{\rho=1,2} \Big(\int_\kappa^\mu \Big\| g_\eta |Z^\eta{-}Z^{\eta T^\kappa}|^{\star\ell}_{T^\lambda} \Big\|^\rho_{L^p} d\lambda \Big)^{1/\rho}$$

by induction hyp.: $\leq (\ell{+}1)C_p^\diamond \cdot C_\ell \|\mathbf{g}\|_{L^p} \max_{\rho=1,2} \Big(\int_\kappa^\mu (\lambda-\kappa)^{\ell\rho/2}\, d\lambda\Big)^{1/\rho}$

$$= (\ell{+}1)C_p^\diamond \cdot C_\ell \|\mathbf{g}\|_{L^p} \max_{\rho=1,2} \cdot \frac{1}{(\ell\rho/2+1)^{1/\rho}} (\mu-\kappa)^{\ell/2+1/\rho}$$

as $\mu-\kappa\leq 1$: $\leq \sqrt{\ell{+}1}\, C_p^\diamond C_\ell (\mu-\kappa)^{(\ell+1)/2} \cdot \|\mathbf{g}\|$

to $\|\, g_\eta |Z^\eta - Z^{\eta T^\kappa}|_{T^\lambda}^{\star\ell+1} \|_{L^p}$. The contribution of the second term (C.2) can be estimated by

$$\frac{(\ell{+}1)\ell}{2} \frac{C_{\ell-1}}{(\ell{-}1)/2+1} \cdot \|\,\mathbf{g}\,\|_{L^p} = \ell C_{\ell-1} \cdot \|\,\mathbf{g}\,\|_{L^p} \ ,$$

due to inequality (C.1). This establishes the claim (i) and gives the recurrence relation

$$C_0 = 1\ ,\ C_1 \leq C_p^\diamond\ ,\ C_{\ell+1} \leq C_\ell \cdot C_p^\diamond \sqrt{2(\ell{+}2)} + C_{\ell-1}\cdot \ell \quad \text{for} \quad \ell{\geq}2\ .$$

4.5.21 Set $Q_1 \stackrel{\text{def}}{=} \mathbb{E}\big[\phi(A)\cdot\psi(\tfrac{1}{A})\big]$. Then

$$\begin{aligned}
\mathbb{E}[\phi(Y)\cdot\psi(\frac{1}{A})] &= \mathbb{E}\big[\phi(Y)\cdot\psi(\frac{1}{A})[Y\leq A]\big] + \mathbb{E}\big[\phi(Y)\cdot\psi(\frac{1}{A})[Y>A]\big] \\
&\leq Q_1 + \mathbb{E}\big[\big(\phi(Y)-\phi(A)\big)\cdot\psi(\frac{1}{A})[Y>A]\big] \\
&= Q_1 + \int [A<y\leq Y]\, d\phi(y)\, \Big[A \leq \frac{1}{a}\Big]\, d\psi(a)\, d\mathbb{P} \\
&\leq Q_1 + \int \mathbb{P}\Big[Y\geq y, A\leq y\wedge\frac{1}{a}\Big]\, d\phi(y)\, d\psi(a) \\
&\leq Q_1 + \int \frac{1}{y}\cdot\big(A\wedge y\wedge\frac{1}{a}\big)\, d\phi(y)\, d\psi(a)\, d\mathbb{P}\ .
\end{aligned}$$

Now

$$\begin{aligned}
q_2 &\stackrel{\text{def}}{=} \int \frac{1}{y}\cdot\big(A\wedge y\wedge\frac{1}{a}\big)\, d\phi(y)\, d\psi(a) \\
&= \int \frac{1}{y}\cdot y\wedge\frac{1}{a}\cdot\Big[A>\frac{1}{a}\Big]\, d\phi(y)\, d\psi(a) + \int \frac{1}{y}\cdot A\wedge y\cdot\Big[A\leq\frac{1}{a}\Big]\, d\phi(y)\, d\psi(a) \\
&= \int \Big[y\leq\frac{1}{a}\ ,\ a>\frac{1}{A}\Big]\, d\phi(y)\, d\psi(a) + \int \frac{1}{ya}\cdot\Big[y>\frac{1}{a}\ ,\ a>\frac{1}{A}\Big]\, d\phi(y)\, d\psi(a) \\
&+ \int \frac{A}{y}\cdot\Big[A<y\ ,\ a\leq\frac{1}{A}\Big]\, d\phi(y)\, d\psi(a) + \int \Big[y<A\ ,\ a\leq\frac{1}{A}\Big]\, d\phi(y)\, d\psi(a) \\
&\leq \int_{\frac{1}{A}}^\infty \phi(1/a)\, d\psi(a) + \int_{\frac{1}{A}}^\infty \Big(\frac{1}{a}\int_{1/a}^\infty \frac{d\phi(y)}{y}\Big)\, d\psi(a) \\
&+ A\int_A^\infty \frac{d\phi(y)}{y}\ \psi(\frac{1}{A}) + \phi(A)\cdot\psi(\frac{1}{A})
\end{aligned}$$

$$= \int_{\frac{1}{A}}^{\infty} \Phi(1/a)\, d\psi(a) + \Phi(A)\psi(\frac{1}{A})\ .$$

Inserting this into the previous inequality results in inequality (4.5.32). An easy manipulation shows that $\phi(y) = y^\beta$ has $\Phi(y) = y^\beta/(1-\beta)$ and that, with $\psi(a) = a^\alpha$,

$$\int_{\frac{1}{A}}^{\infty} \Phi(1/a)\, d\psi(a) = \frac{\alpha}{(1-\beta)(\beta-\alpha)} \cdot A^{\beta-\alpha}\ .$$

Consequently $\phi(Y)\cdot\psi(\frac{1}{A}) = A^{\alpha-\beta}$ and

$$\begin{aligned}
&\big(\Phi(A) + \phi(A)\big)\cdot\psi(\frac{1}{A}) + \int_{\frac{1}{A}}^{\infty} \Phi(1/a)\, d\psi(a) \\
&= \Big(1 + \frac{1}{1-\beta} + \frac{\alpha}{(1-\beta)(\beta-\alpha)}\Big)\cdot A^{\beta-\alpha} \\
&= \frac{(2-\beta)(\beta-\alpha)+\alpha}{(1-\beta)(\beta-\alpha)}\cdot A^{\beta-\alpha} \le \frac{2}{(1-\beta)(\beta-\alpha)}\cdot A^{\beta-\alpha}\ .
\end{aligned}$$

Taking expectations results in inequality (4.5.33).

4.5.26 The underlying filtration $\mathcal{F}_{.}$ is assumed right-continuous. A ***time transformation*** is an increasing collection $T^{\bullet} \overset{\text{def}}{=} \{T^\lambda : 0 \le \lambda < \infty\}$ of stopping times with $T^\lambda \xrightarrow[\lambda\to\infty]{} \infty$. $T^{\bullet}$ is called ***left-continuous*** or ***right-continuous*** provided it equals

$$T^{\bullet-} : \lambda \mapsto T^{\lambda-} \overset{\text{def}}{=} \lim\{T^\kappa : \lambda > \kappa \uparrow \lambda\}$$

or

$$T^{\bullet+} : \lambda \mapsto T^{\lambda+} \overset{\text{def}}{=} \lim\{T^\mu : \lambda < \mu \downarrow \lambda\}\ ,$$

respectively. With the three associated time transformations (see Exercises 1.3.15 and 1.3.30) $T^{\bullet-}$, $T^{\bullet}$, $T^{\bullet+}$ come the filtrations $\mathcal{F}_{T^{\bullet-}}$, $\mathcal{F}_{T^{\bullet}}$, $\mathcal{F}_{T^{\bullet+}}$. Clearly

$$T^{\lambda-} \le T^\lambda \le T^{\lambda+} \ \text{ and } \ \mathcal{F}_{T^{\lambda-}} \subseteq \mathcal{F}_{T^\lambda} \subseteq \mathcal{F}_{T^{\lambda+}}\ , \qquad \lambda \ge 0\ .$$

It is the last filtration, $\underline{\mathcal{F}}_{.} \overset{\text{def}}{=} \mathcal{F}_{T^{\bullet+}} = \{\mathcal{F}_{T^{\lambda+}} : 0 \le \lambda < \infty\}$, that we single out, for its right-continuity (ibidem). By interpreting λ as a new time parameter, the time transformations $T^{\bullet-}$, $T^{\bullet}$, and $T^{\bullet+}$ can also be thought of as $\underline{\mathcal{F}}_{.}$-adapted increasing processes, $T^{\bullet-}$ left-continuous, and $T^{\bullet+}$ right-continuous. The finite random variable

$$\Lambda_t \overset{\text{def}}{=} \inf\{\lambda : T^{\lambda-} > t\} = \inf\{\lambda : T^\lambda > t\} = \inf\{\lambda : T^{\lambda+} > t\} \qquad \text{(C.3)}$$

is, for every $t \ge 0$, an $\underline{\mathcal{F}}_{.}$-stopping time and defines a right-continuous time transformation $\Lambda_{.}$ on $\underline{\mathcal{F}}_{.}$ (*ibidem*). Considered as a process, $t \mapsto \Lambda_t$ is easily seen to be $\mathcal{F}_{.}$-adapted. Indeed, for any $\mu > 0$ we have $[\Lambda_t < \mu]$

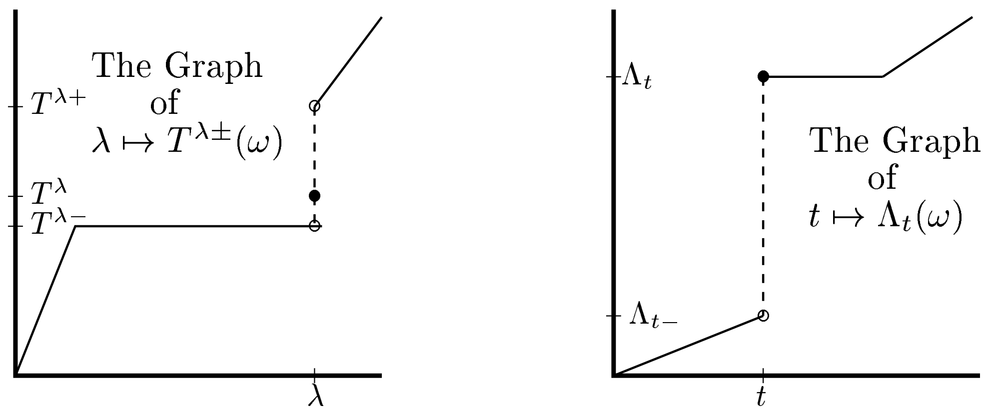

Figure C.19 A Time Transformation

$= \bigcup\{[T^{\lambda+} > t] : \lambda < \mu\} = \bigcup\{[T^q > t] : \mathbb{Q} \ni q < \mu\} \in \mathcal{F}_t$. Its left-continuous version is at $t > 0$ given by

$$\Lambda_{t-} = \inf\{\lambda : T^{\lambda+} \geq t\}\ . \tag{C.4}$$

Lemma C.1 *(i) Equalities (C.3) and (C.4) hold. (ii) For* $\lambda, t \geq 0$

$$T^{\lambda-} = \inf\{t : \Lambda_t \geq \lambda\} \leq T^{\lambda} \leq T^{\lambda+} = \inf\{t : \Lambda_t > \lambda\}\ ; \tag{C.5}$$

$$[T^{\lambda-} \leq t] = [\lambda \leq \Lambda_t] \text{ and } [\Lambda_{t-} \leq \lambda] = [t \leq T^{\lambda+}]\ ; \tag{C.6}$$

and thus
$$T^{\lambda-} \leq t \leq T^{\lambda+} \iff \Lambda_{t-} \leq \lambda \leq \Lambda_t\ . \tag{C.7}$$

(iiia) The following are equivalent: $\Lambda_{\bullet}$ *is strictly increasing; for all* $\lambda \geq 0$, $T^{\lambda-} = T^{\lambda} = T^{\lambda+}$*; one of* $T^{\bullet-}$, $T^{\bullet}$, $T^{\bullet+}$ *is continuous; all of them are.*
(iiib) $T^{\bullet}$ *is strictly increasing if and only if* $\Lambda_{\bullet}$ *is continuous.*
(iv) The T^{λ} *are finite (everywhere, nearly, almost surely) if and only if* $\Lambda_t \xrightarrow[t\to\infty]{} \infty$ *(everywhere, nearly, almost surely). The* T^{λ} *are bounded if and only if* $\inf\{\Lambda_t(\omega) : \omega \in \Omega\} \xrightarrow[t\to\infty]{} \infty$.
(v) If T *is an* $\mathcal{F}_{\bullet}$*-stopping time, then* Λ_T *is an* $\underline{\mathcal{F}}_{\bullet}$*-stopping time; if* L *is an* $\underline{\mathcal{F}}_{\bullet}$*-stopping time, then* $T^{L+} = \underleftarrow{T}^{L}$ *is an* $\mathcal{F}_{\bullet}$*-stopping time.*

(vi)
$$\Lambda_\infty \stackrel{\text{def}}{=} \sup\{\lambda : T^{\lambda} < \infty\} \quad \text{is an } \underline{\mathcal{F}}_{\bullet}\text{-stopping time.}$$

(vii) If the T^{λ} *are nearly finite, then* $\mathcal{F}_{\bullet}$ *and* $\mathcal{F}_{T^{\bullet}}$ *have the same nearly empty sets.*

Proof. (i) If $\inf\{\lambda : T^{\lambda+} > t\} < \mu$, then $T^{\lambda} > t$ for some $\lambda < \mu$ and thus $T^{\mu-} > t$ and $\inf\{\lambda : T^{\lambda-} > t\} \leq \mu$: $\inf\{\lambda : T^{\lambda-} > t\} \leq \inf\{\lambda : T^{\lambda+} > t\}$ follows, and with the reverse inequality being obvious we get equality throughout (C.3). As to (C.4), both sides of the equation define left-continuous functions of t that agree unless the level set $[T^{\bullet+} = t]$ has strictly positive length, which can happen only countably often. (ii) The inequalities in (C.5) are obvious. The equalities follow directly from the right-continuity of $\Lambda_\bullet$ and $T^{\bullet+}$ in conjunction with (C.3) and (C.4). Equation (C.7) is but a summary of (C.6). (iii) Clearly $T^{\bullet-}$ is left-continuous and $T^{\bullet+}$ is right-continuous; they agree iff they are continuous. The equalities in (C.5) make it obvious that they do agree iff $\Lambda_\bullet$ has no level sets $[\Lambda_\bullet = t]$ of strictly positive length, i.e., iff $\Lambda_\bullet$ is strictly increasing. (v) If Λ takes countably many values λ_i, then $[T^\Lambda \le t] = \bigcup [T^\Lambda \le t, \Lambda = \lambda_i] = \bigcup [T^{\lambda_i} \le t] \cap [\Lambda = \lambda_i]$ belongs to $\mathcal{F}_t$ inasmuch as $[T^{\lambda_i} \le t] \in \mathcal{F}_t$ and $[\Lambda = \lambda_i] \in \mathcal{F}_{T^{\lambda_i}}$. In the general case use the stopping times $\Lambda^{(n)}$ of exercise 1.3.20 and the right-continuity of both $\underleftarrow{T}^{\bullet}$ and of $\underleftarrow{\mathcal{F}}_{T^\bullet}$. The same argument shows that Λ_T is an $\underline{\mathcal{F}}_\bullet$-stopping time. (vi) $[\lambda < \Lambda_\infty] = [T^{\lambda+} < \infty] \in \underline{\mathcal{F}}_\lambda$. (vii) Use equation (C.5) and exercise 3.5.19. Let now $\underline{\boldsymbol{B}}$ denote a copy of the base space $\boldsymbol{B} = [0,\infty) \times \Omega$ and denote its typical point by (λ, ω). Since $T^{\lambda-}$ may well be infinite, we need to augment $\boldsymbol{B}$ temporarily: $\boldsymbol{B}'$ is the base space with the graph $[\![\infty]\!] = \{\infty\} \times \Omega$ of the infinite stopping time adjoined:

$$\boldsymbol{B}' \stackrel{\text{def}}{=} [0,\infty] \times \Omega \;=\; \boldsymbol{B} \cup [\![\infty]\!] \,.$$

The *left-continuous* time transformations $T^{\bullet-}$ and $\Lambda_{\bullet-}$ give rise to maps (see figure C.20)

$$T^{\bullet-} : \underline{\boldsymbol{B}} \to \boldsymbol{B}' \quad \text{via} \quad (\lambda, \omega) \mapsto \big(T^{\lambda-}(\omega), \omega\big)$$

and

$$\Lambda_{\bullet-} : \boldsymbol{B} \to \underline{\boldsymbol{B}}' \quad \text{via} \quad (t, \omega) \mapsto \big(\Lambda_{t-}(\omega), \omega\big) \,,$$

in which the image of $[\![0, \Lambda_\infty)\!)$ lies in $\boldsymbol{B}$ and that of $\boldsymbol{B}$ in $[\![0, \Lambda_\infty)\!)$, respectively. If both $\Lambda_\bullet$ and $T^{\bullet-}$ are continuous or are strictly increasing, then $T^{\bullet-}$ and $\Lambda_{\bullet-}$ are inverses of each other.

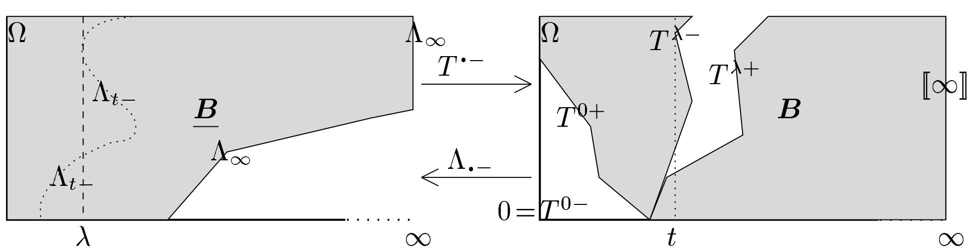

Figure C.20 A Time Transformation

Let us denote by an underscore the composition with $T^{\bullet-}$. To be quite precise, for $X : \boldsymbol{B} \to \mathbb{R}$,

$$\underline{X}_\lambda(\omega) \stackrel{\text{def}}{=} \begin{cases} X_{T^{\lambda-}}(\omega) & \text{where } T^{\lambda-}(\omega) < \infty, \text{ i.e., on } [\![0, \Lambda_\infty)\!) \\ \quad 0 & \text{on } [\![\Lambda_\infty, \infty)\!), \text{ i.e., for } \Lambda_\infty(\omega) \le \lambda < \infty. \end{cases}$$

If X is progressively measurable for $\mathcal{F}_\bullet$, then $\underline{X}$ is adapted to $\underline{\mathcal{F}}_\bullet$ (proposition 1.3.9); if X is left-continuous, then clearly so is $\underline{X}$. The usual sequential closure argument shows that $X \mapsto \underline{X}$ takes $\mathcal{F}_{\bullet}$-predictable processes to $\underline{\mathcal{F}}_{\bullet}$-predictable processes that vanish on $[\![\Lambda_\infty, \infty)\!)$.

