3

Extension of the Integral

Recall our goal: if Z is an L^p-integrator, then there exists an extension of its associated elementary integral to a class of integrands on which the Dominated Convergence Theorem holds.

The reader with a firm grounding in Daniell's extension of the integral will be able to breeze through the next 40 pages, merely identifying the results presented with those he is familiar with; the presentation is fashioned so as to facilitate this transition from the ordinary to the stochastic integral. The reader not familiar with Daniell's extension can use them as a primer.

Daniell's Extension Procedure on the Line

As before we look for guidance at the half-line. Let z be a right-continuous distribution function of finite variation, let the integral be defined on the elementary functions by equation (2.2) on page 44, and let us review step 2 of the integration process, the extension theory. Daniell's idea was to apply Lebesgue's definition of an outer measure of sets to functions, thus obtaining an upper integral of functions. A short overview can be found on page 395 of appendix A. The upshot is this. Given a right-continuous distribution function z of finite variation on the half-line, Daniell first defines the associated elementary integral $\mathfrak{e} \to \mathbb{R}$ by equation (2.2) on page 44, and then defines a seminorm, the ***Daniell mean*** $\| \ \|_z^*$, on all functions $f : [0, \infty) \to \overline{\mathbb{R}}$ by

$$\| f \|_z^* = \inf_{|f| \le h \in \mathfrak{e}^\uparrow} \ \sup_{\phi \in \mathfrak{e}, |\phi| \le h} \left| \int \phi \, dz \right| . \tag{3.1}$$

Here $\mathfrak{e}^\uparrow$ is the collection of all those functions that are pointwise suprema of *countable* collections of elementary integrands. The integrable functions are simply the closure of $\mathfrak{e}$ under this seminorm, and the integral is the extension by continuity of the elementary integral. This is the Lebesgue–Stieltjes integral. The Dominated Convergence Theorem and the numerous beautiful features of the Lebesgue–Stieltjes integral are all due to only two properties of Daniell's mean $\| \ \|_z^*$; it is *countably subadditive*:

$$\left\| \sum_{n=1}^{\infty} f_n \right\|_z^* \le \sum_{n=1}^{\infty} \| f_n \|_z^* , \qquad f_n \ge 0 ,$$

and it is additive on $\mathfrak{e}_+$, as it agrees there with the variation measure $\lvert dz \rvert$.

Let us put this procedure in a general context. Much of modern analysis concerns linear maps on vector spaces. Given such, the analyst will most frequently start out by designing a seminorm on the given vector space, one with respect to which the given linear map is continuous, and then extend the linear map by continuity to the completion of the vector space under that seminorm. The analysis of the extended linear map is generally easier because of the completeness of its domain, which furnishes limit points to many arguments. Daniell's method is but an instance of this. The vector space is $\mathfrak{e}$, the linear map is $x \mapsto \int x\,dz$, and the Daniell mean $\|\ \|_z^*$ is a suitable, in fact superb, seminorm with respect to which the linear map is continuous. The completion of $\mathfrak{e}$ is the space $\mathcal{L}^1(dz)$ of integrable functions.

3.1 The Daniell Mean

We shall extend the elementary stochastic integral in literally the same way, by designing a seminorm under which it is continuous. In fact, we shall simply emulate Daniell's "up-and-down procedure" of equation (3.1) and thence follow our noses.

The first thing to do is to replace the absolute value, which measures the size of the real-valued integral in equation (3.1), by a suitable size measurement of the random variable-valued elementary stochastic integral that takes its place. Any of the means and gauges mentioned on pages 33–34 will suit. Now a right-continuous adapted process Z may be an $L^p(\mathbb{P})$-integrator for some pairs $(p, \mathbb{P})$ and not for others. We will pick a pair $(p, \mathbb{P})$ such that it is. The notation will generally reflect only the choice of p, and of course Z, but not of $\mathbb{P}$; so the size measurement in question is $\|\ \|_p$, $⌈\!⌈\ ⌉\!⌉_p$, or $\|\ \|_{[\alpha]}$, depending on our predilection or need. The stochastic analog of definition (3.1) is

$$⌈\!⌈ F ⌉\!⌉_{Z\text{–}p}^* \stackrel{\text{def}}{=} \inf_{|F| \le H \in \mathcal{E}_+^\uparrow} \ \sup_{X \in \mathcal{E}, |X| \le H} \left⌈\!\!\left⌈ \int X\, dZ \right⌉\!\!\right⌉_p \text{, etc.} \tag{3.1.1}$$

Here $\mathcal{E}_+^\uparrow$ denotes the collection of positive processes that are pointwise suprema of a *sequence* of elementary integrands. Let us write separately the "up-part" and the "down-part" of (3.1.1): for $H \in \mathcal{E}_+^\uparrow$

$$\|H\|_{Z\text{–}p}^* = \sup\left\{ \left\| \int X\, dZ \right\|_p : X \in \mathcal{E}, |X| \le H \right\} \qquad (p \ge 1);$$

$$⌈\!⌈H⌉\!⌉_{Z\text{–}p}^* = \sup\left\{ \left⌈\!\!\left⌈ \int X\, dZ \right⌉\!\!\right⌉_p : X \in \mathcal{E}, |X| \le H \right\} \qquad (p \ge 0);$$

$$\|H\|_{Z\text{–}[\alpha]}^* = \sup\left\{ \left\| \int X\, dZ \right\|_{[\alpha]} : X \in \mathcal{E}, |X| \le H \right\} \qquad (p = 0).$$