Definition C.2 *An L^p-integrator Z is called **compatible** with the time transformation $T^{\bullet}$ if (a) the stopped process $Z^{T^{\mu+}}$ is $\mathcal{I}^p$-bounded for all $\mu < \infty$ and (b) Z is nearly constant in time on every interval of the form $[\![T^{\lambda-}, T^{\lambda+}]\!]$ – by lemma C.1 (iii) this holds in particular when $\Lambda_{\bullet}$ is strictly increasing.*

Exercise C.3 If the $T^{\lambda-}$ are predictable, as is typical, then the compatibility simply means that $\bigcup_{\lambda>0} [\![T^{\lambda-}, T^{\lambda+}]\!]$ is Z-negligible.

Proposition C.4 *Let $0 \le p < \infty$ and suppose the L^p-integrator Z is compatible with the time transformation $T^{\bullet}$. Then $\underline{Z}$ is nearly right-continuous and adapted to $\underline{\mathcal{F}}_{\bullet}$; in fact, it is an L^p-integrator on $\underline{\mathcal{F}}_{\bullet}$ and satisfies*

$$\lceil\!\lceil \underline{Z}^{\mu} \rceil\!\rceil_{\mathcal{I}^p} \le \lceil\!\lceil Z^{T^{\mu}} \rceil\!\rceil_{\mathcal{I}^p} . \tag{C.8}$$

Moreover, for every Z–p-integrable process X the process $\underline{X}$ is $\underline{Z}$–p-integrable and

$$\int_{\boldsymbol{B}} X_s \, dZ_s = \int_{\underline{\boldsymbol{B}}} \underline{X}_\lambda \, d\underline{Z}_\lambda . \tag{C.9}$$

Proof. For every $\lambda \ge 0$ let N_λ be the nearly empty set of $\omega \in \Omega$ so that $t \mapsto Z_t(\omega)$ is not constant for $T^{\lambda-}(\omega) \le t \le T^{\lambda+}(\omega)$. The representation

$$N \stackrel{\text{def}}{=} \bigcup_\lambda N_\lambda = \bigcup_n \Big(\bigcup_\lambda N^\lambda \cap [T^{\lambda+} > T^{\lambda-} + 1/n < \infty] \Big)$$

exhibits their union N as a countable union of nearly empty sets, which is therefore nearly empty. Indeed, there are at most countably many λ's for which the sets in the inner union are non-void. Upon removal of N we are left with a process Z whose paths are càdlàg and constant in time on every interval of the form $[\![T^{\lambda-}, T^{\lambda+}]\!]$ not only nearly but in fact everywhere. If $\lambda_n \downarrow \lambda$, then $\underline{Z}_{\lambda_n} = Z_{T^{\lambda_n -}} = Z_{T^{\lambda_n +}} \xrightarrow[n\to\infty]{} Z_{T^{\lambda+}} = Z_{T^{\lambda-}} = \underline{Z}_\lambda$, therefore $\underline{Z}$ is right-continuous at any $\lambda \ge 0$.

Let then $$X \stackrel{\text{def}}{=} f_0 [\![0]\!] + \sum_{n=1}^{N} f_n \cdot (\!(\lambda_n, \lambda_{n+1}]\!] , \qquad f_n \in L^\infty(\underline{\mathcal{F}}_{\lambda_n}) ,$$

be a typical elementary integrand from $\mathcal{E}[\underline{\mathcal{F}}_{\bullet}]$ with $\lambda_{N+1} \le \mu$ (see (2.1.1)).

Then $$\int X \, d\underline{Z} = f_0 \cdot Z_0 + \textstyle\sum_{n=1}^N f_n \cdot \big(Z_{T^{\lambda_{n+1}}} - Z_{T^{\lambda_n}}\big) = \displaystyle\int X' \, dZ ,$$

where $$X' \stackrel{\text{def}}{=} f_0 \cdot [\![0]\!] + \textstyle\sum_{n=1}^N f_n \cdot (\!(T^{\lambda_n}, T^{\lambda_{n+1}}]\!] \in \mathcal{P}[\mathcal{F}_{\bullet}]$$

From this equation (C.8) is evident.

Let $X = f \cdot (\!(s,t]\!]$ with $f \in \mathcal{F}_s$. Then[3]

$$\underline{(\!(s,t]\!]}_\lambda = [s < T^{\lambda-} \le t] = [\Lambda_s < \lambda \le \Lambda_t] = (\!(\Lambda_s, \Lambda_t]\!]_\lambda$$

[3] In accordance with convention A.1.5 on page 364 sets are identified with their (idempotent) indicator functions. A stochastic interval $(\!(S,T]\!]$, for instance, has at the instant s the value $(\!(S,T]\!]_s = [S < s \le T] = \begin{cases} 1 & \text{if } S(\omega) < s \le T(\omega) \\ 0 & \text{elsewhere} \end{cases}$.

and $$\int_{\boldsymbol{B}} X\, dZ = \int f\cdot (\!(s,t]\!]\, dZ = f\cdot\big(Z_t - Z_s\big)$$

as $T^{\Lambda_t -} \le t \le T^{\Lambda_t}$: $$= f\cdot\big(Z_{T^{\Lambda_t}} - Z_{T^{\Lambda_s}}\big) = \int f\cdot (\!(\Lambda_s,\Lambda_t]\!]\, d\underline{Z}$$

$$= \int \underline{X}_\lambda\, d\underline{Z}_\lambda \ .$$

By linearity, (C.9) is true for $X\in\mathcal{E}$, and then for bounded predictable X. It is a matter of bookkeeping to extend this to Z–0-integrable processes X. ∎

C.5 An integrator Z compatible with $T^{\bullet}$ has $\underline{[Z,Z]} = [\underline{Z},\underline{Z}]$.

Proof. For $0\le\mu<\infty$

$$\int_{0+}^{T^{\mu-}} Z_{\bullet-}\, dZ = \int_{\boldsymbol{B}} (\!(0,T^{\mu-}]\!]\cdot Z_{\bullet-}\, dZ = \int_{[\![0,\Lambda_\infty)\!)} (\!(0,T^{\mu-}]\!]_{T^{\lambda-}}\cdot \underline{Z}_{\lambda-}\, d\underline{Z}_\lambda$$

$$= \int (\!(0,T^{\mu+}]\!]_{T^{\lambda-}}\underline{Z}_{\lambda-}\, d\underline{Z}_\lambda = \int_{0+}^{\mu} \underline{Z}_{\lambda-}\, d\underline{Z}_\lambda \ .$$

Thus $$\underline{[Z,Z]}_\mu = [Z,Z]_{T^{\mu-}} = Z^2_{T^{\mu-}} - Z_0^2 - 2\int_{0+}^{T^{\mu-}} Z_{\bullet-}\, dZ$$

$$= \underline{Z}^2_\mu - \underline{Z}^2_0 - 2\int_{0+}^{\mu} \underline{Z}_{\lambda-}\, d\underline{Z}_\lambda = [\underline{Z},\underline{Z}]_\mu \ .$$

∎

C.6 Let M be a continuous local martingale on $(\Omega,\mathcal{F},\mathbb{P})$, set $\Lambda = [M,M]$, and introduce the time transformations $T^{\lambda-}, T^{\lambda+}$ of (C.5), setting $T^\lambda \stackrel{\text{def}}{=} T^{\lambda-}$. By inequality (4.2.8), M is constant on the intervals $[\![T^{\lambda-},T^{\lambda+}]\!]$, so $\underline{M}$ is a continuous local martingale. If $\Lambda_t \xrightarrow[t\to\infty]{} \infty$, then by item C.5 and corollary 3.9.5, $\underline{M}$ is a standard Wiener process on $\underline{\mathcal{F}}_{\bullet}$. If $\mathbb{P}[\Lambda_\infty<\infty]>0$, let W be a Wiener process on $(\Omega',\mathcal{F}'_{\bullet},\mathbb{P}')$ that is independent of $(\mathcal{F}_\infty,\mathbb{P})$, and set $\underline{M}' \stackrel{\text{def}}{=} [\![0,\Lambda_\infty)\!)*\underline{M} + [\![\Lambda_\infty,\infty)\!)*W$. Show that $\underline{M}'$ is a standard Wiener process on $(\Omega\times\Omega',\mathcal{F}_{\bullet}\times\mathcal{F}'_{\bullet},\mathbb{P}\times\mathbb{P}')$.

Sure Control of Integrators can be had in a manner similar to item C.6. Suppose $\boldsymbol{Z}$ is a vector of L^q-integrators, $q\ge 2$, and $\Lambda = \Lambda^{\langle q\rangle}[\boldsymbol{Z}]$ is its previsible controller from theorem 4.5.1. It is nary a loss of generality to assume that Λ is strictly increasing and $\Lambda_t \xrightarrow[t\to\infty]{} \infty$ (see remark 4.5.2). By lemma C.1, we then have equality of continuous time transformations

$$T^{\lambda-} \stackrel{\text{def}}{=} \inf\{t : \Lambda_t \ge \lambda\} = T^\lambda \stackrel{\text{def}}{=} T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t:\Lambda_t > \lambda\} \ .$$

Since $\Lambda_\infty = \infty$, the map $T^{\bullet-} : \underline{\boldsymbol{B}} \to \boldsymbol{B}$ is continuous and surjective. The controlling estimate (4.5.1) turns into

$$\big\|\, |\underline{\boldsymbol{X}}*\underline{\boldsymbol{Z}}|^\star_\Lambda \big\|_{L^p} \le C_p^\diamond \cdot \max_{\rho=1^\diamond,p^\diamond} \Big\| \Big(\int_0^{\Lambda+} |\underline{\boldsymbol{X}}|^\rho_\lambda\, d\lambda\Big)^{1/\rho} \Big\|_{L^p}$$

for any $\underline{\mathcal{F}}_{\cdot}$-stopping time Λ and any $p \in [2, q]$. Continue on.

4.6.5 (adapt exercise 1.3.47 or theorem 5.7.3 (iii)).

4.6.14 (i) Equation (4.6.19) has the easy consequence

$$\mathbb{E}\Big[\exp\Big(i\int h\, d\jmath_{\boldsymbol{Z}}\Big)\Big] = \mathbb{E}\Big[\exp\Big(i\sum_j \alpha_j \jmath_{\boldsymbol{Z}}(A_j)\Big)\Big]$$
$$= \prod_j \exp\Big((\nu\times\lambda)(A_j)\big(e^{i\alpha_j} - 1\big)\Big) ,$$

showing that the random variables $\jmath_{\boldsymbol{Z}}(A_j)$ are independent Poisson with means $(\nu\times\lambda)(A_j)$.

4.6.21 $\mathcal{A}$ gives rise to a Lévy process $\boldsymbol{Z}$ (page 267). The definition (4.6.31) produces a Feller semigroup whose generator is necessarily dissipative and conservative and is given by $\mathcal{A}$ (equation (4.6.32)) on $\mathcal{S}$. $\mathcal{S}$ is dense in C_0 and invariant under both $T_{\cdot}^{\mathcal{D}}$ and $T_{\cdot}^{\mathcal{J}}$, thus under $T_{\cdot}$, and therefore is a core for $\mathcal{A}$ (exercise A.9.4).

5.1.7 (i) The function $f(x) \stackrel{\text{def}}{=} \int_0^{|x|} s \wedge 1\, ds$ of example A.2.48 serves again: the paths $e^{\bullet}/n$ converge to zero in $\mathfrak{s}_1$, yet the remainder

$$RF(e^{\bullet}/n; 0) = f(e^{\bullet}/n) - f(0) - f'(0)\cdot e^{\bullet}/n = f(e^{\bullet}/n)$$

has $$\|RF(e^{\bullet}/n; 0)\|_1 = \|e^{\bullet}/n\|_1 \neq o(\|e^{\bullet}/n\|_1) .$$

(ii) On the positive side, if $M^\circ > M$, pick $n \in \mathbb{N}$ and let $\epsilon > 0$ be given. There exists a $t > n$ so that $2Le^{(M-M^\circ)t} \le \epsilon$. There exists a $\delta > 0$ so that for all $u, v \in U$ in the ball of radius n and all $x, y \in \mathbb{R}^n$ in the ball of radius $ne^{M^\circ t}$, $|v{-}u| + |y{-}x| \le \delta$ implies $\big\|Df(v, y) - Df(u, x)\big\| \le \epsilon$. Let $|v{-}u| + \|y_{\cdot} - x_{\cdot}\| \le \delta e^{-Mt}$. Then $|y_s - x_s| \le \delta$ for $s \le t$ and so

$$\big|Rf(v, y_s; u, x_s)\big| \le e^{Ms}\int_0^1 \Big\|Df\big((u, x_s) + \lambda(v{-}u, y_s{-}x_s)\big) - Df(u, x_s)\Big\| d\lambda$$
$$\times \|y_{\cdot} - x_{\cdot}\|_M$$
$$\le \begin{Bmatrix} \epsilon e^{Ms} & \text{for } s \le t \\ e^{Ms}2L \le \epsilon e^{M^\circ s} & \text{for } s > t \end{Bmatrix} \times \|y_{\cdot} - x_{\cdot}\|_M ,$$

which implies $$\frac{\|Rf[v, y_{\cdot}; u, x_{\cdot}]\|_{M^\circ}}{\|y_{\cdot} - x_{\cdot}\|_M} \le \epsilon .$$

5.1.8 (i) Both $\xi^f_{\cdot+s}(x)$ and $\xi^f_{\cdot}\big(\xi^f_s(x)\big)$ satisfy $dX_t = f(X_t)\, dt$ with $X_0 = \xi^f_s(x)$. (ii) Hint: Define $D\xi^f_t[x]$ as the solution of equation (5.1.24) on page 278. Then write a differential equation for $\Delta_{\cdot} \stackrel{\text{def}}{=} \xi^f_{\cdot}(x) - \xi^f_{\cdot}(x') - D\xi^f_{\cdot}[x]\cdot(x{-}x')$. Then show, using inequality (5.1.16) on page 276, that $\|\Delta_{\cdot}\|/|x{-}x'| \xrightarrow[x-x'\to 0]{} 0$. This means that $D\xi^f_{\cdot}[x]$ is the Fréchet derivative at

x of $\xi^f_{\bullet} : x' \mapsto \xi_{\bullet}(x')$, map from $\mathbb{R}^n$ to $\mathfrak{s}$ (see definition A.2.49 on page 390). As for equation (5.1.24), both sides answer the same initial value problem.

5.1.10 (ii) By the chain rule, $\sigma \mapsto \Xi^{\boldsymbol{f}}[x, \boldsymbol{z}_\sigma]$ solves (5.1.25).

5.1.11 (i) The exponential limit for the growth in time follows from item 5.1.4.

(ii) Set ${}^0\xi'[x,t] \stackrel{\text{def}}{=} \xi'[x,t] - x$ and $0 \le s \le \delta$. Since ${}^0\xi'[c,0] = 0 \quad \forall\, c \in \mathbb{R}^n$, we have $\xi'_{;\nu}[c,0] = 0$ for all $c \in \mathbb{R}^n$ and, for some c_s between c, c',

$$\begin{aligned} {}^0\xi'[c',s] - {}^0\xi'[c,s] &= {}^0\xi'_{;\nu}[c_s,s](c'-c)^\nu \\ &= \big({}^0\xi'_{;\nu}[c_s,s] - {}^0\xi'_{;\nu}[c_s,0]\big)(c'-c)^\nu \ , \end{aligned}$$

whence $$\big|{}^0\xi'[c',s] - {}^0\xi'[c,s]\big| \le L'\delta \cdot |c'-c| \le \big(e^{L'\delta} - 1\big) \cdot |c'-c|$$

and $$\big|\xi'[c',s] - \xi'[c,s]\big| \le e^{L'\delta} \cdot |c'-c| \ , \qquad 0 \le s \le \delta \ .$$

(iii) For fixed t and δ set $k \stackrel{\text{def}}{=} \lceil t/\delta \rceil$ and $t_i \stackrel{\text{def}}{=} i\delta$ for $i = 0, 1, \ldots, k$. Then $t_{k-1} < t \le t_k$. Let $\Delta^\star_i$ denote the maximal function of the difference of the global solution at t_i, which is $x_{t_i} = \xi[c, t_i]$, from its ξ'-approximate x'_{t_i}. Consider an $s \in [t_i, t_{i+1}]$.

Since $$\begin{aligned} x'_s - x_s &= \xi'\big[x'_{t_i}, s-t_i\big] - \xi'\big[x_{t_i}, s-t_i\big] \\ &\quad + \xi'\big[x_{t_i}, s-t_i\big] - \xi[x_{t_i}, s-t_i] \ , \end{aligned}$$

we have $$\big|\Delta^\star_{i+1}\big| \le \big|\Delta^\star_i\big| \times e^{L'\delta} + (|x|^\star_{t_i}+1) \times (\underline{m}\delta)^r e^{\underline{m}\delta} \ ,$$

which implies $$\big|\Delta^\star_k\big| \le (|x|^\star_{t_k}+1) \times (\underline{m}\delta)^r e^{\underline{m}\delta} \cdot \sum_{0 \le i < k} e^{iL'\delta}$$

for $x_{\bullet} \in \mathfrak{s}_M$, as $t_k = k\delta$: $$\le (\|x_{\bullet}\|_M e^{Mt_k} + 1) \times (\underline{m}\delta)^r e^{\underline{m}\delta} \cdot \frac{e^{L'k\delta} - 1}{e^{L'\delta} - 1}$$

by (5.1.16), as $\delta \to 0$: $$\le \frac{2}{1-\gamma} \Big(\|c\|_M + \frac{|f(0)|}{eM} + 1\Big)\, e^{Mt_k} \times (\underline{m}\delta)^r e^{\underline{m}\delta} \cdot k\, e^{L't_k}$$

since $k = t_k/\delta$: $$\begin{aligned} &\le \mathit{const}\, (\|c\|_M + 1)\underline{m}^r e^{\underline{m}\delta} \delta^{r-1} \times t_k \cdot e^{(M+L')t_k} \\ &\le \overline{b} \cdot (|c|+1) \times \delta^{r-1} \cdot e^{\overline{m}t_k} \end{aligned}$$

for suitable $\overline{b} = \overline{b}[f;\xi']$ and $\overline{m} = \overline{m}[f;\xi'] > M+L'$.