Then on an arbitrary numerical process F,

$$\|F\|^*_{Z\text{–}p} = \inf\Big\{\|H\|^*_{Z\text{–}p} :\ H \in \mathcal{E}^{\uparrow}_+\ ,\ H \ge |F|\Big\} \qquad (p \ge 1)\ ;$$

$$⦀F⦀^*_{Z\text{–}p} = \inf\Big\{⦀H⦀^*_{Z\text{–}p} :\ H \in \mathcal{E}^{\uparrow}_+\ ,\ H \ge |F|\Big\} \qquad (p \ge 0)\ ;$$

$$\|F\|^*_{Z\text{–}[\alpha]} = \inf\Big\{\|H\|^*_{Z\text{–}[\alpha]} :\ H \in \mathcal{E}^{\uparrow}_+\ ,\ H \ge |F|\Big\} \qquad (p = 0)\ .$$

We shall refer to $⦀\ ⦀^*_{Z\text{–}p}$ as ***THE Daniell mean***. It goes with that semivariation which comes from the subadditive functional $⦀\ ⦀_p$ – the subadditivity of $⦀\ ⦀_p$ is the reason for singling it out. $⦀\ ⦀^*_{Z\text{–}p}$, too, will turn out to be subadditive, even countably subadditive. This property makes it best suited for the extension of the integral. If the probability needs to be mentioned, we also write $⦀\ ⦀^*_{Z\text{–}p;\mathbb{P}}$ etc.

As we would on the line we shall now establish the properties of the mean. Here as there, the Dominated Convergence Theorem and all of its beautiful corollaries are but consequences of these. The arguments are standard.

Exercise 3.1.1 $⦀\ ⦀^*_{Z\text{–}p}$ agrees with the semivariation $⦀\ ⦀_{Z\text{–}p}$ on $\mathcal{E}_+$. In fact, for $X \in \mathcal{E}$ we have $⦀X⦀^*_{Z\text{–}p} = ⦀|X|⦀_{Z\text{–}p}$. The same holds for the means associated with the other gauges.

Exercise 3.1.2 The following comes in handy on several occasions: let S, T be stopping times and assume that the projection $[S < T]$ of the stochastic interval $(\!(S, T]\!]$ on Ω has measure less than ϵ. Then any process F that vanishes outside $(\!(S, T]\!]$ has $⦀F⦀^*_{Z\text{–}0} \le \epsilon$.

Exercise 3.1.3 For a standard Wiener process W and arbitrary $F : \boldsymbol{B} \to \overline{\mathbb{R}}$,

$$\|F\|^*_{W\text{–}2} = \left(\int^* F^2(s,\omega)\ ds\times\mathbb{P}(d\omega)\right)^{1/2} .$$

$\|\ \|^*_{W\text{–}2}$ is simply the square mean for the measure $ds \times \mathbb{P}$ on $\mathcal{E}$. It is the mean originally employed by Itô and is still much in vogue (see definition (4.2.9)).

A Temporary Assumption

To start on the extension theory we have to place a temporary condition on the L^p-integrator Z, one that is at first sight rather more restrictive than the mere right-continuity in probability expressed in (IC-0); we have to require

Assumption 3.1.4 *The elementary integral is continuous in p-mean along increasing sequences. That is to say, for every increasing sequence $\big(X^{(n)}\big)$ of elementary integrands whose pointwise supremum X also happens to be an elementary integrand, we have*

$$\lim_n \int X^{(n)}\ dZ = \int X\ dZ \quad \textit{in p-mean.} \qquad \text{(IC-p)}$$

Exercise 3.1.5 This is equivalent with either of the following conditions:
(i) ***σ-continuity at*** 0: for every sequence $\big(X^{(n)}\big)$ of elementary integrands that decreases pointwise to zero, $\lim_{n\to\infty}\int X^{(n)}\,dZ = 0$ in p-mean;
(ii) ***σ-additivity***: for every sequence $\big(X^{(n)}\big)$ of positive elementary integrands whose sum is a priori an elementary integrand,

$$\sum_n \int X^{(n)}\,dZ = \int \sum_n X^{(n)}\,dZ \quad \text{in } p\text{-mean.}$$

Assumption 3.1.4 clearly implies (RC-0). In view of exercise 3.1.5 (ii), it is also reasonably called ***p-mean σ-additivity***. An L^p-integrator actually satisfies (IC-p) automatically; but when this fact is proved in proposition 3.3.2, the extension theory of the integral done under this assumption is needed. The reduction of (IC-p) to (RC-0) in section 3.3 will be made rather simple if the reader observes that

*In the extension theory of the elementary integral below, use is made only of the structure of the set $\mathcal{E}$ of elementary integrands – it is an algebra and vector lattice closed under chopping of bounded functions on some set, which is called the **base space** or **ambient set** – and of the properties* (B-p) *and* (IC-p) *of the vector measure*

$$\int \cdot\, dZ : \mathcal{E} \to L^p\ .$$

In particular, the structure of the ambient set is irrelevant to the extension procedure. The words "process" and "function" (on the base space) are used interchangeably.

Properties of the Daniell Mean

Theorem 3.1.6 *The Daniell mean* $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-p}$ *has the following properties:*
(i) It is defined on all numerical functions on the base space and takes values in the positive extended reals $\overline{\mathbb{R}}_+$.
*(ii) It is **solid**:* $|F| \le |G|$ *implies* $\lceil\!\lceil F \rceil\!\rceil^*_{Z-p} \le \lceil\!\lceil G \rceil\!\rceil^*_{Z-p}$.
*(iii) It is **continuous along increasing sequences*** $\big(\boldsymbol{H^{(n)}}\big)$ ***of*** $\mathcal{E}_+^{\uparrow}$ *:*

$$\Big\lceil\!\!\Big\lceil \sup_n H^{(n)} \Big\rceil\!\!\Big\rceil^*_{Z-p} = \sup_n \Big\lceil\!\!\Big\lceil H^{(n)} \Big\rceil\!\!\Big\rceil^*_{Z-p}\ .$$

*(iv) It is **countably subadditive**: for any sequence* $\big(F^{(n)}\big)$ *of positive functions on the base space*

$$\Big\lceil\!\!\Big\lceil \sum_{n=1}^{\infty} F^{(n)} \Big\rceil\!\!\Big\rceil^*_{Z-p} \le \sum_{n=1}^{\infty} \Big\lceil\!\!\Big\lceil F^{(n)} \Big\rceil\!\!\Big\rceil^*_{Z-p}\ .$$

*(v) Elementary integrands are **finite for the mean**:* $\lim_{r\to 0} \lceil\!\lceil rX \rceil\!\rceil^*_{Z-p} = 0$ *for all* $X \in \mathcal{E}$ *– when* $p > 0$ *this simply reads* $\lceil\!\lceil X \rceil\!\rceil^*_{Z-p} < \infty$.

(vi) For any sequence $\big(X^{(n)}\big)$ *of positive elementary integrands*

$$\left(\lim_{r\to 0}\left\lceil\!\left\lceil r\cdot\sum_{n=1}^{\infty}X^{(n)}\right\rceil\!\right\rceil^{*}_{Z-p}=0\right)\quad \textit{implies}\quad \left(\left\lceil\!\left\lceil X^{(n)}\right\rceil\!\right\rceil^{*}_{Z-p}\xrightarrow[n\to\infty]{}0\right)\qquad\text{(M)}$$

– when $p>0$ *this simply reads:*

$$\left(\left\lceil\!\left\lceil\sum_{n=1}^{\infty}X^{(n)}\right\rceil\!\right\rceil^{*}_{Z-p}<\infty\right)\quad \textit{implies}\quad \left(\left\lceil\!\left\lceil X^{(n)}\right\rceil\!\right\rceil^{*}_{Z-p}\xrightarrow[n\to\infty]{}0\right).$$

[It is this property which distinguishes the Daniell mean from an ordinary sup-norm and which is responsible for the Dominated Convergence Theorem and its beautiful consequences.]

(vii) The mean $\lceil\!\lceil\ \rceil\!\rceil^{*}_{Z-p}$ ***majorizes the elementary stochastic integral****:*

$$\left\lceil\!\left\lceil\int X\,dZ\right\rceil\!\right\rceil_{p}\le\lceil\!\lceil X\rceil\!\rceil^{*}_{Z-p}\qquad\forall\,X\in\mathcal{E}\,.$$

Proof. The first property that is possibly not obvious is (iii). To prove it let $(H^{(n)})$ be an increasing sequence of $\mathcal{E}_{+}^{\uparrow}$. Its pointwise supremum H clearly belongs to $\mathcal{E}_{+}^{\uparrow}$ as well. From the solidity,

$$\lceil\!\lceil H\rceil\!\rceil^{*}_{Z-p}\ge\sup\left\lceil\!\left\lceil H^{(n)}\right\rceil\!\right\rceil^{*}_{Z-p}.$$

To show the reverse inequality assume that $\lceil\!\lceil H\rceil\!\rceil^{*}_{Z-p}>a$. There exists an $X\in\mathcal{E}$ with $|X|\le H$ and

$$\left\lceil\!\left\lceil\int X\,dZ\right\rceil\!\right\rceil_{p}>a\,.$$

Write X as the difference $X=X_{+}-X_{-}$ of its positive and negative parts. For every n there is a sequence $(X^{(n,k)})$ with pointwise supremum $H^{(n)}$. Set

$$X^{(N)}=\bigvee_{n,k\le N}X^{(n,k)}\quad\text{and}\quad X_{\pm}^{(N)}=X^{(N)}\wedge X_{\pm}\,.$$

Clearly $X_{\pm}^{(N)}\uparrow X_{\pm}$, and therefore, with $\overline{X}^{(N)}=X_{+}^{(N)}-X_{-}^{(N)}$,

$$\int\overline{X}^{(N)}\,dZ\to\int X\,dZ\qquad\text{in }p\text{-mean.}$$

It is here that assumption 3.1.4 is used. Thus $\lceil\!\lceil\int\overline{X}^{(N)}\,dZ\rceil\!\rceil_{p}>a$ for sufficiently large N. As $|\overline{X}^{(N)}|\le H^{(N)}$, $\lceil\!\lceil H^{(N)}\rceil\!\rceil^{*}_{Z-p}>a$ eventually. This argument applies to the Daniell extension of any other semivariation – associated with any other solid and continuous functional on L^p – as well and shows that $\|\ \|^{*}_{Z-p}$ and $\|\ \|^{*}_{Z-[\alpha]}$, too, are continuous along increasing sequences of $\mathcal{E}_{+}^{\uparrow}$.