5.2.3 We know already from inequality (5.2.6) that

$$\Big\|\big(\boldsymbol{F}^\nu * \boldsymbol{Z}\big)^\star_{T^\mu-}\Big\|^*_{L^p} \le C^\diamond_p \max_{\rho=1,p} \Big\|\Big(\int_0^\mu |\boldsymbol{F}^\nu_{T^\lambda-}|^\rho_\infty \, d\lambda\Big)^{1/\rho}\Big\|^*_{L^p}$$

by exercise A.3.29: $$\le C^\diamond_p \max_{\rho=1,p} \Big(\int_0^\mu \big\||\boldsymbol{F}^{\nu\star}_{T^\lambda-}|_\infty\big\|^{*\rho}_{L^p} \, d\lambda\Big)^{1/\rho} .$$

Taking the p-norm for counting measure and using Fubini's theorem gives

$$\Big\|\big|\big(\boldsymbol{F} * \boldsymbol{Z}\big)^\star_{T^\mu-}\big|_p\Big\|^*_{L^p} \le C^\diamond_p \max_{\rho=1,p} \Big(\int_0^\mu \big\||\boldsymbol{F}^\star_{T^\lambda-}|_{\infty p}\big\|^{*\rho}_{L^p} \, d\lambda\Big)^{1/\rho}$$

Measuring in $L^p\big(Me^{-M\mu}\,d\mu\big)$ the part that appears for $\rho = p$ on the right-hand side gives a number less than $M^{-1/p}\cdot \big\| |\boldsymbol{F}|_\infty \big\|^{\bullet}_{p,M}$. For the case $\rho = 1$ we set $f(\lambda) \stackrel{\text{def}}{=} \| |\boldsymbol{F}_{T^\lambda -}|^\star_{\infty p} \|^{*\rho}_{L^p}$ and $\alpha \stackrel{\text{def}}{=} M/(p+p')$ and estimate

$$\begin{aligned}
\int_0^\mu f(\lambda)\,d\lambda &= \int [\lambda \le \mu] e^{\alpha\lambda}\cdot[\lambda\le\mu] f(\lambda) e^{-\alpha\lambda} d\lambda \\
&\le \Big(\int_0^\mu e^{\alpha p'\lambda} d\lambda\Big)^{1/p'} \cdot \Big(\int_0^\mu f(\lambda)^p e^{-\alpha p\lambda} d\lambda\Big)^{1/p} \\
&< \Big(\frac{1}{\alpha p'}\Big)^{1/p'} e^{\alpha\mu}\cdot \Big(\int [\lambda\le\mu] f(\lambda)^p e^{-\alpha p \lambda}\,d\lambda\Big)^{1/p}
\end{aligned}$$

whence

$$\begin{aligned}
\int\Big(\int_0^\mu f(\lambda)\,d\lambda\Big)^p M e^{-M\mu} d\mu &< \Big(\frac{1}{\alpha p'}\Big)^{\frac{p}{p'}}\cdot \iint [\lambda\le\mu] f(\lambda)^p e^{-\alpha p\lambda} M e^{(\alpha p - M)\mu} d\mu d\lambda \\
&< \Big(\frac{1}{\alpha p'}\Big)^{p/p'}\cdot \int f(\lambda)^p e^{-\alpha p \lambda} \frac{M}{\alpha p - M} e^{(\alpha p - M)\mu}\Big|_\lambda^\infty\, d\lambda \\
&= \Big(\frac{1}{\alpha p'}\Big)^{p/p'} \frac{1}{M-\alpha p}\cdot \int_0^\infty f(\lambda)^p\, M e^{-M\lambda}\,d\lambda \\
&\le \Big(\frac{1}{\alpha p'}\Big)^{p/p'} \frac{1}{M-\alpha p}\cdot \big\| |\boldsymbol{F}|_\infty \big\|^{\bullet p}_{p,M} \\
&= \Big(\frac{p}{M}\Big)^p \cdot \big\| |\boldsymbol{F}|_\infty \big\|^{\bullet p}_{p,M}
\end{aligned}$$

Thus $$\big\| \boldsymbol{F}_{.-}*\boldsymbol{Z} \big\|^{\bullet}_{p,M} \le C_p^\diamond \Big(\frac{1}{M^{1/p}} \vee \frac{p}{M}\Big)\cdot \big\| |\boldsymbol{F}|_\infty \big\|^{\bullet p}_{p,M} .$$

5.2.16 needed

5.2.17 First consider the equation $X = 1 + X{*}W$, whose solution is the Doléans–Dade exponential $\mathscr{E}_t = e^{W_t - t/2}$ of W (see proposition 3.9.2 on page 159). Here $\Lambda_t^{\langle q\rangle}[W] = t \quad \forall\, q$, and by exercise 4.5.6 on page 240, $d\Lambda_t^{\langle q\rangle}[\mathscr{E}] = e^{qW_t - qt/2}\,dt$. Now if $\Lambda_t^{\langle q\rangle}[\mathscr{E}]$ were dominated by a sure controller ξ, i.e., $d\Lambda_t^{\langle q\rangle}[\mathscr{E}] \le d\xi$, then the absolutely continuous part $d\xi^{\|}$ of $d\xi$ would have to have a locally integrable Radon–Nikodym Derivative $h(t) = d\xi^{\|}(t)/dt < \infty$, which would have to satisfy $e^{qW_t - qt/2} \le h(t)$ almost surely; since $\mathbb{P}[W_t > K] > 0$ for all $K \in \mathbb{N}$ and $t > 0$, this is impossible. We see that some boundedness assumption on the value ${}^0\boldsymbol{F}[\boldsymbol{X}]$ of ${}^0\boldsymbol{F}$ at the solution $\boldsymbol{X}$ of theorem 5.2.15 on page 291 is needed.

Assume then that $\Lambda^{\langle q\rangle}$ is dominated by the sure controller η: $d\Lambda^{\langle q\rangle}[\boldsymbol{Z}] \le d\eta$, and that ${}^0\boldsymbol{F}[\boldsymbol{X}]$ is a bounded process. By exercise 4.5.6 on page 240, the controllers $\Lambda^{\langle q\rangle}[X^\eta]$ of the components X^η of $\boldsymbol{X}$ satisfy $d\Lambda^{\langle q\rangle}[X^\eta] \le$ *const* $\times\, d\eta$, and then so does $d\Lambda^{\langle q\rangle}[\boldsymbol{X}]$.

5.2.20 needed

5.2.21 ${}^0\mathfrak{U}[X] \stackrel{\text{def}}{=} \mathfrak{U}[X] - C$ has the same contractivity modulus $\gamma < 1$ as $\mathfrak{U}$. Thus $\|C\|^\star_{p,M} = \|X - {}^0\mathfrak{U}[X]\|^\star_{p,M} \le \|X\|^\star_{p,M} + \|{}^0\mathfrak{U}[X] - {}^0\mathfrak{U}[0]\|^\star_{p,M} \le (1+\gamma)\|X\|^\star_{p,M}$.

5.2.22 Apply (5.2.34) with $C = C[u], C' = C[v]$ and $\boldsymbol{F}[\,\cdot\,] = \boldsymbol{F}[u,\,\cdot\,], \boldsymbol{F}'[\,\cdot\,] = \boldsymbol{F}[v,\,\cdot\,]$.

5.2.24 : Let $t < \infty$. $\boldsymbol{Z}^t$ is an L^p-integrator for some $p > \dim U$ and an equivalent probability $\mathbb{P}'$. The assumed Lipschitz conditions imply that the stopped processes $X[u]^t$ satisfy (5.2.41) for $\mathbb{P}'$ (proposition 5.2.22). Corollary 5.2.23 shows that they are nearly continuous in $u \in U$. Then let $t \to \infty$.

5.3.11 (i) Let A be a symmetric bilinear form on euclidean space $\mathbb{R}^n$. Then $\sup\{|A(\xi,\eta)| : |\xi|_2 \le 1, |\eta|_2 \le 1\} = \sup\{|A(\xi,\xi)| : |\xi|_2 \le 1\}$. This is easily seen by diagonalizing the symmetric matrix A; in fact, $|A(\xi,\eta)|$ is ***strictly less*** than the supremum above unless ξ and η are collinear and of unit euclidean length. (ii) Let now A be a symmetric k-linear form on euclidean space $\mathbb{R}^n$. Then $\sup\{|A(\xi_1,\ldots,\xi_k)| : |\xi_i|_2 \le 1, 1{\le}i{\le}k\}$ is taken at a k-tuple $(\xi_1,\ldots,\xi_k)$ of unit vectors. Considering $A(\xi_1,\xi_2,\ldots)$ a bilinear form in (ξ_1,ξ_2) and using (i) shows that $\xi_1 = \pm\xi_2$. Similarly $\xi_i = \pm\xi_j$ for $1 \le i < j \le k$. Thus the supremum is taken also at a k-tuple of the form $(\xi,\xi,\ldots,\xi)$. (iii) Next let D^* be a k-linear scalar form on a seminormed space $(E, \|\ \|_E)$. Let $x_i \in E_1$, $i = 1,\ldots,k$, with $a < \|D^*(x_1,\ldots,x_k)\|_S$. Define the k-linear form A on euclidean space $\mathbb{R}^k$ by $A(\xi_1,\ldots,\xi_k) \stackrel{\text{def}}{=} D^*\big(\sum \xi_1^\kappa x_\kappa, \ldots, \sum \xi_k^\kappa x_\kappa\big)$. By (ii) there is a $\xi \in \mathbb{R}^k$ so that, with $x \stackrel{\text{def}}{=} \sum \xi^\kappa x_\kappa$, $a < D^*(x,\ldots,x)$. Now $\|x\|_E \le \sum_\kappa |\xi_\kappa| \le k^{1/2}$. Thus $a < k^{k/2}\{\sup D^*(x,\ldots,x) : \|x\|_E \le 1\}$. (iv) Finally let D be a k-linear map from a seminormed space $(E,\|\ \|_E)$ to another seminormed space $(S,\|\ \|_S)$. Let $a < \sup\{\|D(x_1,\ldots,x_k)\|_S : \|x_i\|_E \le 1\}$. There are a linear form y^* in the dual S^* that has norm $\|y^*\|_{S^*} \le 1$ and elements $x_1,\ldots,x_k$ in E so that $D^* \stackrel{\text{def}}{=} y^* \circ D$ has $a < \|D^*(x_1,\ldots,x_k)\|_S$. By (iii), there is an $x \in E_1$ with $a < k^{k/2} D^*(x,\ldots,x)$:
$\sup\{\|D(x_1,\ldots,x_k)\|_S : \|x_i\|_E \le 1\} \le k^{k/2}\sup\{\|D(x,\ldots,x)\|_S : \|x\|_E \le 1\}$.

5.3.18 Let us pick a $u \in U$, and write D^λ for $D^\lambda F[u]$ and $T^\lambda[v]$ for $T^\lambda F[u](v)$. For $v, v' \in U$

$$
\begin{aligned}
T^l[v'] - T^l[v] &= \sum_{0\le\lambda\le l} \frac{D^\lambda}{\lambda!} \cdot \Big[(v'-u)^{\otimes\lambda} - (v-u)^{\otimes\lambda}\Big] \\
&= \sum_{0\le\lambda\le l} \frac{D^\lambda}{\lambda!} \cdot \Big[\sum_{0\le i\le\lambda} \binom{\lambda}{i} (v-u)^{\otimes\lambda-i} \otimes (v'-v)^{\otimes i} - (v-u)^{\otimes\lambda}\Big] \\
&= \sum_{0\le\lambda\le l} \frac{D^\lambda}{\lambda!} \cdot \sum_{0<i\le\lambda} \binom{\lambda}{i} (v-u)^{\otimes\lambda-i} \otimes (v'-v)^{\otimes i}
\end{aligned}
$$

$$= \sum_{0<i\leq l} \sum_{i\leq\lambda\leq l} \frac{\binom{\lambda}{i} D^\lambda}{\lambda!} \cdot (v{-}u)^{\otimes\lambda-i} \otimes (v'{-}v)^{\otimes i} ,$$

whence

$$D^i T^l[v]{\cdot}\xi^{\otimes i} = \sum_{i\leq\lambda\leq l} \frac{i!\binom{\lambda}{i} D^\lambda}{\lambda!} \cdot (v{-}u)^{\otimes\lambda-i} \otimes \xi^{\otimes i} \qquad , \ i = 1, \cdots, l$$

$$= D^i F[u]{\cdot}\xi^{\otimes i} + \sum_{i<\lambda\leq l} \frac{D^\lambda}{(\lambda-i)!} \cdot (v{-}u)^{\otimes\lambda-i} \otimes \xi^{\otimes i}$$

$$= D^i T^l[u]{\cdot}\xi^{\otimes i} \ + \ D^{i+1} \cdot \xi^{\otimes i} \otimes (v{-}u) \ + \ r(v{-}u) ,$$

where each summand of $r(v{-}u)$ contains a power $(v{-}u)^{\otimes p}$ with $p \geq 2$. Hence $v \mapsto D^i T^l F[u](v){\cdot}\xi^{\otimes i}$ has i^{th} derivative $\eta \mapsto D^{i+1} F[u] \cdot \xi^{\otimes i} \otimes \eta$ at $v = u$, for $i = 1, \ldots, l-1$. In fact, since the derivatives appearing in $r(v{-}u)$ are bounded, $v \mapsto D^i T^l F[u](v){\cdot}\xi^{\otimes i}$ is uniformly differentiable.
Let us define $G[v] \stackrel{\text{def}}{=} F[v] - T^l F[u](v)$. This function is l-times weakly uniformly differentiable with $D^0 G[u] = \ldots = D^l G[u] = 0$ and $\| G[u] \|^\circ_S = o\big(\| v{-}u \|^l_E\big)$. It is left to prove that $v \mapsto D^i G[v] \cdot \xi^{\otimes i}$ has vanishing derivative at u, for $i = 1, \ldots, l-1$. Let $\xi, \xi' \in E$ be unit vectors and $\tau \in \mathbb{R}$ and set $v \stackrel{\text{def}}{=} u + \tau\xi$, $v' \stackrel{\text{def}}{=} u + \tau\xi'$

Then both $G[v'] - G[v] = \big(G[u + \tau\xi]{-}G[u]\big) - \big(G[u + \tau\xi']{-}G[u]\big)$

$$= \sum_{0<i\leq l} \tau^i D^i G[v] \cdot (\xi' - \xi)^{\otimes i} + R^l G[v; v']$$

and $R^l G[v; v']$ are $o(\tau^l)$ when measured with $\| \ \|^\circ_S$, independently of the location of $u \in U$. Hence so is the sum above. This implies

$$\frac{D^i G[\tau v] \cdot (\xi' - \xi)^{\otimes i}}{\tau^{l-i}} \xrightarrow[\tau\to 0]{} 0 \qquad , \ i = 0, \ldots, l ;$$

and therefore $\qquad D\big\{v \mapsto D^i G[v] \cdot (\xi' - \xi)^{\otimes i}\big\} = 0$ at $u \quad , \ i = 0, \ldots, l{-}1$.

5.3.19 Answer forthcoming.
5.4.8 Since the mesh of $\mathcal{T}$ goes to zero, we have $\big|\boldsymbol{F}'[X]{-}\boldsymbol{F}[X]\big|_\infty \to 0$ pointwise on $\boldsymbol{H} \times \boldsymbol{B}$, after composition with THE time transformation $T^\bullet$ on $\{1, \ldots, n\} \times \{\lambda : 0 \leq \lambda < \infty\} \times \Omega$. The Dominated Convergence Theorem produces $\big\| F^{\mathcal{T}}[X]{-}F[X] \big\|^\bullet_{p,M} \xrightarrow[\delta\to 0]{} 0$ for all $M > 0$ after integration suitable powers over $d\mathbb{P}$ and $d\lambda$, and exercise 5.2.3 on page 285 yields $\big\| F^{\mathcal{T}}[X]{-}F[X] \big\|^\star_{p,M} \xrightarrow[\delta\to 0]{} 0$.
5.4.12 Apply theorem 3.9.24 on page 170.
5.4.17 (i) Let P be such that $\big|\Phi_{;\nu}[x, \boldsymbol{z}]\big| \leq P(|\boldsymbol{z}|)$ for $\Phi = \Xi_{;\eta}$ and $\Phi = \Xi'_{;\eta\nu} \Xi'^{\nu}_{;\theta}$. In fact, let Φ be one of these (Itô-) coefficients, say the one with

index ι, and, given λ, pick k so that $\kappa \overset{\text{def}}{=} k\delta \le \lambda < (k+1)\delta$. Write $\boldsymbol{\zeta}^T \overset{\text{def}}{=} \boldsymbol{Z} - \boldsymbol{Z}^T$. Then

$$\left\| \overline{F}'_\iota[Y]_{T^\lambda} - \overline{F}'_\iota[X]_{T^\lambda} \right\|_{L^p} = \left\| \Phi[Y_{T^\kappa}, \boldsymbol{\zeta}^{T^\kappa}_{T^\lambda}] - \Phi[X_{T^\kappa}, \boldsymbol{\zeta}^{T^\kappa}_{T^\lambda}] \right\|_{L^p}$$

by the mean value theorem: $$\le \left\| \left|Y - X\right|^\star_{T^\kappa} P\big(\left|\boldsymbol{Z}_{T^\lambda} - \boldsymbol{Z}_{T^\kappa}\right|\big) \right\|_{L^p}$$

by inequality (4.5.29): $$\le \left\| \left|Y - X\right|^\star_{T^\kappa} \right\|_{L^p} \times P'\big(\sqrt{\lambda - \kappa}\big)$$

with $L \overset{\text{def}}{=} P'(\sqrt{\delta})$: $$\le L \cdot \left\| \left|Y - X\right|^\star_{T^\kappa} \right\|_{L^p} \le L \cdot \left\| \left|Y - X\right|^\star_{T^\lambda} \right\|_{L^p} .$$

(ii) From $\Xi'[C, \boldsymbol{z}] = C + \Xi'_{;\eta}[C, 0] z^\eta + \Xi'_{;\eta\theta}[C, \overline{\boldsymbol{z}}] z^\eta z^\theta$ and the assumption $\Xi'_{;\eta}, \Xi'_{;\eta\theta} \in \mathcal{BP}$, we see that ${}^0\Xi'[C, \boldsymbol{z}] \overset{\text{def}}{=} \Xi'[C, \boldsymbol{z}] - C$ has $\left|{}^0\Xi'[C, \boldsymbol{z}]\right| \le P(|\boldsymbol{z}|)$, P being a polynomial with zero constant term. Therefore, by (4.5.27), $\| \left|{}^0\Xi'[C, \boldsymbol{\zeta}^{T^\kappa}]\right|^\star_{T^\lambda} \|_{L^p} \le P'\big(\sqrt{\lambda - \kappa}\big)$, P' being another polynomial with zero constant term. From this, inequality (5.4.31) is evident. Also, we see here that $\| \left|{}^0\Xi'[C', \boldsymbol{\zeta}^{T^\kappa}_{\bullet}] - {}^0\Xi'[C, \boldsymbol{\zeta}^{T^\kappa}_{\bullet}]\right|^\star_{T^\lambda} \|_{L^p}$ is bounded by a multiple of $\sqrt{\lambda - \kappa}$ for λ in some neighborhood of κ, whence inequality (5.4.32) for $\lambda \approx \kappa$.

For large λ we write $\left|\Xi'[C', \boldsymbol{z}] - \Xi'[C, \boldsymbol{z}]\right| \le |C' - C| \cdot P(|\boldsymbol{z}|)$.