(iv) We start by proving the subadditivity of $\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p}$ on the class $\mathcal{E}_+^\uparrow$. Let $H^{(i)} \in \mathcal{E}_+^\uparrow$, $i = 1, 2$. There is a sequence $\big(X^{(i,n)}\big)_n$ in $\mathcal{E}_+$ whose pointwise supremum is $H^{(i)}$. Replacing $X^{(i,n)}$ by $\sup_{\nu \le n} X^{(i,\nu)}$, we may assume that $\big(X^{(i,n)}\big)$ is increasing. By (iii) and proposition 2.2.1,

$$\Big\lceil\!\Big\lceil H^{(1)} + H^{(2)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} = \lim_n \Big\lceil\!\Big\lceil X^{(1,n)} + X^{(2,n)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p}$$

$$\le \lim_n \left(\Big\lceil\!\Big\lceil X^{(1,n)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} + \Big\lceil\!\Big\lceil X^{(2,n)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} \right) = \Big\lceil\!\Big\lceil H^{(1)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} + \Big\lceil\!\Big\lceil H^{(2)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} .$$

To prove the countable subadditivity in general let $\big(F^{(n)}\big)$ be a sequence of numerical functions on the base space with $\sum \lceil\!\lceil F^{(n)} \rceil\!\rceil^*_{Z\text{–}p} < a < \infty$ – if the sum is infinite, there is nothing to prove. There are $H^{(n)} \in \mathcal{E}_+^\uparrow$ with $F^{(n)} \le H^{(n)}$ and $\sum \lceil\!\lceil H^{(n)} \rceil\!\rceil^*_{Z\text{–}p} < a$. The process $H = \sum H^{(n)}$ belongs to $\mathcal{E}_+^\uparrow$ and exceeds F. Consequently

$$\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}p} \le \lceil\!\lceil H \rceil\!\rceil^*_{Z\text{–}p} = \sup_N \Big\lceil\!\Big\lceil \sum_{n=1}^N H^{(n)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p}$$

from first part of proof: $$\le \sup_N \sum_{n=1}^N \Big\lceil\!\Big\lceil H^{(n)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} = \sum_{n=1}^\infty \Big\lceil\!\Big\lceil H^{(n)} \Big\rceil\!\Big\rceil^*_{Z\text{–}p} < a .$$

(v) follows from condition (B-p) on page 53, in view of exercise 3.1.1.

It remains to prove (M), which is the substitute for the additivity that holds in the scalar case. Note that it is a statement about the behavior of the mean $\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p}$ on $\mathcal{E}_+$, where it equals the semivariation $\lceil\!\lceil \ \rceil\!\rceil_{Z\text{–}p}$ (see definition (2.2.1)).

We start with the case $p > 0$. Since $\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p} = (\| \ \|^*_{Z\text{–}p})^{p \wedge 1}$, it suffices to show that

$$\left(\Big\| \sum_{n=1}^\infty X^{(n)} \Big\|^*_{Z\text{–}p} < \infty \right) \quad \text{implies} \quad \left(\Big\| X^{(n)} \Big\|_{Z\text{–}p} \xrightarrow[n\to\infty]{} 0 \right) . \tag{$*$}$$

Now $\| X^{(n)} \|_{Z\text{–}p} \xrightarrow[n\to\infty]{} 0$ means that for any sequence $\big(X'^{(n)}\big)$ of elementary integrands with $|X'^{(n)}| \le X^{(n)}$

$$\Big\| \int X'^{(n)} \, dZ \Big\|_p \xrightarrow[n\to\infty]{} 0 . \tag{$**$}$$

For if $\| X^{(n)} \|_{Z\text{–}p} \not\to 0$, then the very definition of the semivariation would produce a sequence violating $(**)$.

Let $\epsilon_1(t), \epsilon_2(t), \ldots$ be independent identically distributed Bernoulli random variables, defined on a probability space $(D, \mathcal{D}, \tau)$, with $\tau([\epsilon_\nu = \pm 1]) = 1/2$. Then, with $f_n \overset{\text{def}}{=} \int X'^{(n)} \, dZ$,

$$\sum_{n \le N} \epsilon_n(t) f_n = \int \sum_{n \le N} \epsilon_n(t) X'^{(n)} \, dZ , \qquad t \in D .$$

The second of Khintchine's inequalities, proved as theorem A.8.26, provides a universal constant $K_p = K_p^{(A.8.5)}$ such that

$$\Big(\sum_{n\le N} f_n^2\Big)^{1/2} \le K_p \cdot \Big(\int \Big|\sum_{n\le N} \epsilon_n(t) f_n\Big|^p \,\tau(dt)\Big)^{1/p} .$$

Applying $\|\cdot\|_p$ and using Fubini's theorem A.3.18 on this results in

$$\begin{aligned}\Big\|\Big(\sum_{n\le N} f_n^2\Big)^{1/2}\Big\|_p &\le K_p \cdot \Big(\int\int \Big|\int \sum_{n\le N} \epsilon_n(t) X'^{(n)}\, dZ\Big|^p \,d\mathbb{P}\,\tau(dt)\Big)^{1/p} \\ &\le K_p \cdot \Big(\int \Big\|\sum_{n\le N} \epsilon_n(t) X'^{(n)}\Big\|_{Z-p}^p \,\tau(dt)\Big)^{1/p} \\ &\le K_p \cdot \sup_N \Big\|\sum_{n\le N} X^{(n)}\Big\|_{Z-p} \le K_p \cdot \Big\|\sum_{n=1}^{\infty} X^{(n)}\Big\|_{Z-p}^* < \infty .\end{aligned}$$

The function $h \stackrel{\text{def}}{=} \big(\sum_{n\in\mathbb{N}} f_n^2\big)^{1/2}$ therefore belongs to L^p. This implies that $f_n \to 0$ a.s. and dominatedly (by h); therefore $\|f_n\|_p \xrightarrow[n\to\infty]{} 0$.

If $p = 0$, we use inequality (A.8.6) instead:

$$\Big(\sum_{n\le N} f_n^2\Big)^{1/2} \le K_0 \cdot \Big\|\sum_{n\le N} \epsilon_n(t) f_n\Big\|_{[\kappa_0;\tau]} ;$$

and thus, applying $\|\ \|_{[\alpha;\mathbb{P}]}$ and exercise A.8.16,

$$\begin{aligned}\Big\|\Big(\sum_{n\le N} f_n^2\Big)^{\frac12}\Big\|_{[\alpha;\mathbb{P}]} &\le K_0 \cdot \Big\|\Big\|\sum_{n\le N} \epsilon_n(t) f_n\Big\|_{[\kappa_0;\tau]}\Big\|_{[\alpha;\mathbb{P}]} \\ &\le K_0 \cdot \Big\|\Big\|\int \sum_{n\le N} \epsilon_n X'^{(n)}\, dZ\Big\|_{[\gamma;\mathbb{P}]}\Big\|_{[\alpha\kappa_0-\gamma;\tau]} \\ &\le K_0 \cdot \Big\|\Big\|\sum_{n\le N} X^{(n)}\Big\|_{Z-[\gamma]}\Big\|_{[\alpha\kappa_0-\gamma;\tau]} \le K_0 \cdot \Big\|\sum_{n\le N} X^{(n)}\Big\|_{Z-[\gamma]} .\end{aligned}$$

This holds for all $\gamma < \alpha\kappa_0$, and therefore

$$\Big\|\Big(\sum_{n\le N} f_n^2\Big)^{1/2}\Big\|_{[\alpha]} \le K_0 \cdot \Big\|\sum_{n\le N} X^{(n)}\Big\|_{Z-[\alpha\kappa_0]} \le K_0 \cdot \Big\|\sum_{n=1}^{\infty} X^{(n)}\Big\|_{Z-[\alpha\kappa_0]}^* .$$

It is left to the reader to show that $(*)$ implies

$$\Big\|\sum_{n=1}^{\infty} X^{(n)}\Big\|_{Z-[\alpha\kappa_0]}^* < \infty \qquad\qquad \forall \alpha > 0 ,$$

which, in conjunction with the previous inequality, proves that

$$\Big(\sum_{n=1}^{\infty} f_n^2\Big)^{1/2} < \infty \quad \text{a.s.}$$

Thus clearly $f_n \to 0$ in L^0. ▮

It is worth keeping the quantitative information gathered above for an application to the square function on page 148.

Corollary 3.1.7 *Let Z be an adapted process and $X^{(1)}, X^{(2)}, \ldots \in \mathcal{E}$. Then*

$$\Big\| \Big(\sum_n \Big(\int X^{(n)}\, dZ\Big)^2\Big)^{1/2} \Big\|_p \le K_p \cdot \Big\| \sum_n |X^{(n)}| \Big\|^*_{Z\text{–}p}, \qquad p > 0;$$

and

$$\Big\| \Big(\sum_n \Big(\int X^{(n)}\, dZ\Big)^2\Big)^{1/2} \Big\|_{[\alpha]} \le K_0 \cdot \Big\| \sum_n |X^{(n)}| \Big\|^*_{Z\text{–}[\alpha\kappa_0]}, \qquad p = 0.$$

The constants K_p, κ_0 are the Khintchine constants of theorem A.8.26.

Exercise 3.1.8 The $\| \ \|^*_{Z\text{–}p}$ for $0 < p < 1$, and the $\| \ \|^*_{Z\text{–}[\alpha]}$ for $0 < \alpha$, too, have the properties listed in theorem 3.1.6, except countable subadditivity.