This implies

$$\left\| \left|\Xi'[C', \boldsymbol{\zeta}^{T^\kappa}_{\bullet}] - \Xi'[C, \boldsymbol{\zeta}^{T^\kappa}_{\bullet}]\right|^\star_{T^\lambda} \right\|_{L^p} \le \left\|C' - C\right\|_{L^p} \cdot P'(\sqrt{\lambda - \kappa})$$

for some (other) L': $$\le \left\|C' - C\right\|_{L^p} \cdot e^{L'(\lambda - \kappa)} .$$

5.4.20 $X - X'$ solves $\Delta = \big\{\big(\overline{F}[X']_\iota - \overline{F}'_\iota[X']_\iota\big) * \overline{Z}^\iota\big\} + \big(\overline{F}_\iota[\Delta + X'] - \overline{F}_\iota[X']\big) * \overline{Z}^\iota$, so by exercise 5.2.21, $\big\| \big(\overline{F}[X']_\iota - \overline{F}'_\iota[X']_\iota\big) * \overline{Z}^\iota \big\|_{p,M} = o(\delta)$. By theorems 4.3.1 and 2.3.6 the martingale parts $(\bullet\bullet) * \widetilde{Z}^\eta$ satisfy the same estimate; if $\boldsymbol{Z}$ happens to be a martingale, this reads $\big\| \big(f_\eta(X') - F'_\eta[X']\big) * Z^\eta \big\|_{p,M} = o(\delta)$. If it so happens that $\boldsymbol{Z}$ is a standard Wiener process, then it follows from theorem 4.2.12 that $\| \big\{ \int_0^\delta \big(f_\eta(X'_s) - F'_\eta[X']_s\big)^2 ds \big\}^{1/2} \|_{L^p} = o(\delta)$. In particular for $p = 2$, $\int_0^\delta \| f_\eta\big(C + {}^0\Xi'(s)\big) - {}^0\Xi'_{;\eta}(s) \|^2_{L^2} \, ds = o(\delta^2)$. Now the integrand is the square of a function ϕ that obeys $|\phi(t) - \phi(s)| \le const \cdot \sqrt{|t - s|}$ (see inequality (5.5.9) on page 333). If such ϕ also has $\int_0^\delta \phi^2(s) ds = o(\delta^2)$, it must have $\phi(\delta) = o(\sqrt{\delta})$. Else there exists an a with $\phi(t) \ge a\sqrt{t}$ arbitrarily close to 0. With b the Hölder 1/2-continuity modulus, $\phi(s) \ge a\sqrt{t} - b\sqrt{t - s} \ge 0$ for $\gamma t \le s \le t$. $\gamma = (b^2 - a^2)/b^2$. Then $\int_0^t \phi^2(s)\, ds \ge \int_{\gamma t}^t \phi^2(s) \ge C(a, b) t^2$. Contradiction. This is equation (5.4.36).

5.4.31 Set $K \overset{\text{def}}{=} t/\delta$. Then the number of evaluations of ξ' is

$$\begin{aligned} N &= |\boldsymbol{Z}_\delta|/\delta + |\boldsymbol{Z}_{2\delta}|/\delta + \ldots + |\boldsymbol{Z}_{K\delta}|/\delta \\ &= (\delta + |\boldsymbol{W}_\delta|) + (2\delta + |\boldsymbol{W}_{2\delta}|) + \ldots (K\delta + |\boldsymbol{W}_{K\delta}|) \end{aligned}$$

and has expectation

$$N_1 = \frac{const}{\delta}\Big(\delta K(K+1)/2 + const\textstyle\sum_1^K \sqrt{k\delta}\Big)$$
$$\approx \frac{const}{\delta}\Big(t^2/\delta + const\sqrt{\delta}\cdot K^{3/2}\Big)$$
$$\approx \frac{b_1 t^2 + b_2 t^{3/2}}{\delta^2} = \frac{B_1(t)}{\delta^2}\ .$$

By inequality (5.4.50) on page 328, $\delta^{-2} \approx B_1^{2/r} e^{2M_1 t/r}$, whence the claim.

5.4.32 We pick an exponent $p \geq 2$ and a growth constant $M > M^{\diamond(5.2.20)}_{p,L}$ to our liking, and let $\gamma \stackrel{\text{def}}{=} \gamma^{(5.2.21)}_{p,M,L} < 1$ denote the modulus of contractivity of $Y \mapsto \mathfrak{U}[Y] \stackrel{\text{def}}{=} C + \overline{\boldsymbol{f}}(Y) * \boldsymbol{Z}$ in $\mathfrak{S}^{\star n}_{p,M}$ that goes with these choices. Having picked a $\delta > 0$, we fix the partition by $T_k \stackrel{\text{def}}{=} k\cdot\delta$, set $X'_0 \stackrel{\text{def}}{=} c$, and continue recursively: when at time T_k the approximate solution X'_t has been defined for $0 \leq t \leq T_k$ and the signal $\boldsymbol{Z}_t$ has been observed at time $t \in [T_k, T_{k+1}]$ set[4],

$$\big(\boldsymbol{f}\cdot(\boldsymbol{Z}_t - \boldsymbol{Z}^{T_k}_t)\big)(x) \stackrel{\text{def}}{=} \textstyle\sum_\eta f_\eta(x)\zeta^{k\eta}_t\ ,$$

and
$$X'_t \stackrel{\text{def}}{=} \xi'[X'_{T_k}, 1; \boldsymbol{f}\cdot(\boldsymbol{Z}_t - \boldsymbol{Z}^{T_k}_t)]\ .$$

The next and last item on the agenda is of course the estimation of the deviation of the exact solution X to (5.4.25) from its Ξ'-approximate X' made with step size δ. We set the stage. X is the solution to the fixed point problem

$$X_t = C + \mathfrak{U}[X]_t \quad \text{, where } \ \mathfrak{U}[Y] \stackrel{\text{def}}{=} \overline{f}_\eta(Y) * Z^\eta\ ;$$

the Ξ'-approximate X' is the one and only *continuous* process with

$$X'_t = C + \mathfrak{U}'[X']_t \stackrel{\text{def}}{=} C + \sum_k (\!(T_k, T_{k+1}]\!]_t\, \xi'[X'_{T_k}, 1; \boldsymbol{f}\cdot(\boldsymbol{Z}_t - \boldsymbol{Z}^{T_k}_t)]\ ;$$

and X'' shall be the auxiliary process

$$X''_t = C + \mathfrak{U}''[X'']_t \stackrel{\text{def}}{=} C + \sum_k (\!(T_k, T_{k+1}]\!]_t\, \xi[X'_{T_k}, 1; \boldsymbol{f}\cdot(\boldsymbol{Z}_t - \boldsymbol{Z}^{T_k}_t)]\ .$$

Then
$$\Delta_t \stackrel{\text{def}}{=} X'_t - X_t = \mathfrak{U}'[X']_t - \mathfrak{U}[X]_t$$
$$= \underbrace{\mathfrak{U}'[X']_t - \mathfrak{U}[X'']_t}_{} + \underbrace{\mathfrak{U}[X'']_t - \mathfrak{U}[X']_t}_{} + \underbrace{\mathfrak{U}[\Delta + X]_t - \mathfrak{U}[X]_t}_{}$$
$$= \quad D^{(1)}_t \quad + \quad D^{(2)}_t \quad + \int_0^t \overline{G}_\eta[\Delta]_s\, dZ^\eta_s\ .$$

[4] In practice one would not want to do this for all $t \in [\![T_k, T_{k+1}]\!]$, of course, only for $t = T_{k+1}$, $k = 0, 1, \ldots$. The generality adopted is in keeping with our definition 5.4.14.

This exhibits Δ as the solution of a stochastic differential equation whose (endogenous but non-autologous) coupling coefficients $\overline{G}_\eta[\Delta_t] \stackrel{\text{def}}{=} \overline{f}_\eta(\Delta_t+X_t) - \overline{f}_\eta(X_t)$ are Lipschitz with constant L and vanish at $\Delta = 0$, and with non-constant initial condition $D^{(1)} + D^{(2)}$. According to inequality (5.1.16) on page 276,

$$\|\Delta\|^\star_{p,M} \leq \frac{1}{1-\gamma} \cdot \left(\|D^{(1)}\|^\star_{p,M} + \|D^{(2)}\|^\star_{p,M} \right) .$$

To estimate Δ it suffices therefore to estimate $D^{(1)}$ and $D^{(2)}$. We start with $D^{(2)}$. For $T_k \leq s \leq T_{k+1}$ we have

$$\begin{aligned} |X''_s - X'_s| &= |\xi[X'_{T_k}, 1; \boldsymbol{f}\cdot(\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s)] - \xi'[X'_{T_k}, 1; \boldsymbol{f}\cdot(\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s)]| \\ &\leq \left(\underline{m}[\boldsymbol{f}\cdot(\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s); \xi'] \cdot s\right)^{r+1} e^{\underline{m}[\boldsymbol{f}\cdot(\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s); \xi'] \cdot s} . \end{aligned}$$

With
$$m \stackrel{\text{def}}{=} \sup\{\underline{m}[f_\eta z^\eta; \xi'] : |\boldsymbol{z}| \leq 1\} \tag{C.10}$$

this gives

$$|X''_s - X'_s| \leq \left(m|\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s|\right)^{r+1} e^{m|\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s|} . \tag{C.11}$$

There is of course an assumption hidden in definition (C.10), namely
Condition C.7 *If ξ' is locally of order $r+1$ on $f_1, \ldots, f_d$, then $\underline{m}[\,\cdot\,, \xi']$ is bounded on their convex hull.* (If, as is often the case, $\underline{m}[f; \xi']$ can be estimated by polynomials in the uniform bounds of various derivatives of f, then the present condition is easily verified.)

Therefore

$$D^{(2)}_t = \int_0^t \overline{f}_\eta(X''_s) - \overline{f}_\eta(X'_s)\, dZ^\eta$$

has size

$$\begin{aligned} \Big\| |D^{(2)}|^\star_{T_k} \Big\|_{L^p} &\leq C_p^{\diamond(4.5.1)} d \cdot \max_{\rho=1^\diamond,2} \left\| \left(\int_0^{T_k} \left| \overline{f}_\eta(X''_s) - \overline{f}_\eta(X'_s) \right|^\rho ds \right)^{1/\rho} \right\|_{L^p} \\ &\leq C_p^\diamond dL \cdot \max_{\rho=1^\diamond,2} \left\| \left(\int_0^{T_k} |X''_s - X'_s|^\rho\, ds \right)^{1/\rho} \right\|_{L^p} \\ \text{by A.3.29:} \quad &\leq C_p^\diamond dL \cdot \max_{\rho=1^\diamond,2} \left(\int_0^{T_k} \|X''_s - X'_s\|^\rho_{L^p}\, ds \right)^{1/\rho} \\ &= C_p^\diamond dL \cdot \max_{\rho=1^\diamond,2} \left(\sum_{0\leq\kappa<k} \int_{T_\kappa}^{T_{\kappa+1}} \|X''_s - X'_s\|^\rho_{L^p}\, ds \right)^{1/\rho} \\ \text{by (C.11):} \quad &\leq C_p^\diamond dL \cdot \max_{\rho=1^\diamond,2} \left(\sum_{0\leq\kappa<k} \int_{T_\kappa}^{T_{\kappa+1}} \left\| \left(m|\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s|\right)^{r+1} e^{m|\boldsymbol{Z}_s - \boldsymbol{Z}^{T_k}_s|} \right\|^\rho_{L^p} ds \right)^{\frac{1}{\rho}} \\ &= C_p^\diamond dLm^{r+1} \cdot \max_{\rho=1^\diamond,2} k^{1/\rho} \cdot \left(\int_0^\delta \left\| |\boldsymbol{Z}_s|^{r+1} e^{m|\boldsymbol{Z}|_s} \right\|^\rho_{L^p} ds \right)^{\frac{1}{\rho}} \end{aligned} \tag{C.12}$$

by 5.2.18: $$\le C_p^\diamond dLm^{r+1}B'\cdot\max_{\rho=1^\diamond,2}(T_k/\delta)^{1/\rho}\cdot\Big(\int_0^\delta\big(s^{(r+1)/2}\,e^{M's}\big)^\rho ds\Big)^{\frac{1}{\rho}}$$

$$\le C_p^\diamond dLm^{r+1}B'(T_k\vee\sqrt{T_k})e^{M'\delta}\cdot\Big(\frac{\delta^{-1}\delta^{\frac{(r+1)}{2}+1}}{(r+1)/2+1}\vee\frac{\delta^{\frac{-1}{2}}\delta^{r/2+1}}{(r/2+1)/2}\Big)$$

$$\le C_p^\diamond dLm^{r+1}B'e^{M'\delta}\delta^{(r+1)/2}\times\big(T_k\vee\sqrt{T_k}\big)\ .$$

From this it is evident that there is a constant $B^{(2)}=B^{(2)}_{d,m,p,r}$ such that

$$\Big\|D_t^{(2)\star}\Big\|_{L^p}\le B^{(2)}\cdot\delta^{(r+1)/2}\times\big(t\vee\sqrt{t}\big)\qquad\forall\,t\ge0$$

for small δ. Multiplying this by e^{-Mt} and taking the supremum over $t\ge0$ gives

$$\big\|D^{(2)}\big\|^\star_{p,M}\le B^{(2)}\cdot\delta^{(r+1)/2}\ .$$

for sufficiently small $\delta>0$ and suitable constant $B^{(2)}$. Equality (C.12) is justified by the observation that $\boldsymbol{Z}_s$ and $\boldsymbol{Z}_{T_k+s}-\boldsymbol{Z}_{T_k}$ have the same distribution (exercise 3.9.7).

It is time to estimate $D^{(1)}$. On an interval $(\!(T_k,T_{k+1}]\!]$ we have

$$\mathfrak{U}'[X']_t=\xi'[X'_{T_k},1;\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']$$

$$=\mathfrak{U}'[X']_{T_k}+\Big(\xi'[X'_{T_k},1;\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']-X'_{T_k}\Big)$$

and $$\mathfrak{U}[X'']_t=\mathfrak{U}[X'']_{T_k}+\int_{T_k}^t\overline{f}_\eta(X''_s)\,dZ_s^\eta$$

$$=\mathfrak{U}[X'']_{T_k}+\Big(\xi[X'_{T_k},1;\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']-X'_{T_k}\Big)\ .$$

Thus $$\big|D_t^{(1)}-D_{T_k}^{(1)}\big|=\big|\xi'[X'_{T_k},1;\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']-\xi[X'_{T_k},1;\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']\big|$$

$$\le\big(\underline{m}[\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']\cdot1\big)^{r+1}\times e^{m[\boldsymbol{f}\cdot(\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k});\xi']\cdot1}\ ,$$

using (C.10): $$\le(m\cdot|\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k}|)^{r+1}\,e^{m|\boldsymbol{Z}_t-\boldsymbol{Z}_t^{T_k}|}\ ,$$

whence

$$\sup_{T_k\le t\le T_{k+1}}\big|D_t^{(1)}-D_{T_k}^{(1)}\big|\le(m\cdot|\boldsymbol{Z}-\boldsymbol{Z}^{T_k}|^\star_{T_{k+1}})^{r+1}\,e^{m|\boldsymbol{Z}-\boldsymbol{Z}^{T_k}|^\star_{T_{k+1}}}$$

and $D_{T_{k+1}}^{(1)\star}-D_{T_k}^{(1)\star}\le(m\cdot|\boldsymbol{Z}-\boldsymbol{Z}^{T_k}|^\star_{T_{k+1}})^{r+1}\,e^{m|\boldsymbol{Z}-\boldsymbol{Z}^{T_k}|^\star_{T_{k+1}}}$.

Hence $$D_{T_k}^{(1)\star}\le\sum_{0\le\kappa<k}(m\cdot|\boldsymbol{Z}-\boldsymbol{Z}^{T^\kappa}|^\star_{T_{\kappa+1}})^{r+1}\,e^{m|\boldsymbol{Z}-\boldsymbol{Z}^{T^\kappa}|^\star_{T_{\kappa+1}}}\ .$$

Now the distribution of $|\boldsymbol{Z}-\boldsymbol{Z}^{T^\kappa}|^\star_{T_{\kappa+1}}$ is exactly that of $\boldsymbol{Z}^\star_\delta$, thus

$$\Big\|D_{T_k}^{(1)\star}\Big\|_{L^p}\le m^{r+1}k\cdot\Big\||\boldsymbol{Z}|^\star_\delta\,e^{m|\boldsymbol{Z}|^\star_\delta}\Big\|_{L^p}$$

by 5.2.18: $\leq m^{r+1} T_k \delta^{-1} \cdot B' \delta^{(r+1)/2} e^{M'\delta}$.

From this it is evident that there is a constant $B^{(1)} = B^{(1)}_{d,m,p,r}$ such that for small δ

$$\left\| D_t^{(1)\star} \right\|_{L^p} \leq B^{(1)} \cdot \delta^{r/2-1/2} \times t \quad \forall\, t \geq 0 \, .$$

5.5.5 Let $\mathbb{L}^{(n)}$ be a sequence of laws in $\mathfrak{L} \stackrel{\text{def}}{=} \big\{(\boldsymbol{Z}_{\boldsymbol{.}}, X_{\boldsymbol{.}})[\mathbb{P}] : X_{\boldsymbol{.}} \in \mathbb{X}_{AB}\big\}$. There is a subsequence, which we will again denote by $\mathbb{L}^{(n)}$, such that $\mathbb{L}^{(n)}(\phi) \xrightarrow[n\to\infty]{} \mathbb{L}^{(\infty)}(\phi)$ for all $\phi \in C_b\big(\mathscr{C}^{d+n}[0,\infty)\big)$, defining a positive linear functional $\mathbb{L}^{(\infty)}$ of mass one on $C_b\big(\mathscr{C}^{d+n}[0,\infty)\big)$. Due to Kolmogorov's theorem A.3.17 and the established tightness of the projection of $\mathbb{L}^{(\infty)}$ on the $\mathscr{C}^{d+n}[0,u]$, $u \in \mathbb{N}$, $\mathbb{L}^{(\infty)}$ is σ-additive; due to the polish nature of $\mathscr{C}^{d+n}[0,\infty)$ it is even tight. Thus it belongs to the weak closure of $\{\mathbb{L}^{(n)}\}$ in the set of probabilities on $\mathscr{C}^{d+n}[0,\infty)$. This argument shows that $\mathfrak{L}$ is relatively compact in the topology of weak convergence of measures. The uniform tightness now follows from exercise A.4.8.

5.6.2 (i)

$$\begin{aligned}
dD\overline{D} &= D_{\boldsymbol{.}-} d\overline{D} + dD\overline{D}_{\boldsymbol{.}-} + [D,\overline{D}] \\
&= D_{\boldsymbol{.}-} d\overline{Y} \overline{D}_{\boldsymbol{.}-} + D_{\boldsymbol{.}-} dY \overline{D}_{\boldsymbol{.}-} + D_{\boldsymbol{.}-} d[Y,\overline{Y}]\overline{D}_{\boldsymbol{.}-} \\
&= D_{\boldsymbol{.}-}(-dY + d^c[Y,Y] + dJ)\overline{D}_{\boldsymbol{.}-} + D_{\boldsymbol{.}-} dY \overline{D}_{\boldsymbol{.}-} \\
&\qquad + D_{\boldsymbol{.}-}(-d[Y,Y] + d[Y,{}^c[Y,Y]] + d[Y,J])\overline{D}_{\boldsymbol{.}-} \\
&= D_{\boldsymbol{.}-}(d^c[Y,Y] + dJ)\overline{D}_{\boldsymbol{.}-} + D_{\boldsymbol{.}-}(-d[Y,Y] + d[Y,J])\overline{D}_{\boldsymbol{.}-} \\
&= D_{\boldsymbol{.}-}(dJ - {}^j[Y,Y] + d[Y,J])\overline{D}_{\boldsymbol{.}-} = D_{\boldsymbol{.}-}(\Delta J - (\Delta Y)^2 + \Delta Y \Delta J)\overline{D}_{\boldsymbol{.}-} \\
&= D_{\boldsymbol{.}-}\big((I + \Delta Y)\Delta J - (\Delta Y)^2\big)\overline{D}_{\boldsymbol{.}-} = 0 \, .
\end{aligned}$$

(ii) is evident.