3.2 The Integration Theory of a Mean

Any functional $[\![\ ]\!]^*$ satisfying (i)–(vi) of theorem 3.1.6 is called a mean on $\mathcal{E}$. This notion is so useful that a little repetition is justified:

Definition 3.2.1 *Let $\mathcal{E}$ be an algebra and vector lattice closed under chopping of bounded functions, all defined on some set $\boldsymbol{B}$. A **mean on** $\mathcal{E}$ is a positive $\overline{\mathbb{R}}$-valued functional $[\![\ ]\!]^*$ that is defined on all numerical functions on $\boldsymbol{B}$ and has the following properties:*

(i) It is solid: $|F| \le |G|$ implies $[\![F]\!]^* \le [\![G]\!]^*$.

(ii) It is continuous along increasing sequences $\big(X^{(n)}\big)$ of $\mathcal{E}_+$:

$$\Big[\!\Big[\sup_n X^{(n)} \Big]\!\Big]^* = \sup_n \Big[\!\Big[X^{(n)} \Big]\!\Big]^* .$$

(iii) It is countably subadditive: for any sequence $\big(F^{(n)}\big)$ of positive functions on $\boldsymbol{B}$

$$\Big[\!\Big[\sum_{n=1}^{\infty} F^{(n)} \Big]\!\Big]^* \le \sum_{n=1}^{\infty} \Big[\!\Big[F^{(n)} \Big]\!\Big]^* . \qquad (CSA)$$

(iv) The functions of $\mathcal{E}$ are finite for the mean: for every $X \in \mathcal{E}$

$$\lim_{r \to 0} [\![rX]\!]^* = 0 .$$

(v) For any sequence $\left(X^{(n)}\right)$ *of positive functions in* $\mathcal{E}$

$$\left(\lim_{r\to 0}\left\lceil\!\left\lceil r\cdot\sum_{n=1}^{\infty}X^{(n)}\right\rceil\!\right\rceil^{*}=0\right)\quad\text{implies}\quad\left(\left\lceil\!\left\lceil X^{(n)}\right\rceil\!\right\rceil^{*}\xrightarrow[n\to\infty]{}0\right)\,. \tag{M}$$

Let $\mathcal{V}$ *be a topological vector space with a gauge* $\lceil\!\lceil\ \rceil\!\rceil_{\mathcal{V}}$ *defining its topology, and let* $\mathcal{I}:\mathcal{E}\to\mathcal{V}$ *be a linear map. The mean* $\lceil\!\lceil\ \rceil\!\rceil^{*}$ *is said to* ***majorize*** $\mathcal{I}$ *if* $\lceil\!\lceil\mathcal{I}(X)\rceil\!\rceil_{\mathcal{V}}\le\lceil\!\lceil X\rceil\!\rceil^{*}$ *for all* $X\in\mathcal{E}$. $\lceil\!\lceil\ \rceil\!\rceil^{*}$ *is said to* ***control*** $\mathcal{I}$ *if there is a constant* $C<\infty$ *such that for all* $X\in\mathcal{E}$

$$\lceil\!\lceil\mathcal{I}(X)\rceil\!\rceil_{\mathcal{V}}\le C\cdot\lceil\!\lceil X\rceil\!\rceil^{*}\,. \tag{3.2.1}$$

The crossbars on top of the symbol $\lceil\!\lceil\ \rceil\!\rceil^{*}$ are a reminder that the mean is subadditive but possibly not homogeneous. THE Daniell mean was constructed so as to majorize the elementary stochastic integral.

This and the following two sections will use only the fact that the elementary integrands $\mathcal{E}$ form an algebra and vector lattice closed under chopping, of bounded functions, and that $\lceil\!\lceil\ \rceil\!\rceil^{*}$ is a mean on $\mathcal{E}$. The nature of the underlying set $\boldsymbol{B}$ in particular is immaterial, and so is the way in which the mean was constructed. In order to emphasize this point we shall develop in the next few sections the integration theory of a general mean $\lceil\!\lceil\ \rceil\!\rceil^{*}$. Later on we shall meet means other than Daniell's mean $\lceil\!\lceil\ \rceil\!\rceil^{*}_{Z\text{–}p}$, so that we may then use the results established here for them as well. In fact, Daniell's mean is unsuitable as a controlling device for Picard's scheme, which so far was the motivation for all of our proceedings. Other "pathwise" means controlling the elementary integral have to be found (see definition (4.2.9) and exercise 4.5.18).

Exercise 3.2.2 A mean is automatically continuous along increasing sequences $(H^{(n)})$ of $\mathcal{E}_{+}^{\uparrow}$:

$$\left\lceil\!\left\lceil\sup_n H^{(n)}\right\rceil\!\right\rceil^{*}=\sup_n\left\lceil\!\left\lceil H^{(n)}\right\rceil\!\right\rceil^{*}\,. \tag{$\uparrow$}$$

(Every mean that we shall encounter in this book, including Daniell's, is actually continuous along arbitrary increasing sequences. That is to say, ($\uparrow$) holds for any increasing sequence of positive numerical functions. See proposition 3.6.5.)