(5.6.10) Let $\mathbb{R}^n \ni x_n \to x$. Then $X_t^{x_n} \to X_t^x$ in probability (use inequality (5.2.36), if necessary after a change of measure that turns $\boldsymbol{Z}^t$ into an L^2-integrator). Next let $\phi \in C_{00}(\mathbb{R}^n)$, supported on the ball $B_r(0)$ of radius r, say. Then the probability that $X_{\boldsymbol{.}}^x$ enters $B_r(0)$ before time t is less than $C^{(5.2.25)} \big|\boldsymbol{f}\big|_{\infty p} e^{Mt}/(|x| - r) \xrightarrow[x\to\infty]{} 0$. This implies $T_t\phi(x) \xrightarrow[x\to\infty]{} 0$. Approximating an arbitrary $\phi \in C_0$ uniformly by functions of compact support will establish the claim $T_t\phi \in C_0$.

5.7.2 (ii) Let $s < t$, say.

$$\mathbb{E}^x\Big[|\phi(X_t) - \phi(X_s)|^2\Big] = \mathbb{E}^x\Big[\phi^2(X_s) - 2\phi(X_s)\cdot\phi(X_t) + \phi^2(X_t)\Big]$$

condition on $\mathcal{F}_s$ and use (5.7.1): $= \mathbb{E}^x\Big[\big(\phi^2 - 2\phi \cdot T_{t-s}(\phi) + T_{t-s}(\phi^2)\big) \circ X_s\Big]$

by equation (5.7.1): $= T_s\Big(\phi^2 - 2\phi \cdot T_{t-s}(\phi) + T_{t-s}(\phi^2)\Big)(x)$.

This converges to zero uniformly in x as $(t-s) \to 0$ on the grounds that the argument of T_s converges uniformly to 0 and the operator norm $\|T_s\|$ is less than 1. For the second claim observe that

$$T_{u-\tau}\phi \circ X_\tau - (T_{u-t}\phi \circ X_t = T_{u-\tau}\phi \circ X_\tau - T_{u-t}\phi \circ X_\tau \\ + T_{u-t}\phi \circ X_\tau - T_{u-t}\phi \circ X_t$$

can be made arbitrarily small in $\mathcal{L}^2(\mathbb{P}^x)$ if τ is chosen sufficiently close to t: given $\epsilon > 0$ choose $t' \in (t,u)$ so that $t < \tau < t'$ implies that $T_{u-\tau}\phi$ and $T_{u-t}\phi$ differ uniformly by less that $\epsilon/2$; this will make the $\mathcal{L}^2(\mathbb{P}^x)$-mean of the first summand less than $\epsilon/2$; then use the continuity of the curve $\tau \mapsto (T_{u-t}\phi)(X_\tau)$ to make the $\mathcal{L}^2(\mathbb{P}^x)$-mean of the second summand less than $\epsilon/2$ as well.
(iii) Let $0 \le s < t$. Since

$$e^{-\alpha t}T_{t-s}U_\alpha\gamma = e^{-\alpha t}T_{t-s}\int_0^\infty e^{-\alpha\sigma}T_\sigma\gamma\, d\sigma = e^{-\alpha t}\int_0^\infty e^{-\alpha\sigma}T_{\sigma+t-s}\gamma\, d\sigma$$

$$\sigma \to \sigma - t + s: \qquad = e^{-\alpha s}\int_{t-s}^\infty e^{-\alpha\sigma}T_\sigma\gamma\, d\sigma \le e^{-\alpha s}U_\alpha\gamma\ ,$$

equation (5.7.1) gives

$$\mathbb{E}^x\big[Z_t|\mathcal{F}_s\big] = \mathbb{E}^x\big[e^{-\alpha t}U_\alpha\gamma \circ X_t|\mathcal{F}_s\big] = e^{-\alpha t}T_{t-s}U_\alpha\gamma \circ X_s \qquad \mathbb{P}^x\text{-a.s.} \\ \le e^{-\alpha s}U_\alpha\gamma \circ X_s = Z_s\ .$$

5.7.6 There is an increasing sequence $\psi_n \in C_{00}(E)$ with pointwise limit $|\psi|$. Since $\mathbb{E}^x[\psi_n \circ X_t] = T_t\psi_n(x) \le \breve{T}_t|\psi|(x) < \infty$, the random variables $\psi(X_t)$ etc. are all $\mathbb{P}^x$-integrable. Now

$$\begin{aligned} \mathbb{E}^x[\psi(X_t) - \psi(X_s)|\mathcal{F}_s] &= \big(\breve{T}_{t-s}\psi - \psi\big) \circ X_s && \mathbb{P}^x\text{-a.s.} \\ &= \int_s^t \Big(\breve{T}_{\tau-s}\breve{A}\psi\Big) \circ X_s\, d\tau \\ &= \int_s^t \mathbb{E}^x[\breve{A}\psi \circ X_\tau|\mathcal{F}_s]\, d\tau && \mathbb{P}^x\text{-a.s.} \\ &= \mathbb{E}^x\Big[\int_s^t \breve{A}\psi \circ X_\tau\, d\tau\Big|\mathcal{F}_s\Big] && \mathbb{P}^x\text{-a.s.}\ . \end{aligned}$$

5.7.8 A differs $\mathbb{P}^x$-negligibly from a set $A^{\mathbb{P}^x} \in \mathcal{F}_0[X_.]$. $\mathcal{F}_0[X_.]$ is generated by X_0, which equals x $\mathbb{P}^x$-almost surely, and so equals the σ-algebra of $\mathbb{P}^x$-negligible sets and their complements. For the second claim observe that T^B is an $\mathcal{F}_{.+}^{\mathfrak{P}}[X_.]$-stopping time (corollary A.5.12); so $A \stackrel{\text{def}}{=} [T^B = 0]$ belongs to $\underleftarrow{\mathcal{F}}_0^{\mathfrak{P}}[X_.] = \mathcal{F}_0^{\mathfrak{P}}[X_.]$.
5.7.12 (i) There is an increasing sequence of compact sets K_n whose interiors exhaust E. The paths ω which are in K_n at all times form a compact set

$\mathcal{K}_n$ in the product topology, which is the topology of pointwise convergence. The bounded continuous cylinder functions based on finite sets $\sigma \in [0,\infty)$ form an algebra that separates the points of $\mathcal{K}_n$. By theorem A.2.2 there is one of them, say $F_n = \phi_n \circ X_{\sigma_n}$, with $|F - F_n| < 1/n$ uniformly on $\mathcal{K}_n$. Set $\tau = \bigcup \sigma_n \cup \mathbb{Q}$ and regard F_n as a continuous bounded cylinder function based on τ: $F_n = f_n \circ X_\tau$, where $f_n = \phi_n \circ \pi^\tau_{\sigma_n}$. Clearly $f \stackrel{\text{def}}{=} \lim f_n$ exists uniformly on $\bigcup_n X_\tau(\mathcal{K}_n)$ and $F = f \circ X_\tau$ on bounded paths. To see that f is continuous on E_τ, let $x^\alpha_{\boldsymbol{.}} \to x_{\boldsymbol{.}}$ pointwise. Pick a bounded path $x^0_{\boldsymbol{.}}$ on all of $[0,\infty)$ and define $\overline{x}^\alpha_t$ to equal x^α_t for $t \in \tau$, x^0_t otherwise, and similary define $\overline{x}$. Then $\overline{x}^\alpha_{\boldsymbol{.}} \to \overline{x}_{\boldsymbol{.}}$, and therefore $f(x^\alpha) = F(\overline{x}^\alpha) \to F(\overline{x}) = f(x)$.

(ii) Since τ is dense in $[0,\infty)$ and the paths of $\mathscr{D}_b(E)$ are right-continuous, the cylinder functions Φ based on finite subsets of τ (and restricted to $\mathscr{D}_b(E)$) form an algebra $\mathcal{A}$ that generates $\mathcal{F}_\infty$. Since $\mathcal{A}$ is countably generated $\mathbb{P}^x$ is order-continuous on it. Since $x \mapsto \mathbb{E}^x[\Phi]$ is continuous for $\Phi \in \mathcal{A}$ and F is continuous in the topology generated by $\mathcal{A}$ on $\mathscr{D}_b(E)$, $x \mapsto \mathbb{E}^x[F]$ is continuous as well (proposition A.4.1). The argument for (iii) is similar.

5.7.13 Let K_n be an increasing sequence of compacta whose interiors exhaust E, and set $\Omega_n = \{\omega : \omega_q \in K_n \text{ for } \mathbb{Q} \ni q \le t+1\}$. Since $\mathscr{D}_b(E) = \bigcup \Omega_n$, $\mathbb{P}^x[\Omega_n] > 1 - \epsilon$ for sufficiently large n. The right-continuity of ω causes $\omega_s \in K_n \quad \forall s \le t$.

A.2.3 (i) The condition is clearly sufficient. For the necessity suppose then that the sequence (ϕ_n) in $\mathcal{E}$ or $\overline{\mathcal{E}}$ converges uniformly on $\boldsymbol{A}$ to f. By taking a subsequence we may assume that $|f - \phi_n| \le 2^{-n-1}$ on $\boldsymbol{A}$. Then $|\phi_{n+1} - \phi_n| \le 2^{-n}$ on $\boldsymbol{A}$, and the function $\overline{\psi}_n \stackrel{\text{def}}{=} -2^{-n} \vee \big(\phi_{n+1} - \phi_n\big) \wedge 2^{-n}$, which by theorem A.2.2 (i) belongs to $\overline{\mathcal{E}}$, equals $\phi_{n+1} - \phi_n$ on A_0. Therefore the sum $\overline{\phi} \stackrel{\text{def}}{=} \phi_1 + \sum_{n=1}^\infty \overline{\psi}_n$ converges uniformly, with limit in $\overline{\overline{\mathcal{E}}} = \overline{\mathcal{E}}$. Clearly $f = \overline{\phi}$ on A_0.

(ii) Given $\epsilon > 0$ find $\phi_1, \phi_2 \in \mathcal{E}$ with $\overline{\rho}(f_i, \phi_i) < \epsilon$ on $\boldsymbol{A}$, $i = 1,2$. Then set $M \stackrel{\text{def}}{=} \sup(|\phi_1| \vee |\phi_2|$, and on $[-M,M] \times [-M,M]$ approximate $(x,y) \mapsto \overline{\rho}(x,y)$ uniformly to within ϵ by a polynomial $p(x,y)$. Then $\big|\overline{\rho}(f_1,f_2) - p(\phi_1,\phi_2)\big| \le \big|\overline{\rho}(f_1,f_2) - \overline{\rho}(\phi_1,\phi_2)\big| + \big|\overline{\rho}(\phi_1,\phi_2) - p(\phi_1,\phi_2)\big| \le 3\epsilon$ on $\boldsymbol{A}$.

A.2.6 $\widehat{\mathcal{E}}$ consists of functions of compact support, and $\widehat{\overline{\phi}}$ is a function of compact support K as well. There exists $\widehat{\psi} \in \widehat{\mathcal{E}} = C_{00}(\widehat{\boldsymbol{B}})$ with $\widehat{\psi} \ge K$. Tietze's extension theorem provides a function $\widehat{\rho} \in \overline{\mathcal{E}}$ equal to $1/\widehat{\psi}$ on K. If $\widehat{\mathcal{E}} \ni \widehat{\rho}_n \to \widehat{\rho}$ and $\widehat{\mathcal{E}} \ni \widehat{\phi}_n \to \widehat{\overline{\phi}}$ uniformly, then the $(\widehat{\phi}_n \cdot \widehat{\psi}\widehat{\rho}_n) \circ j \in \mathcal{E}$ form a $\mathcal{E}$-confined sequence converging uniformly to $\overline{\phi}$.

A.2.9 We check the σ-continuity at 0 (see exercise 3.1.5 on page 90).
$\Leftarrow$: Let $\mathcal{E} \ni X_n \downarrow 0$. Then $\widehat{k} \stackrel{\text{def}}{=} \inf \widehat{X}_n$ vanishes on $j(\boldsymbol{B})$ and is the pointwise infimum of a sequence in $\widehat{\mathcal{E}}$. Consequently $\theta(X_n) = \widehat{\theta}(\widehat{X}_n) \to \int \widehat{k}\, d\widehat{\theta} = 0$: θ is indeed σ-additive.
$\Rightarrow$: If $\widehat{k} : \widehat{\boldsymbol{B}} \to \mathbb{R}$ vanishes on $j(\boldsymbol{B})$ and is the pointwise infimum of a sequence $(\widehat{X}_n)$ in $\widehat{\mathcal{E}}$ then $X_n \stackrel{\text{def}}{=} \widehat{X}_n \circ j \downarrow 0$ on $\boldsymbol{B}$ and so $\int \widehat{k}\, d\widehat{\mathcal{I}} = \lim \widehat{\mathcal{I}}(\widehat{X}_n) =$

$\lim \mathcal{I}(X_n) = 0$.

A.2.10 The assumption of weak σ-additivity means that for all continuous linear functionals $g : L^p \to \mathbb{R}$, $\theta \stackrel{\text{def}}{=} g \circ \mathcal{I}$ is σ-additive. The extension theory of Sections 3.1–3.2 furnishes an integral $\int \cdot \, d\widehat{\mathcal{I}}$ of the σ-additive Gelfand transform $\widehat{\mathcal{I}}$ (see assumption 3.1.4). Let then $\mathcal{E} \ni X_n \downarrow 0$, and set $\widehat{k} \stackrel{\text{def}}{=} \inf \widehat{X}_n$. Then $f \stackrel{\text{def}}{=} \lim \mathcal{I}(X_n) = \lim \widehat{\mathcal{I}}(\widehat{X}_n) = \int \widehat{k} \, d\widehat{\mathcal{I}}$ exists in the norm topology L^p, due to the Dominated Convergence Theorem. Since $\langle g|f\rangle = \lim \langle g|\mathcal{I}(X_n)\rangle = \lim \theta(X_n) = 0$ for all g in the dual of L^p, $f = 0$ thanks to the Hahn–Banach theorem A.2.25.

A.2.16 part (iii): In view of part (i) we may as well assume that $\mathcal{E}$ contains the constants and is uniformly closed. Let $\mathcal{E}_0$,

$$\Pi = \prod_{\psi \in \mathcal{E}_0} \left[-\|\psi\|_u, +\|\psi\|_u \right] ,$$

$j : E \to \Pi$, and $\overline{E} = \overline{j(E)}$ be as in the proof of theorem A.2.2 on page 369. Π and $\overline{E}$ are compact. Let us denote by $\mathfrak{u}$ the $\mathcal{E}$-uniformity on E and by $\overline{\mathfrak{u}}$ the unique uniformity of the compact space $\overline{E}$, the $C(\overline{E})$-uniformity.

(a) Observe first that $\overline{\phi} \mapsto \overline{\phi} \circ j$ is an isometric (for the sup–norm) algebra isomorphism of $C(\overline{E})$ with $\mathcal{E}$. (If j is injective this says that $\mathcal{E}$ consists exactly of the restrictions to E of the continuous functions on $\overline{E}$.)

(b) For a real valued function ϕ denote by d_ϕ the pseudometric

$$(x, y) \mapsto d_\phi(x, y) \stackrel{\text{def}}{=} |\phi(x) - \phi(y)| .$$

Consider the bases $\mathfrak{u}_0 \stackrel{\text{def}}{=} \{d_\phi : \phi \in \mathcal{E}\}$ and $\overline{\mathfrak{u}}_0 \stackrel{\text{def}}{=} d_{\overline{\phi}} : \overline{\phi} \in C(\overline{E})\}$ for the uniformities $\mathfrak{u}$ and $\overline{\mathfrak{u}}$, respectively. They are in one–to–one correspondence via

$$d_\phi = d_{\overline{\phi}} \circ j{\times}j : (x, y) \mapsto d_{\overline{\phi}}\big(j(x), j(y)\big) , \qquad \phi = \overline{\phi} \circ j .$$

It is easy to see that, for *any* $\overline{d}$ in the saturation $\overline{\mathfrak{u}}$ of $\overline{\mathfrak{u}}_0$, $\overline{d} \circ j{\times}j$ belongs to $\mathfrak{u}$. Conversely, for $d \in \mathfrak{u}$ set

$$\overline{d}(\xi, \eta) \stackrel{\text{def}}{=} \lim d\big(j^{-1}(\mathfrak{V}_\xi), j^{-1}(\mathfrak{V}_\eta)\big) , \qquad \xi, \eta \in \overline{E} ,$$

where $\mathfrak{V}_\xi$ denotes the neighborhood filter of ξ, considered as usual as a family of idempotent functions (convention A.1.5 on page 364). Since $j(E)$ is dense in $\overline{E}$, $j^{-1}(\mathfrak{V}_\xi)$ and $j^{-1}(\mathfrak{V}_\eta)$ are filters, in fact Cauchy filters on E, so the limit $\overline{d}(\xi, \eta)$ exists. $\overline{d}$ is a pseudometric and belongs to $\overline{\mathfrak{u}}$. Indeed, let $\epsilon > 0$. There are $d_1, \ldots, d_n \in \mathfrak{u}_0$ and $\delta > 0$ so that $max_i d_i(x, y) < \delta \implies d(x, y) < \epsilon$. The d_i are of the form $d_i = \overline{d}_i \circ j{\times}j$ with $\overline{d}_i \in \overline{\mathfrak{u}}_0$; then clearly $max_i \overline{d}_i(\xi, \eta) < \delta \implies \overline{d}(\xi, \eta) \le \epsilon$. Now $d = \overline{d} \circ j{\times}j$:

$$\mathfrak{u} = \overline{\mathfrak{u}} \circ j{\times}j .$$

From this it is obvious that $j : E \to \overline{E}$ is uniformly continuous. (If j is injective this says that $\mathfrak{u}$ is the uniformity induced on E from the uniformity

of $\overline{E}$.
(c) Being compact, $\overline{E}$ is complete: a Cauchy filter on $\overline{E}$ is contained in some ultrafilter, which converges; its limit is the limit of the given Cauchy filter.
(d) To see that the compact Hausdorff space $\overline{E}$ with its unique uniformity is the completion of $(E, \mathfrak{u}_{\mathcal{E}})$, let $f : E \to Y$ be some uniformly continuous map into a complete uniform space $(Y, \mathfrak{v})$. We define the extension $\overline{f} : \overline{E} \to Y$ as follows: For $\xi \in \overline{E}$, the neighborhood filter $\overline{\mathfrak{V}_\xi}$ is Cauchy and $j^{-1}\big(\overline{\mathfrak{V}_\xi}\big)$ is a Cauchy filter on E. Therefore its forward image $f(\mathfrak{V})$ is a Cauchy filter in Y and has a limit, and that shall be $\overline{f}(\xi)$. In other words,

$$\overline{f}(\xi) \stackrel{\text{def}}{=} \lim f\big(j^{-1}(\mathfrak{V}_\xi\big) .$$

Clearly $\overline{f} \circ j = f$.
It is left to be shown that $\overline{f} : \overline{E} \to Y$ is uniformly continuous. Let then $d' \in \mathfrak{v}$ and $\epsilon > 0$. There are a $\delta > 0$ and a $d = \overline{d} \circ j{\times}j \in \mathfrak{u}$ such that $d(x,y) < 3\delta$ implies $d'\big(f(x), f(y)\big) < \epsilon/3$. Let then ξ, η be any two points in $\overline{E}$ with $\overline{d}(\xi,\eta) < \delta$. There are points $x, y \in E$ with $j(x), j(y)$ so close to ξ, η, respectively, that $\overline{d}\big(\xi, j(x)\big) < \delta$, $\overline{d}\big(\eta, j(y)\big) < \delta$, $d'\big(\overline{f}(\xi), f(x)\big) < \epsilon/3$, and $d'\big(\overline{f}(\eta), f(y)\big) < \epsilon/3$. Then $d(x,y) < 3\delta$ and therefore

$$\begin{aligned} d'\big(\overline{f}(\xi), \overline{f}(\eta)\big) &\le d'\big(\overline{f}(\xi), f(x)\big) + d'\big(f(x), f(y)\big) + d'\big(f(y), \overline{f}(\eta)\big) \\ &\le \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \ : \end{aligned}$$

$\overline{f}$ is uniformly continuous. This establishes that $j : (E, \mathfrak{u}) \to (\overline{E}, \overline{\mathfrak{u}})$ has the universality property of a Hausdorff completion.
A.2.19 (iv) implies (ii), for lower semicontinuity: Let $x \in E$, set $r = f(x)$, and let $\epsilon > 0$. The function that has on $[f \le r - \epsilon]$ the value $\inf f$ and at x the value $r - \epsilon$ is continuous on the closed set $[f \le r - \epsilon] \cup \{x\}$. Tietze's extension theorem provides a continuous extension ϕ to all of E with values in $[\inf f, r - \epsilon]$. Clearly $\phi \le f$ and $\phi(x) \ge f(x) - \epsilon$.
A.2.24 Let $\mathcal{U}[P]$ be the algebra of lemma A.2.20, and $j : P \to \widehat{P}$ as furnished by theorem A.2.2 (ii). Since $1 \in \mathcal{U}[P]$, $\widehat{P}$ is compact; since $\mathcal{U}[P]$ is countably generated, $\widehat{P}$ is metrizable; since $\mathcal{U}[P]$ generates the topology of P, j is a homeomorphism of P with its image $j(P)$. By exercise A.2.23, $j(P)$ is a $\mathcal{G}_\delta$-set of $\widehat{P}$; and in the compact metric space $\widehat{P}$ every open set is a $\mathcal{K}_\sigma$-set.
A.2.25 (i) We start with the case that A is a linear subspace and B is open. (In this case the result is known as the **theorem of Mazur**.) Zorn's lemma provides a maximal linear subspace $\mathcal{M}$ containing A and disjoint from B. $\mathcal{M}$ is evidently closed. We have to show that the quotient $\widetilde{\mathcal{V}} \stackrel{\text{def}}{=} \mathcal{V}/\mathcal{M}$ is one–dimensional and therefore linearly homeomorphic with $\mathbb{R}$. Let $p : \mathcal{V} \to \widetilde{\mathcal{V}}$ be the quotient map and assume by way of contradiction that $\widetilde{\mathcal{V}}$ has dimension > 1. $\widetilde{\mathcal{V}}$ contains no linear subspace L disjoint from the open convex set $\widetilde{B} \stackrel{\text{def}}{=} p(B)$ other than $\{0\}$; else $p^{-1}(L)$ would be a linear subspace of $\mathcal{V}$

disjoint from B and properly containing $\mathcal{M}$. Now the punctured space $\widetilde{\mathcal{V}}_* \overset{\text{def}}{=} \widetilde{\mathcal{V}} \setminus \{0\}$ is connected, since its intersection with every two–dimensional subspace of $\widetilde{\mathcal{V}}$ is homeomorphic with $\mathbb{R}^2_*$, and it contains the open convex cone $\widetilde{C} \overset{\text{def}}{=} \bigcup_{\lambda>0} \lambda\widetilde{B}$, which is therefore not closed in $\widetilde{\mathcal{V}}_*$; there exists a boundary point x of $\widetilde{C}$ in $\widetilde{\mathcal{V}}_*$. No positive scalar multiple λx, $\lambda \geq 0$ belongs to $\widetilde{C}$, and neither does $-\lambda x$: if it did then $-x \in \widetilde{C}$ and the whole segment $[-x, x) \overset{\text{def}}{=} \{\lambda x : \lambda \in [-1, 1)\}$ would lie in $\widetilde{C}$, in particular then $0 = 0x \in \widetilde{C}$, which is false. That is to say, the whole closed linear subspace $\mathbb{R}x \overset{\text{def}}{=} \{\lambda x : \lambda \in \mathbb{R}\}$ is disjoint from $\widetilde{B}$, and there is a contradiction: we must indeed have $\dim \widetilde{\mathcal{V}} = 1$.

We compose p with a suitable linear homeomorphism $\widetilde{\mathcal{V}} \to \mathbb{R}$ to arrive at a continuous linear functional $f : \mathcal{V} \to \mathbb{R}$ which vanishes on A and has $f(B) \subset (0, \infty)$, then pick $c = 0$. The pair (f, c) answers the description.

In the case that A is an arbitrary closed convex set and B is still open, we apply Mazur's theorem with B replaced by the open convex set $B - A \overset{\text{def}}{=} \{b - a : b \in B\ ,\ a \in A\}$ and with $A = \{0\}$. The continuous functional f found above has $f(b) > f(a)$ for all $b \in B$ and $a \in A$. Taking $c \overset{\text{def}}{=} \inf\{f(b) : b \in B\}$ yields the claim.

In the case that A is an arbitrary closed convex set and B is compact, consider the open sets $A + V \overset{\text{def}}{=} \{a + x : a \in A\ ,\ x \in V\}$, where V runs through the convex open symmetric ($V = -V$) neighborhoods of zero. Their intersection A with B is void, so one of them, say $A+V$, has void intersection with B. Apply the previous result to A and the open convex set $B - V$.

(ii) Let f be a linear functional defined and continuous on the linear subspace $\mathcal{M} \subset \mathcal{V}$. Let A be the closure of $f^{-1}(\{0\})$ and $B \overset{\text{def}}{=} \{b\}$, where $b \in \mathcal{M}$ has $f(b) = 1$ — such b exists and lies outside A except in the trivial case that $f = 0$. Now (i) provides a continuous linear functional $f' : \mathcal{V} \to \mathbb{R}$ and c a scalar so that $f'(a) \leq c$ for $a \in A$ and $f'(b) > c$. Since A is a subspace, $f' = 0$ on A, which implies $c \geq 0$, and we may assume $c = 0$. Then $F \overset{\text{def}}{=} f'/f'(b)$ is the desired extension of f to all of $\mathcal{V}$.

(iii) Suppose A is convex and closed in $\mathcal{V}$. For every $b \notin A$ there are, by (i), a continuous linear functional f_b and a constant c_b so that $A \subseteq [f_b \leq c_b]$. In other words, A is the intersection of the weakly closed halfspaces $[f_b \leq c_b]$, $b \notin A$, and is thus weakly closed itself.

(iv) A set $\mathcal{F}$ of linear functionals on $\mathcal{V}$ is equicontinuous if $\bigcap_{f\in\mathcal{F}} \big[|f| \leq 1\big]$ is a neighborhood of zero in $\mathcal{V}$. For instance, if $\mathcal{V}$ is a seminormed space with seminorm $\|\ \|_{\mathcal{V}}$, then $\mathcal{F}$ is equicontinuous if and only if it is uniformly bounded on the unit ball of $\mathcal{V}$: $s \overset{\text{def}}{=} \sup\{|f(x)| : f \in \mathcal{F}\ ,\ \|x\|_{\mathcal{V}} \leq 1\} < \infty$; then the $\|\ \|_{\mathcal{V}}$-ball of radius s^{-1} is contained in $\bigcap_{f\in\mathcal{F}} \big[|f| \leq 1\big]$. In partivular the unit ball $\{x^* \in \mathcal{V}^* : \|x^*\|_{\mathcal{V}^*} \leq 1\}$ is equicontinuous (here $\|x^*\|_{\mathcal{V}^*} \overset{\text{def}}{=} \sup |x^*(x)| : \|x\|_{\mathcal{V}} \leq 1\}$.)

Let then $\mathcal{F} \subset \mathcal{V}$ be equicontinuous and set $V \overset{\text{def}}{=} \bigcap_{f\in\mathcal{F}} \big[|f| \leq 1\big]$. Then every $x \in \mathcal{V}$ is absorbed by V, say $x \in \lambda_x V$, and therefore $f(x) \in I_x \overset{\text{def}}{=} [-\lambda_x, \lambda_x]$.

Let now $\mathcal{U}$ be an ultrafilter on $\mathcal{F}$. Then the sets $U(x) \stackrel{\text{def}}{=} \{f(x) : f \in \mathcal{F}\}$, $U \in \mathcal{U}$, form an ultrafilter in the compact interval I_x that has a limit $g(x)$ (see exercise A.2.13 on page 374). It is easy to see that $x \mapsto g(x)$ is linear and that $g(V) \subseteq [-1,1]$, so that g is continuous. That is to say, g is in the weak *–closure of $\mathcal{F}$ (see item A.2.32 on page 381), showing that that closure is weak *–compact.

As a corollary, the unit ball of a reflexive Banach space E, being the unit ball of the dual of E^*, is weakly compact.

A.2.29 Set $\Phi_0(r) = \sup\{\llbracket f \rrbracket' : \llbracket f \rrbracket < r\}$. This is clearly an increasing positive numerical function on $\mathbb{R}_+$ satisfying

$$\llbracket f \rrbracket' \le \Phi_0(\llbracket f \rrbracket) , \qquad f \in \mathcal{V} .$$

Given an $\epsilon > 0$, we can find a $\llbracket\ \rrbracket$–δ-ball $\{f : \llbracket f \rrbracket < \delta\}$ contained in the $\llbracket\ \rrbracket'$–ϵ-ball $\{f : \llbracket f \rrbracket' < \epsilon\}$. Clearly $r < \delta$ implies $\Phi_0(r) < \epsilon$: Φ_0 has the desired property $\Phi(r) \xrightarrow[r\to 0]{} 0$. Now Φ_0 may not be right-continuous, but

$$\Phi(r) \stackrel{\text{def}}{=} \inf\big(\Phi_0(s) : s > r\big)$$

is, and it retains the other properties of Φ_0.

A.2.36 The right-continuous version of $x_{\boldsymbol{.}}$ will satisfy the same inequality, so we may as well assume $x_{\boldsymbol{.}}$ to be right-continuous to start with. Set $\alpha \stackrel{\text{def}}{=} 2(A/B+C)$ and $\beta \stackrel{\text{def}}{=} \max_{\rho=p,q}(2B)^\rho/\rho$. Then $\xi_\lambda \stackrel{\text{def}}{=} A/B + x_\lambda$ satisfies

$$\xi_\mu \le \frac{\alpha}{2} + \max_{\rho=p,q} B\Big(\int_0^\mu \xi_\lambda^\rho \, d\lambda\Big)^{1/\rho} , \qquad \mu \ge 0 ,$$

and the claim follows from $\xi_\lambda \le \alpha e^{\beta\lambda} \quad \forall\, \lambda$. This is certainly true in some neighborhood of $\lambda = 0$. If $\Lambda \stackrel{\text{def}}{=} \{\inf \lambda : \xi_\lambda > \alpha e^{\beta\lambda}\} < \infty$, then by right-continuity

$$\begin{aligned} \alpha e^{\beta\Lambda} \le \xi_\Lambda &\le \frac{\alpha}{2} + \max_{\rho=p,q} \alpha B\Big(\int_0^\Lambda e^{\beta\rho\lambda}\, d\lambda\Big)^{1/\rho} \\ &< \frac{\alpha}{2} + \max_{\rho=p,q} \frac{\alpha B e^{\beta\Lambda}}{(\beta\rho)^{1/\rho}} \le \frac{\alpha}{2} + \frac{\alpha}{2} e^{\beta\Lambda} \le \alpha e^{\beta\Lambda} . \end{aligned}$$

This contradiction shows that $\Lambda = \infty$.

A.2.40 Cover U by bounded sets.

A.2.50 Chain Rule: If $\Phi : D \to E$ is differentiable and $F : E \to S$ weakly differentiable, then $F \circ \Phi : D \to S$ is weakly differentiable and $DF \circ \Phi[u] = DF[\Phi(u)]\, D\Phi[u]$. For c) apply the chain rule and the mean value theorem to $t \mapsto F(u + t(v-u))$.

A.3.1 (i): Let d be a metric defining the topology. A closed set F is the zero-set of the function $x \mapsto d(x,F) \stackrel{\text{def}}{=} \inf\{d(x,y) : y \in F\}$, which is continuous. Therefore F is a Baire set. Then so is every open set.

(ii): First the case that G is metrizable. Let $U \subset G$ be open and set

$U_k \stackrel{\text{def}}{=} \{x \in G : d(x, U^c) \geq 1/k\} \subseteq U$. The interiors of these closed sets exhaust U. Therefore $f^{-1}(U) = \bigcup_k \bigcup_N \bigcap_{n>N} f_n^{-1}(U_k) \in \mathcal{F}$ – since $\{B : f^{-1}(B) \in \mathcal{F}\}$ is a σ-algebra containing the open sets, it contains the Borels. In the general case consider $\Gamma(B) \in \mathcal{F}\}$. This is a σ-algebra. If $\phi \circ f$ is $\mathcal{F}$-measurable for all $\phi \in C_{\mathbb{R}}(G)$ then Γ contains the sets $\{\phi^{-1}(B_0)\}$ for any $\phi \in C_{\mathbb{R}}(G)$ and any $B_0 \in \mathcal{B}^*(G)$, if $\phi \circ f$ is $\mathcal{F}/\mathcal{B}^*(\mathbb{R})$-measurable for all $\phi \in C_{\mathbb{R}}(G)$. and thus contains $\mathcal{G}$. That is to say, $f : F \to G$ is $\mathcal{F}/\mathcal{B}^*(G)$-measurable iff $\phi \circ f$ is $\mathcal{F}$-measurable for all $\phi \in C_{\mathbb{R}}(G)$. So if the f_n are $\mathcal{F}/\mathcal{B}^*(G)$-measurable and converge pointwise to f, then $\phi \circ f = \lim \phi \circ f_n$ is measurable for all $\phi \in C_{\mathbb{R}}(G)$, and so f is $\mathcal{F}/\mathcal{B}^*(G)$-measurable.
(iii): This counterexample is from [27], page 96, and was pointed out to me by Oliver Diaz–Espinoza. Let $f = I$ be the unit interval, and equip $G = I^I$ with the topology of pointwise convergence. For every $x \in I$ let $f_n(x) \in I^I$ be the function $y \mapsto max(0, 1 - n|x - y|)$. The maps $f_n : I \to I^I$ are continuous, but their pointwise limit f, which maps every $x \in I$ to $1_{\{x\}} : y \mapsto [x = y]$ is not Borel measurable.

A.3.2 (i) The functions f of this description form a sequentially closed family.
(ii) Suppose $\sup_n f_n > 0$. Then $f^{(n)} \stackrel{\text{def}}{=} 1 \wedge n \sup_{\nu<n} |f_\nu| \in \mathcal{E}$ and $1 = \lim f^{(n)} \in \mathcal{E}^\sigma$. Conversely, if $1 \in \mathcal{E}^\sigma$, then there is a countable family $\{f_1, f_2, \ldots\} \subset \mathcal{E}$ whose sequential closure contains 1. The f_n cannot all vanish at any one point because then so would 1, thus $\sup |f_n| > 0$.
(iii) The $\mathcal{E}_\alpha$ above are the same whether the sequences occurring in their definition are considered as $\mathbb{R}$-valued sequences that converge pointwise in $\mathbb{R}$ or as $\overline{\mathbb{R}}$-valued sequences that happen to have a real-valued pointwise limit.

A.3.6 The sequential closure of the class of differences of bounded lower semicontinuous functions contains the topology[13], which forms a multiplicative class; it therefore contains all Borel functions. Conversely, a lower semicontinuous function h is the supremum of the countable collection $q \cdot [h > q]$, $q \in \mathbb{Q}$, and so is Borel measurable. Then so is a bounded lower semicontinuous function, a difference of such, and every function in the sequential closure of such differences.

A.3.7 Suppose $f \in \mathcal{E}^\sigma$ is $\mathcal{E}$-confined. There is a $\psi \in \mathcal{E}$ with $|f| \leq \psi$. The collection of functions $g \in \mathcal{E}^\sigma$ such that $-\psi \vee g \wedge \psi$ is the limit of a bounded $\mathcal{E}$-confined sequence in $\mathcal{E}^\sigma$ is sequentially closed and thus contains $\mathcal{E}^\sigma$.

A.3.8 (i) This is just the definition of inner measure, which agrees with μ on $\mathcal{A}^\sigma$. (ii) Any idempotent member (set) of the class $\inf \{ \dot{f} \in L^0(\mathcal{A}^\sigma, \mu) : \exists f \in \dot{f} \text{ with } f \geq \Omega' \}$ will do.

A.3.10 Due to lemma A.2.20 any decreasingly directed collection $\Phi \subset C_b(E)$ contains a decreasing sequence (ϕ_n) with the same pointwise infimum; then $\inf \mu(\Phi) = \inf \mu(\phi_n)$. For if $\mu(\phi) < a$ for some $\phi \in \Phi$, then $\phi \wedge \phi_n \downarrow \phi$ and consequently $\inf \mu(\phi_n) \leq \lim \mu(\phi \wedge \phi_n) < a$.

A.3.21 See [5, page 286 ff.].

A.3.25 There is a collection $\mathcal{L}$ of affine functions $\mathbb{R}_+ \ni x \mapsto \ell(x) = ax + b$ whose pointwise infimum is ϕ. For each one of them b is positive and $\int \phi(|z|)\, d\mu \le \int \ell(|z|)\, d\mu = a \int |z|\, d\mu + b \int 1\, d\mu \le a \int |z|\, d\mu + b = \ell\big(\int |z|\, d\mu\big)$. Taking the infimum over $\ell \in \mathcal{L}$ yields the claim: $\int \phi(|z|)\, d\mu \le \phi\big(\int |z|\, d\mu\big)$.