Negligible Functions and Sets

In this subsection only the solidity and countable subadditivity of the mean $\lceil\!\lceil\ \rceil\!\rceil^{*}$ are exploited.

Definition 3.2.3 *A numerical function* F *on the base space (a process) is called* $\lceil\!\lceil\ \rceil\!\rceil^{*}$***-negligible*** *or* ***negligible*** *for short if* $\lceil\!\lceil F\rceil\!\rceil^{*}=0$. *A subset of the base space is negligible if its indicator function is negligible.*[1]

[1] In accordance with convention A.1.5 on page 364 we write variously A or 1_A for the indicator function of A; the $\lceil\!\lceil\ \rceil\!\rceil^{*}$-size of the set A, $\lceil\!\lceil 1_A\rceil\!\rceil^{*}$, is mostly written $\lceil\!\lceil A\rceil\!\rceil^{*}$ when A is a subset of the ambient space.

A property of the points of the underlying set is said to hold ***almost everywhere****, or* ***a.e.*** *for short, if the set of points where it fails to hold is negligible.*

If we want to stress the point that these definitions refer to $\lceil\!\lceil\ \rceil\!\rceil^*$*, we shall talk about* $\lceil\!\lceil\ \rceil\!\rceil^*$*-negligible processes and* $\lceil\!\lceil\ \rceil\!\rceil^*$***-a.e. convergence****, etc. If we want to stress the point that these definitions refer in particular to Daniell's mean* $\lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$*, we shall talk about* $\lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$*-negligible processes and* $\lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$*-a.e. convergence, or also about* ***Z–p-negligible*** *processes,* ***Z–p-a.e. convergence****, etc.*

These notions behave as one expects from ordinary integration:

Proposition 3.2.4 *(i) The union of countably many negligible sets is negligible. Any subset of a negligible set is negligible.*

(ii) A process F *is negligible if and only if it vanishes almost everywhere, that is to say, if and only if the set* $[F \neq 0]$ *is negligible.*

(iii) If the real-valued functions F *and* F' *agree almost everywhere, then they have the same mean.*

Proof. For ease of reading we use the same symbol for a set and its indicator function. For instance, $A_1 \cup A_2 = A_1 \vee A_2$ in the sense that the indicator function on the left[1] is the pointwise maximum of the two indicator functions on the right.

(i) If N_n, $n = 1, \ldots,$ are negligible sets, then due to inequality (CSA)[1]

$$\lceil\!\lceil N_1 \cup N_2 \cup \ldots \rceil\!\rceil^* = \left\lceil\!\!\left\lceil \bigvee_{n=1}^{\infty} N_n \right\rceil\!\!\right\rceil^*$$
$$\leq \left\lceil\!\!\left\lceil \textstyle\sum_{n=1}^{\infty} N_n \right\rceil\!\!\right\rceil^* \leq \textstyle\sum_{n=1}^{\infty} \lceil\!\lceil N_n \rceil\!\rceil^* = 0 \ ,$$

due to the countable subadditivity of $\lceil\!\lceil\ \rceil\!\rceil^*$.

(ii) Obviously[1] $[F \neq 0] \leq \sum_{n=1}^{\infty} |F|$. Thus if $\lceil\!\lceil F \rceil\!\rceil^* = 0$, then

$$\lceil\!\lceil [F \neq 0] \rceil\!\rceil^* \leq \textstyle\sum_{n=1}^{\infty} \lceil\!\lceil F \rceil\!\rceil^* = 0.$$

Conversely, $|F| \leq \sum_{n=1}^{\infty} [F \neq 0]$, so that $\lceil\!\lceil [F \neq 0] \rceil\!\rceil^* = 0$ implies

$$\lceil\!\lceil F \rceil\!\rceil^* \leq \textstyle\sum_{n=1}^{\infty} \lceil\!\lceil [F \neq 0] \rceil\!\rceil^* = 0.$$

(iii) Since by the previous argument $\lceil\!\lceil F \cdot [F \neq F'] \rceil\!\rceil^* \leq \lceil\!\lceil \infty \cdot [F \neq F'] \rceil\!\rceil^* = 0$,

$$\lceil\!\lceil F \rceil\!\rceil^* \leq \left\lceil\!\!\left\lceil F \cdot [F = F'] \right\rceil\!\!\right\rceil^* + \left\lceil\!\!\left\lceil F \cdot [F \neq F'] \right\rceil\!\!\right\rceil^* = \left\lceil\!\!\left\lceil F \cdot [F = F'] \right\rceil\!\!\right\rceil^*$$
$$= \left\lceil\!\!\left\lceil F' \cdot [F = F'] \right\rceil\!\!\right\rceil^* \leq \left\lceil\!\!\left\lceil F' \right\rceil\!\!\right\rceil^* \text{ and vice versa.}$$

Exercise 3.2.5 The filtration being regular, an evanescent process is *Z–p*-negligible.