A.3.26 This is evident if $\Phi(x, \omega)$ is a product of the form $\phi(x)\psi(\omega)$, then if it is the linear combination of such functions, then if it belongs to the sequential closure of the algebra formed by the latter.

A.3.28 Let f^* be an element in the unit ball E_1^* of the dual of E. Then

$$\langle \int f\, d\nu | f^* \rangle = \int \langle f | f^* \rangle\, d\nu \le \int \| f \|_E \, d\lvert \nu \rvert \ .$$

Taking the supremum over $f \in E_1^*$ yields the claim.

A.3.29 The cases $q = \infty$ and $p = q$ are trivial.

The case $1 = p < q$: If $\| \| f \|_{L^1(\mu)} \|_{L^q(\nu)} > 1$, then

$$\int \left(\int |f(x,y)|\, \mu(dx) \right)^q \nu(dy) > 1$$

and there is a function $g \in L_+^{q'}(\nu)$ of norm one in that space with

$$\begin{aligned} 1 &< \int \left(\int |f(x,y)|\, \mu(dx) \right) \cdot g(y)\, \nu(dy) \\ &= \int \left(\int |f(x,y)| \cdot g(y)\, \nu(dy) \right) \mu(dx) \\ &\le \int \left(\int |f(x,y)|^q\, \nu(dy) \right)^{1/q} \cdot \left(\int |g(y)|^{q'}\, \nu(dy) \right)^{1/q'} \mu(dx) \\ &= \Big\| \|f\|_{L^q(\nu)} \Big\|_{L^1(\mu)} \ . \end{aligned}$$

In the remaining case $0 < p < q < \infty$ write

$$\begin{aligned} \Big\| \|f\|_{L^p(\mu)} \Big\|_{L^q(\nu)} &= \Big\| \| |f|^p \|_{L^1(\mu)}^{1/p} \Big\|_{L^q(\nu)} = \Big\| \| |f|^p \|_{L^1(\mu)} \Big\|_{L^{q/p}(\nu)}^{1/p} \\ &\le \Big\| \| |f|^p \|_{L^{q/p}(\nu)} \Big\|_{L^1(\mu)}^{1/p} = \Big\| \|f\|_{L^q(\nu)} \Big\|_{L^p(\mu)} \ . \end{aligned}$$

A.3.34 Only the sufficiency may not be obvious. Assume then that $\big(e^{i\langle \boldsymbol{\xi} | \boldsymbol{x}_n \rangle}\big)$ converges for almost all $\boldsymbol{\xi}$. Then $e^{i\langle \boldsymbol{\xi} | \boldsymbol{x}_n - \boldsymbol{x}_m \rangle} \xrightarrow[m,n\to\infty]{} 1$.

With $\qquad \gamma(\boldsymbol{\xi}) = e^{-|\boldsymbol{\xi}|^2/2} \big/ \sqrt{2\pi}$,

$$Q_{m,n} \stackrel{\text{def}}{=} \int e^{i\langle \boldsymbol{\xi} | \boldsymbol{x}_n - \boldsymbol{x}_m \rangle} \gamma(\boldsymbol{\xi})\, d\boldsymbol{\xi} \xrightarrow[m,n\to\infty]{} 1 \ .$$

Now $\qquad Q_{m,n} = e^{-|\boldsymbol{x}_n - \boldsymbol{x}_m|/2} \int_{\mathbb{R}^d} e^{-|\boldsymbol{\xi} - i(\boldsymbol{x}_n - \boldsymbol{x}_m)|^2/2} \big/ \sqrt{2\pi}\, d\boldsymbol{\xi}$

$$= e^{-|\boldsymbol{x}_n - \boldsymbol{x}_m|/2} \int_{\mathbb{R}^d} e^{-|\boldsymbol{\xi}|^2/2} / \sqrt{2\pi}\, d\boldsymbol{\xi}$$
$$= e^{-|\boldsymbol{x}_n - \boldsymbol{x}_m|/2} .$$

From the resulting $e^{-|\boldsymbol{x}_n - \boldsymbol{x}_m|/2} \xrightarrow[m,n\to\infty]{} 1$ we conclude that $(\boldsymbol{x}_n)$ is Cauchy.

A.3.37 (i) The continuity of $h_1{\star}\mu_2$ is an immediate consequence of the metrizability of G and the Dominated Convergence Theorem. To see that $h_1{\star}\mu_2$ vanishes at ∞, let $\epsilon > 0$ be given. There exist a compact set K_2 so that $\mu_2(K_2^c) < \epsilon$ and a compact set K_1 outside which $|h_1| < \epsilon$. If g lies outside the compact algebraic sum $K_1 + K_2 \subset G$, then $h_1(g - g_2) < \epsilon$ for $g_2 \in K_1$ and therefore $\int_{K_2} h_1(g - g_2)\, \mu_2(dg_2) < \epsilon \|\mu_2\|$. Clearly $\int_{K_2^c} h_1(g - g_2)\, \mu_2(dg_2) < \epsilon \|h_1\|$. Thus $h_1{\star}\mu_2 \xrightarrow[g\to\infty]{} 0$.

A.3.44 Since μ is order-continuous (exercise A.3.10), it makes sense to talk about the support C of μ (exercise A.3.13). Let S be a lifting, Υ a countable uniformly dense subset of $\mathcal{U}[E]$ (see lemma A.2.20), and N the negligible set $\bigcup\{[Sv \ne v] \cap C : v \in \Upsilon\}$. For $x \notin N$ set $Tf(x) = Sf(x)$, $f \in \mathcal{L}^\infty$. If $x \in N$, consider the ideal $\mathcal{I}_x$ of functions $f \in \mathcal{L}^\infty$ that differ negligibly from a function v' that is continuous (in the metric topology) at x and has $v'(x) = 0$. As in the proof of lemma A.3.40 one checks that there is an $\widehat{x} \in \widehat{E}$ at which all the functions of $\widehat{\mathcal{I}}_x$ vanish. Set $Tf(x) = \widehat{f}(\widehat{x})$ and check that T is a lifting with $Tv(x) = v(x)$ for $x \in C$ and $v \in \mathcal{U}[E]$. For $x \in C$ and $\phi \in C_b$ we have by lemma A.2.20 $T\phi(x) \ge \sup\{Tv(x) : \phi \ge v \in \mathcal{U}[E]\} = \sup\{v(x) : \phi \ge v \in \mathcal{U}[E]\} = \phi(x)$. Applying this to $-\phi$ gives $T\phi(x) = \phi(x)$.

A.3.45 Equation (A.3.17): Integrating $e^{-(x^2+y^2)/2}$ over the plane in polar coordinates establishes first that

$$\int_{-\infty}^{+\infty} e^{-x^2/2} = \sqrt{2\pi} .$$

The variable substitution $u = (x - i\xi t)/\sqrt{t}$ turns the integral

$$\mathbb{E}\Big[e^{i\xi X}\Big] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{i\xi x} \cdot e^{-x^2/2t}\, dx$$

into

$$\frac{e^{-t\xi^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-u^2/2}\, du = e^{-t\xi^2/2} .$$

A.3.49 Apply functional calculus to the self-adjoint operator B. Or apply the power series for $\|B\|^{1/2}\sqrt{1-x}$ to the matrix $I - B/\|B\|$, $\|B\|$ being the operator norm of B.

A.4.8 See [12], ch. IX, §5: Treat first the case that E is also locally compact. Suppose the locally compact space E has a countable basis, and $\mathfrak{P}$ is a family of probabilities on E that is relatively compact in the topology

$\sigma\big(\mathfrak{P}^{\bullet}(E), C_{00}(E)\big)$ of convergence on continuous functions of compact support. There exists an increasing sequence of compacta $K_n \subset E$ whose interiors cover E. By way of contradiction assume $\mathfrak{P}$ is not uniformly tight. Then there exist an $\alpha > 0$ and measures $\mu_n \in \mathfrak{P}$ with $\mu_n(K_n) \le 1-\alpha$. Extracting a subsequence we may assume that $\mu_n \to \mu \in \mathfrak{P}^{\bullet}(E)$ on all $\phi \in C_{00}(E)$. Now $\mu(K) > 1-\alpha$ for some compact $K \subset E$, since μ is a tight probability. There exist a $\phi \in C_{00}(E)$ with $\phi = 1$ on K and an $N \in \mathbb{N}$ so that $\mathring{K}_N$ contains the support of ϕ. Now as $\mu_n(\phi) \to \mu(\phi) > 1-\alpha$ we have $\mu_n(K_n) \ge \mu_n(K_N) \ge \mu(\phi) > 1-\alpha$ for sufficiently large $n \ge N$, a contradiction.

An arbitrary polish space is the intersection of a decreasing sequence of open subsets E_n of some compact space (exercise A.6.1). We view the measures of $\mathfrak{P}$ as measures on the E_n, which are locally compact. Due to the first part of the argument there are compacta $K_n \subset E_n$ with $\mu(E_n - K_n) < \alpha 2^{-n} \quad \forall\, \mu \in \mathfrak{P}$. $K \stackrel{\text{def}}{=} \bigcap_n K_n \subset E$ is compact and has $\mu(E-K) \le \alpha$.

A.5.6 (ii) Suppose that $\mathcal{K}' \subset \mathcal{K}^{\cup_f}$ has the property that no finite subcollection has void intersection. Then there is an ultrafilter $\mathcal{U}$ containing $\mathcal{K}'$. Every set $K' \in \mathcal{K}'$ is the finite union $K' = \bigcup_{i=1}^{I(K')} K'_i$ of sets in $\mathcal{K}$. At least one of the K'_i, say $K'_{i(K')}$, belongs to $\mathcal{U}$. No intersection of finitely many such $K'_{i(K')}$ is void, and therefore neither is $\bigcap \big\{K'_{i(K')} : K' \in \mathcal{K}'\big\} \subset \bigcap \mathcal{K}'$.

(iii) Take for the closed sets the sets in $\mathcal{K}^{\cup_f \cap_a}$.

A.5.16 Apply theorem 2.4.7.

A.5.18 Proposition 3.5.2 shows that $\mathcal{O}$ contains $\mathcal{P}$. Let $X \in \mathfrak{D}$. Let $\delta > 0$ and define the stopping times $S_0 = 0$ and

$$S_{k+1} = \inf\{t > S_k : \, |X - X_{S_k}|^\star_t \ge \delta\,\}\ , \qquad k = 1, 2, \ldots$$

and

$$X^{(\delta)} = X_0[\![0]\!] + \sum_k X_{S_k} \cdot [\![S_k, S_{k+1})\!)\ . \qquad (*)$$

Since X has no oscillatory discontinuities, $S_k \uparrow \infty$. Evidently, X and $X^{(\delta)}$ differ uniformly by less than δ. The process $X^{(\delta)}$ differs from the *previsible* process

$$X_0[\![0]\!] + \sum_k X_{S_k} \cdot (\!(S_k, S_{k+1}]\!]$$

only in the set $\bigcup_k [\![S_k]\!]$. The processes $(*)$ form therefore a generator of $\mathcal{O}$ for which the claim is true. It is now easy to check that the optional processes for which the second claim holds is a monotone class. An application of theorem A.3.4 finishes the proof.

A.5.20 The family of processes that have an optional projection is a monotone class. It contains the processes of the form $[t,\infty) \times g$, $g \in \mathcal{F}^*_\infty$ bounded, which generate the measurable σ-algebra. Indeed, let M^g be the right-continuous martingale $\mathbb{E}[g|\mathcal{F}^{\mathbb{P}}_{t+}]$ (proposition 2.5.13). It follows from Doob's optional stopping theorem 2.5.22 that $[\![t,\infty)\!) \cdot M^g$ is an optional projection of $[t,\infty)\times g$. If $X^{\mathcal{O},\mathbb{P}}$ and $\overline{X}^{\mathcal{O},\mathbb{P}}$ are two optional projections of X, then $B \overset{\text{def}}{=} \big[X^{\mathcal{O},\mathbb{P}} \neq \overline{X}^{\mathcal{O},\mathbb{P}}\big]$ is an optional set. If it were not $\mathbb{P}$-evanescent, one could find a stopping time T whose graph is contained in B and has non-negligible projection on Ω. This however is clearly impossible.

A.5.21 $X^{\mathcal{O},\mathbb{P}}$ meets the description. Indeed, for any t and $A\in\mathcal{F}_t=\mathcal{F}^{\mathbb{P}}_{t+}$

$$\mathbb{E}\big[A\cdot X^{\mathcal{O},\mathbb{P}}_t\big]=\mathbb{E}\big[X^{\mathcal{O},\mathbb{P}}_{t_A}[t_A<\infty]\big]=\mathbb{E}\big[X_{t_A}[t_A<\infty]\big]=\mathbb{E}\big[A\cdot X_t\big]\ :$$

the $\mathcal{F}_t$-measurable random variables $X^{\mathcal{O},\mathbb{P}}_t$ and X_t differ negligibly.

A.6.1 (i) See exercise A.2.24. (ii) Let $p:P\to S$ be a continuous map from the polish space onto the Suslin set S, and $C\subset S$ closed. Then $p:p^{-1}(C)\to C$ exhibits C as a Suslin set. (iii) Let S_n be Suslin subsets of a Hausdorff space and $p_n:P_n\to S_n$ continuous surjections. In the product space $\prod_n P_n$, which is polish, let $P\overset{\text{def}}{=}\{(x_n):p_n(x_n)=p_m(x_m) \text{ for } m\neq m\}$. This is a closed and therefore polish subspace of $\prod_n P_n$. The continuous map $p:P\to\bigcap_n S_n$ exhibits the intersection of the S_n as Suslin. For the union, consider the disjoint topological sum $P\overset{\text{def}}{=}\biguplus_n P_n$. Its is again polish, and the obvious map $p:P\to\bigcup_n S_n$ shows that the union of the S_n is polish. (iv) By (i), a closed ball of the given Suslin space S is Suslin. Since S is separable as the continuous image of a separable space and metrizable by assumption, the complement of a closed ball is the countable union of closed balls and is therefore also Suslin. The collection of sets which together with their complements are Suslin is closed under taking complements and by (ii) is closed under countable unions and contains the closed balls. It contains therefore the σ-algebra generated by the closed balls, the Borels.

A.8.1 (i) If $\lceil\!\lceil f\rceil\!\rceil_0\le a$, then there exists a decreasing sequence (λ_n) with $\inf\lambda_n\le a$ and $\mathbb{P}[|f|>\lambda_n]\le\lambda_m$ for $1\le m\le n$. Then $\mathbb{P}[|f|>a]\le\mathbb{P}[|f|>\lambda]\le\lambda_m$ for all m, and therefore $\mathbb{P}[|f|>a]\le\lambda\le a$. The reverse implication is obvious.

(ii), for $p=0$: Let $a=\lceil\!\lceil f\rceil\!\rceil_0$ and $b=\lceil\!\lceil g\rceil\!\rceil_0$. Since $[|f+g|>a+b]\subset[|f|>a]\cup[|g|>b]$, we have by (i) $\mathbb{P}[|f+g|>a+b]\le a+b$, i.e., $\lceil\!\lceil f+g\rceil\!\rceil_0\le\lceil\!\lceil f\rceil\!\rceil_0+\lceil\!\lceil g\rceil\!\rceil_0$.

(iii), for $p=0$: $\lim_{r\to0}\lceil\!\lceil rf\rceil\!\rceil_p=0$ clearly implies $\mathbb{P}[|f|=\infty]=0$.

(iv), for $p=0$: The algebraic properties are obvious. Given a $\lceil\!\lceil\ \rceil\!\rceil_0$-Cauchy sequence (f_n) extract a subsequence (f_{n_k}) with $\lceil\!\lceil f_{n_{k+1}}-f_{n_k}\rceil\!\rceil\le 2^{-k}$. Then $\sum_k|f_{n_{k+1}}-f_{n_k}|$ is a.s. finite, and therefore (f_{n_k}) converges a.s. The limit is a $\lceil\!\lceil\ \rceil\!\rceil_0$-mean limit of (f_n) by a standard argument.

A.8.2 This is clear if $p\ge1$. For $0<p<1$ Hölder's inequality (theorem A.8.4 on page 449) with conjugate exponents $1/p,1/(1-p)$ gives $a_1+\ldots+a_n\le n^{1-p}\big(a_1^{1/p}+\ldots+a_n^{1/p}\big)$ for $a_\nu\ge0$. Apply this to $a_\nu=(\int|f_\nu|^p)$ to obtain

$$\int|f_1+\ldots+f_n|^p\le\int|f_1|^p+\ldots+\int|f_n|^p$$

$$\leq n^{1-p}\Big(\Big(\int |f_1|^p\Big)^{1/p} + \ldots + \Big(\int |f_n|^p\Big)^{1/p}\Big)^p$$

and take p^{th} roots.

A.8.4 Hölder's inequality for the special case $r = 1$ of conjugate exponents is in any textbook on integration. By applying that to the function $|fg|^r$, with conjugate exponents p/r and q/r, the general case follows. For the second claim reduce to the case $\|f\|_p = 1$ and take $g = f^{p-1}$.

A.8.5 Let $1/p$ and $1/q$ be points in I_f and $1/s = \theta/p + (1-\theta)/q$ $(0 < \theta < 1)$ a point in between. Set $a = s\theta/p$; then $1 - a = (1-\theta)s/q$ and $0 < a < 1$. Apply Hölder's inequality with conjugate exponents $1/a$ and $1/(1-a)$ to $|f|^p$ and $|f|^q$:

$$\int |f|^s \, d\mu = \int |f|^{pa} \cdot |f|^{q(1-a)} \leq \Big(\int |f|^p \, d\mu\Big)^a \cdot \Big(\int |f|^q \, d\mu\Big)^{(1-a)}$$

$$= \|f\|_{L^p}^{ap} \cdot \|f\|_{L^q}^{(1-a)q} = \|f\|_{L^p}^{\theta s} \cdot \|f\|_{L^q}^{(1-\theta)s} ,$$

and so $\quad \|f\|_{L^s} \leq \|f\|_{L^p}^{\theta} \cdot \|f\|_{L^q}^{1-\theta}$

and $\ln\big(\|f\|_{L^s}\big) \leq \theta \ln\big(\|f\|_{L^p}\big) + (1-\theta)\ln\big(\|f\|_{L^q}\big)$.

A.8.11 The notions of $\mathcal{F}$-measurability and almost sure finiteness coincide, whether $\mathbb{P}$ or $\mathbb{P}'$ is the measure: $L^0(\mathbb{P})$ and $L^0(\mathbb{P}')$ coincide as sets. Suppose $f_n \to 0$ in $\mathbb{P}$-measure, and let $\epsilon > 0$. Now

$$\begin{aligned}\mathbb{P}'\big[|f_n| > \epsilon\big] &= \int \big[|f_n| > \epsilon\big] \cdot g' \, d\mathbb{P} \\ &= \int \big[|f_n| > \epsilon\big]\big[g' > M\big] \cdot g' \, d\mathbb{P} + \int \big[|f_n| > \epsilon\big]\big[g' \leq M\big] \cdot g' \, d\mathbb{P} \\ &\leq \int \big[g' > M\big] \cdot g' \, d\mathbb{P} + M \cdot \mathbb{P}\big[[|f_n| > \epsilon]\big] .\end{aligned}$$

Choosing first M so large that the first summand on the right is less than $\epsilon/2$ and then n so large that the second summand also is less than $\epsilon/2$, we see that eventually $\lceil\!\lceil f_n \rceil\!\rceil_{L^0(\mathbb{P}')} \leq \epsilon$: $f_n \to 0$ in $\mathbb{P}'$-measure. Interchanging the roles of $\mathbb{P}$ and $\mathbb{P}'$ and replacing g' by $g \stackrel{\text{def}}{=} (g')^{-1}$ shows that the converse implication $f_n \to 0$ in $\mathbb{P}'$-measure $\implies$ $f_n \to 0$ in $\mathbb{P}$-measure is also true: the two topologies are, indeed, the same.

A.8.12 Set $\Phi(r) = \sup\{\lceil\!\lceil f \rceil\!\rceil_{L^0(\mathbb{P}')} : \lceil\!\lceil f \rceil\!\rceil_{L^0(\mathbb{P})} \leq r\}$. To see that $\Phi(r) \xrightarrow[r\to 0]{} 0$ let $\epsilon > 0$ be given and denote by g a Radon–Nikodym derivative $d\mathbb{P}'/d\mathbb{P}$. There is a $K > 1$ so that $\mathbb{P}'[g > K] < \epsilon/2$. Set $\delta \stackrel{\text{def}}{=} \epsilon/(2K)$. If $\lceil\!\lceil f \rceil\!\rceil_{L^0(\mathbb{P})} < \delta$, then $\mathbb{P}[|f| > \epsilon] < \epsilon/(2K)$ and so $\mathbb{P}'[|f| > \epsilon] \leq \mathbb{P}'[g > K] + K\mathbb{P}[|f| > \epsilon] < \epsilon$. To get Φ right-continuous replace it by its right-continuous version.

A.8.15 (i) $\mathbb{E}[|f|^p] = \int_0^\infty t^p \, d\mathbb{P}[|f| \leq t] = \int_0^\infty \big(T^{\lambda+}\big)^p [T^{\lambda+} < \infty] \, d\lambda$ by the change-of-variable theorem 2.4.7, where $T^{\lambda+} \stackrel{\text{def}}{=} \inf\{t : \mathbb{P}[|f| \leq t] > \lambda\} =$

$\inf\{t : \mathbb{P}[|f| > t] \le 1 - \lambda\} = \|f\|_{[1-\lambda]}$ and $[T^{\lambda+} < \infty] = [0 \le \lambda < 1]$. Hence $\mathbb{E}[|f|^p] = \int_0^1 \|f\|_{[1-\lambda]}^p \, d\lambda = \int_0^1 \|f\|_{[\lambda]}^p \, d\lambda$. The last claim is done similarly, with $t \mapsto t^p$ replaced by $t \mapsto \Phi(t)$.

(ii) If $\lambda < \|f + g\|_{[\alpha+\beta]}$, then

$$\begin{aligned}\alpha + \beta &\le \mathbb{P}[|f + g| > \lambda] \le \mathbb{P}[|f| + |g| > \lambda] \\ &\le \mathbb{P}[|f| > \|f\|_{[\alpha]}] + \mathbb{P}[|f| + |g| > \lambda \,,\, |f| \le \|f\|_{[\alpha]}] \\ &\le \alpha + \mathbb{P}[|g| > \lambda - \|f\|_{[\alpha]}] \,.\end{aligned}$$

Thus $\beta \le \mathbb{P}[|g| > \lambda - \|f\|_{[\alpha]}]$, i.e., $\|g\|_{[\beta]} \ge \lambda - \|f\|_{[\alpha]}$ and $\lambda \le \|f\|_{[\alpha]} + \|g\|_{[\beta]}$.

A.8.16 Let $\rho > 0$. Then

$$\begin{aligned} & \rho < \Big\| \|f\|_{[\beta;\tau]} \Big\|_{[\alpha;\mathbb{P}]} \\ \Longrightarrow \quad & \alpha < \mathbb{P}\Big[\||f|\|_{[\beta;\tau]} > \rho \Big] = \mathbb{P}\big[\tau[|f| > \rho] > \beta\big] \\ \Longrightarrow \quad & \alpha\beta < \int \tau[f(\omega, \cdot) > \rho] \, \mathbb{P}(d\omega) = \int \mathbb{P}[f(\cdot, t) > \rho] \tau(dt) \,. \end{aligned}$$

With $g(t) \stackrel{\text{def}}{=} \mathbb{P}[f(\cdot, t) > \rho]$ we have $0 \le g \le 1$ and

$$\begin{aligned}\alpha\beta &< \int g(t) \, \tau(dt) \\ &= \int g(t)[g(t) \le \gamma] \, \tau(dt) + \int g(t)[g(t) > \gamma] \, \tau(dt) \\ &\le \gamma + \tau[g > \gamma]\end{aligned}$$

so that $\qquad \alpha\beta - \gamma \le \tau[g > \gamma] = \tau\big[\mathbb{P}[|f| > \rho] > \gamma\big]$,

i.e., $\qquad \alpha\beta - \gamma \le \tau\big[\|f\|_{[\gamma;\mathbb{P}]} > \rho\big]$,

which reads $\qquad \rho \le \Big\| \|f\|_{[\gamma;\mathbb{P}]} \Big\|_{[\alpha\beta-\gamma;\tau]}$.

A.8.17 (A.8.1): Set $\lambda = \|g\|_{[\alpha]}$ and denote the right-hand side of the first inequality by Λ. Then

$$\begin{aligned}\mathbb{P}[f > \Lambda] &\le \mathbb{P}[f > \Lambda \,;\, g \le \lambda] + \mathbb{P}[g > \lambda] \le \mathbb{P}[f^r/\Lambda^r > 1 \,;\, \lambda/g \le 1] + \alpha \\ &\le \frac{\lambda}{\Lambda^r}\mathbb{E}[f^r/g] + \alpha = \frac{\lambda}{E^r \Lambda^r} + \alpha = \beta + \alpha \,.\end{aligned}$$

(A.8.2): Set $\lambda = \|g\|_{[\alpha/2]}$ and denote the right-hand side of inequality (A.8.2) by Λ. Then

$$\mathbb{P}[fg > \Lambda] \le \mathbb{P}[fg > \Lambda \,;\, g < \lambda] + \mathbb{P}[g \ge \lambda] \le \mathbb{P}[f > \Lambda/\lambda \,;\, \lambda/g > 1] + \alpha/2$$

$$\leq \lambda \mathbb{E}\big[[f > \Lambda/\lambda] \cdot 1/g\big] + \alpha/2 \leq \Big\| \frac{\lambda f}{\Lambda} \Big\|^r_{L^r(\mathbb{P}/g)} + \alpha/2$$

$$\leq \frac{\lambda^{r+1} E^r}{\Lambda^r} + \alpha/2 = \alpha/2 + \alpha/2 = \alpha \, .$$

A.8.22 $p = 0$: If $\|f_n\|_{[\alpha]} \leq a \quad \forall\, n$, then $\mathbb{P}[f_n > a] \leq \alpha \quad \forall\, n$ and consequently $\mathbb{P}[f > a] = \sup_n \mathbb{P}[f_n > a] \leq \alpha$, which says that $\|f\|_{[]}\alpha \leq a$.

A.8.27 In the proof of inequality (A.8.6) replace the constant 2 by any $A > 1$. The argument gives $\|f\|_2 \leq AK_1 \cdot \|f\|_{[(A-1/AK_1)^2]}$. Solving $(A - 1/AK_1)^2 = \kappa$ and using the value $K_1 = \sqrt{2}$ from remark A.8.28 produces the claim.

A.8.29 Since $p \mapsto \Gamma\big((p+1)/2\big)$ is convex, there are two points at which $\Gamma\big((p+1)/2\big)$ equals $\sqrt{\pi}/2$. One of them is $p = 2$; inspection of a table shows that the other is $p_0 \approx 1.85$. To the left of p_0 equation (A.8.8) gives $K_p = 2^{1/p\,-\,1/2}$. Calculations on a hand-held calculator show that the ratio

$$\Big(\sqrt{\pi}/\Gamma((p+1)/2)\Big)\Big/2$$

takes its maximum at $p_m \approx 1.92175$ and that its p_m^{th} root there is approximately 1.000366283.

A.8.30 By the Central Limit Theorem

$$\lim_n \frac{1}{n^p} \int_T |\epsilon_1 + \ldots + \epsilon_n|^p \, d\tau = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^p e^{-x^2/2} \, dx$$
$$= 2^{p/2} \frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \, .$$

Thus

$$K_p \geq \Big(\frac{\sqrt{\pi}}{\Gamma(\frac{p+1}{2})} \Big)^{1/p} \Big/ \sqrt{2} \, .$$

Also, the choice $n = 2$ and $a_1 = a_2 = 1$ implies $K_p \geq 2^{\frac{1}{p} - \frac{1}{2}}$.

A.8.32 (Suggested by Roger Sewell) Show that

$$b(p) = \frac{2 \sin(p\pi/2)\Gamma(p)}{\pi}$$

and then analyze the right hand side or have the computer draw a graph.

A.8.35 Let $x_1, \ldots, x_n \in E$ and $\gamma_1^{(q)}, \ldots, \gamma_n^{(q)}$ symmetric q-stable.

Then $$\Big\|\Big\| \sum_{\nu=1}^n v \circ u(x_\nu)\, \gamma_\nu^{(q)} \Big\|_G \Big\|_{L^p(dx)} \leq T_{p,q}(v) \cdot \Big(\sum_{\nu=1}^n \|u(x_\nu)\|_F^q \Big)^{1/q}$$

$$\leq T_{p,q}(v) \cdot \|u\| \cdot \Big(\sum_{\nu=1}^n \|x_\nu\|_E^q \Big)^{1/q} \, .$$

and $$\Big\|\Big\|\sum_{\nu=1}^{n} v\circ u\big(x_\nu\big)\,\gamma_\nu^{(q)}\Big\|_G\Big\|_{L^p(dx)} \le \|v\|\cdot\Big\|\Big\|\sum_{\nu=1}^{n} u\big(x_\nu\big)\,\gamma_\nu^{(q)}\Big\|_G\Big\|_{L^p(dx)}$$

$$\le \|v\|\cdot T_{p,q}(u)\cdot\Big(\sum_{\nu=1}^{n}\|x_\nu\|_E^q\Big)^{1/q}.$$

A.9.2

$$\begin{aligned}\frac{T_t\psi-\psi}{t} &= \frac{1}{t}\Big(\int_0^s T_{\sigma+t}\phi\,d\sigma - \int_0^s T_\sigma\phi\,d\sigma\Big)\\ &= \frac{1}{t}\Big(\int_t^{s+t} T_\sigma\phi\,d\sigma - \int_0^s T_\sigma\phi\,d\sigma\Big)\\ &= \frac{1}{t}\Big(\int_s^{s+t} T_\sigma\phi\,d\sigma - \int_0^t T_\sigma\phi\,d\sigma\Big) \quad \xrightarrow[t\to 0]{} T_s\phi-\phi\,,\end{aligned}$$

and so $\psi\in\mathcal{D}[\mathcal{A}]$

and $$\mathcal{A}\psi = T_s\phi-\phi$$

or $$T_s\phi-\phi = \mathcal{A}\int_0^s T_\sigma\phi\,d\sigma \quad \text{for all } \phi\in C\,.$$

Since $\int_0^s T_\sigma\phi\,d\sigma/s \xrightarrow[s\to 0]{} \phi$, $\mathcal{D}[\mathcal{A}]$ is dense in $C_0(E)$. We have used here that $T_s\circ\mathcal{A}\phi = \mathcal{A}\circ T_s\phi$ for $\phi\in\mathcal{D}[\mathcal{A}]$, which is left for the reader to establish [34, chapter I].

A.9.4 See [54, page 320].

A.9.8 For every $s\in[0,\infty)$, $x\in E$, and $\beta<1$ there is a function $\phi_{s,x,\beta}\in C_0(E)$ with $0\le\phi\le 1$ and $T_s\phi_{s,x,\beta}(x)>\beta$. Since $s\mapsto T_s\phi(x)$ is continuous, we have $T_s\phi_{s,x,\beta}(x)>\beta$ on a whole neighborhood U_s of s. Any compact interval $[0,k]$ can be covered by finitely many of them; the supremum $\phi^k_{x,\beta}$ of the corresponding $\phi_{s,x,\beta}$'s has $T_s\phi^k_{x,\beta}(x)>\beta$ for all $s\in[0,k]$. For any fixed $\alpha>0$ the choice of a sufficiently large k_α will result in $\alpha U_\alpha\phi^{k_\alpha}_{x,\beta}(x)>\beta$. This inequality will by continuity hold in a whole neighborhood of x. Now let $K\subset E$ be compact. Taking a finite supremum of such functions $\phi^{k_\alpha}_{x,\beta}$ we find a function $\phi^K_{\alpha,\beta}\in C_0(E)$ with $0\le\phi^K_{\alpha,\beta}\le 1$ such that $\alpha U_\alpha\phi^K_{\alpha,\beta}>\beta$ on K. Let K_n be an increasing sequence of compacta whose interiors cover E, take $\alpha_n=1/n$, and choose $\phi_n\in C_0(E)$ such that $\phi_n\le 1$ and $\psi_n\overset{\text{def}}{=}\alpha_n U_{\alpha_n}\phi_n> 1-2^{-n}$ on K_n. Now observe that $\mathcal{A}\psi_n=\alpha_n\psi_n-\alpha_n\phi_n\to 0$.

A.9.9 Applying the identity $(\alpha I-\mathcal{A})U_\alpha=I$ to the ψ_n of A.9.8 (iii) gives

$$\alpha U_\alpha\psi_n - U_\alpha\mathcal{A}\psi_n = \psi_n$$

and in the limit $$\alpha\int_E U_\alpha(x,dy)=1\,.$$

i.e., $$\alpha\int_0^\infty e^{-\alpha s}T_s(x,E)\,ds = 1\,,$$

which shows that the measures $T_s(x,\cdot)$ all have total mass one.

A.9.11 (i), $\Leftarrow$: Urysohn's lemma provides positive continuous functions $\rho_n \le 1$ of compact support so that $[\rho_n \ne 0] \subset [\rho_{n+1} = 1]$ and $\sup_n \rho_n(x) = 1$ at every point $x \in E$. Let $\psi_n \in C$ be so that $|\phi| \le \psi_n$, and $(s,x) \mapsto \int_E^* T_s(x,dy)\,\psi_n(y)$ is finite and continuous on $[0,n] \times K_n$. Then

$$\|\,\phi - \phi\cdot\rho_k\,\|_{n,K_n}^{\smile} = \|\,\phi\cdot(1-\rho_k)\,\|_{n,K_n}^{\smile} \le \|\,\psi_n - \psi_n\cdot\rho_k\,\|_{n,K_n} \xrightarrow[k\to\infty]{} 0\,,$$

since the continuous functions $(s,x) \mapsto \int_E^* T_s(x,dy)\,\big(\psi_n - \psi_n\cdot\rho_k\big)(y)$ decrease pointwise, and by Dini's lemma A.2.1 uniformly on $[0,n]\times K_n$, to zero. There is therefore a k_n so that $\phi_n \stackrel{\text{def}}{=} \phi\cdot\rho_{k_n} \in C_{00}(E)$ has $\|\,\phi - \phi_n\,\|_{n,K_n}^{\smile} < 2^{-n}$. Clearly $\lceil\!\lceil\, \phi - \phi_n \,\rceil\!\rceil^{\smile} \le (n+1)2^{-n} \xrightarrow[n\to\infty]{} 0$.

(i), $\Rightarrow$: Let $\phi_n \in C_{00}(E)$ have $\|\,\phi - \phi_n\,\|_{n,K_n}^{\smile} < 2^{-n}$, $n = 1,2,\ldots$, and set $\phi_0 = 0$. The function $\psi \stackrel{\text{def}}{=} \sum_{n\ge1} |\phi_n - \phi_{n-1}|$ serves simultaneously for all $t < \infty$ and compact $K \subset E$.

(ii) Let $\phi \in \breve C$ and $0 \le s \le u$. Then for every $t < \infty$ and compact $K \subset E$

$$\begin{aligned}
\|\breve T_s\phi\|_{t,K}^{\smile} &\le \sup\Big\{ \int_E^* T_\tau(x,dy)\;\breve T_s|\phi|\,(y) : 0 \le \tau \le t, x \in K \Big\} \\
&\le \sup\Big\{ \int_E^* T_{\tau+s}(x,dy)\;|\phi|(y) : 0 \le \tau \le t, x \in K \Big\} \\
&\le \sup\Big\{ \int_E^* T_\tau(x,dy)\;|\phi|(y) : 0 \le \tau \le t+u, x \in K \Big\} \\
&= \|\phi\|_{t+u,K}^{\smile}\ ,
\end{aligned}$$

showing that $\breve T_u : \breve C \to \breve C$ is continuous. Next let $\breve\phi \in \breve C$ and $\phi \in C_{00}(E)$. Then $\breve T_{\bullet}\breve\phi - T_{\bullet}\phi = \breve T_{\bullet}(\breve\phi - \phi)$ has $\|\breve T_s\breve\phi - T_s\phi\|_{t,K}^{\smile} \le \|\breve\phi - \phi\|_{t+u,K}^{\smile}$ for $0 \le s \le u$. This shows that the curve $\breve T_{\bullet}\breve\phi$ is on bounded intervals $[0,u]$ the uniform limit of continuous curves $T_{\bullet}\phi$ in $\breve C$ and therefore is continuous itself.

A.9.13 See exercise A.3.16 on page 401.

A.9.15 It is evident that $T_t^{\vdash}$ is linear and has operator norm ≤ 1. To see that it maps $C_0^{\vdash}$ into itself it suffices to check its behavior on functions of the form $(\tau,x) \mapsto \phi_1(\tau)\phi_2(x)$, $\phi_1 \in C_0(\mathbb{R}_+), \phi_2 \in C_0(E)$, on the grounds that the linear combinations of these form an algebra $\mathcal{A}_0$ uniformly dense in $C_0^{\vdash}$; so $T_t^{\vdash}\big(C_0^{\vdash}\big) \subset C_0^{\vdash}$ is obvious. By the same token $t \mapsto T_t^{\vdash}\phi$ is continuous for $\phi \in \mathcal{A}_0$ and then for $\phi \in C_0^{\vdash}$. The multiplicativity follows from

$$\begin{aligned}
(T_s^{\vdash}(T_t^{\vdash}\psi))(\tau,x) &= \int (T_t^{\vdash}\psi)(s+\tau,y)\;T_{s+\tau,\tau}(x,dy) \\
&= \int\!\!\int \psi(s+t+\tau,y')\;T_{s+t+\tau,s+\tau}(y,dy')\;T_{s+\tau,\tau}(x,dy) \\
&= \int \psi(t+s+\tau,y)\;T_{s+t+\tau,\tau}(x,dy) = (T_{s+t}^{\vdash}\psi)(\tau,x)\,.
\end{aligned}$$