Processes Finite for the Mean and Defined Almost Everywhere

Definition 3.2.6 *A process F is* ***finite for the mean*** $\lceil\!\lceil\ \rceil\!\rceil^*$ *provided*

$$\lceil\!\lceil r\cdot F \rceil\!\rceil^* \xrightarrow[r\to 0]{} 0\,.$$

The collection of processes finite for the mean $\lceil\!\lceil\ \rceil\!\rceil^*$ is denoted by $\mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$, or simply by $\mathfrak{F}$ if there is no need to specify the mean.

If $\lceil\!\lceil\ \rceil\!\rceil^*$ is the Daniell mean $\lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$ for some $p>0$, then F is finite for the mean if and only if simply $\lceil\!\lceil F \rceil\!\rceil^* < \infty$. If $p=0$ and $\lceil\!\lceil\ \rceil\!\rceil^* = \lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}0}$, though, then $\lceil\!\lceil F \rceil\!\rceil^* \le 1$ for all F, and the somewhat clumsy looking condition $\lceil\!\lceil rF \rceil\!\rceil^* \xrightarrow[r\to 0]{} 0$ properly expresses finiteness (see exercise A.8.18).

Proposition 3.2.7 *A process F finite for the mean $\lceil\!\lceil\ \rceil\!\rceil^*$ is finite $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e.*

Proof.[1] $[|F|=\infty] \le |F|/n$ for all $n\in\mathbb{N}$, and the solidity gives

$$\lceil\!\lceil\, \big[|F|=\infty\big]\, \rceil\!\rceil^* \le \lceil\!\lceil F/n \rceil\!\rceil^* \qquad \forall\, n\in\mathbb{N}.$$

Let $n\to\infty$ and conclude that $\lceil\!\lceil\, \big[|F|=\infty\big]\, \rceil\!\rceil^* = 0$. ∎

The only processes of interest are, of course, those finite for the mean. We should like to argue that the sum of any two of them has finite mean again, in view of the subadditivity of $\lceil\!\lceil\ \rceil\!\rceil^*$. A technical difficulty appears: even if F and G have finite mean, there may be points ϖ in the base space where $F(\varpi)=+\infty$ and $G(\varpi)=-\infty$ or vice versa; then $F(\varpi)+G(\varpi)$ is not defined. The solution to this tiny quandary is to notice that such ambiguities may happen at most in a negligible set of $\varpi's$. We simply extend $\lceil\!\lceil\ \rceil\!\rceil^*$ to processes that are defined merely $\lceil\!\lceil\ \rceil\!\rceil^*$-almost everywhere:

Definition 3.2.8 (Extending the Mean) *Let F be a process defined almost everywhere, i.e., such that the complement of* $\mathrm{dom}(F)$ *is $\lceil\!\lceil\ \rceil\!\rceil^*$-negligible. We set $\lceil\!\lceil F \rceil\!\rceil^* \stackrel{\text{def}}{=} \lceil\!\lceil F' \rceil\!\rceil^*$, where F' is any process defined everywhere and coinciding with F almost everywhere in the points where F is defined.*

Part (iii) of proposition 3.2.4 shows that this definition is good: it does not matter which process F' we choose to agree $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e. with F; any two will differ negligibly and thus have the same mean. Given two processes F and G finite for the mean that are merely almost everywhere defined, we define their sum $F+G$ to equal $F(\varpi)+G(\varpi)$ where both $F(\varpi)$ and $G(\varpi)$ are finite. This process is almost everywhere defined, as the set of points where F or G are infinite or not defined is negligible. It is clear how to define the scalar multiple $r\cdot F$ of a process F that is a.e. defined.

From now on, "***process***" will stand for "almost everywhere defined process" if the context permits it. It is nearly obvious that propositions 3.2.4 and 3.2.7 stay. We leave this to the reader.

Exercise 3.2.9 $\big|\, \lceil\!\lceil F \rceil\!\rceil^* - \lceil\!\lceil G \rceil\!\rceil^* \,\big| \le \lceil\!\lceil F - G \rceil\!\rceil^*$ for any two $F, G \in \mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$.

Theorem 3.2.10 *A process finite for the mean is finite almost everywhere. The collection $\mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$ of processes finite for $\lceil\!\lceil\ \rceil\!\rceil^*$ is closed under taking finite linear combinations, finite maxima and minima, and under chopping, and $\lceil\!\lceil\ \rceil\!\rceil^*$ is a solid and countably subadditive functional on $\mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$. The space $\mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$ is complete under the translation-invariant pseudometric*

$$dist(F, F') \stackrel{\text{def}}{=} \lceil\!\lceil F - F' \rceil\!\rceil^* .$$

Moreover, any mean-Cauchy sequence in $\mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$ has a subsequence that converges $\lceil\!\lceil\ \rceil\!\rceil^*$-almost everywhere to a $\lceil\!\lceil\ \rceil\!\rceil^*$-mean limit.

Proof. The first two statements are left as exercise 3.2.11. For the last two let (F_n) be a mean-Cauchy sequence in $\mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$; that is to say

$$\sup_{m,n \ge N} \lceil\!\lceil F_m - F_n \rceil\!\rceil^* \xrightarrow[N\to\infty]{} 0 .$$

For $n = 1, 2, \ldots$ let F'_n be a process that is everywhere defined and finite and agrees with F_n a.e. Let N_n denote the negligible set of points where F_n is not defined or does not agree with F'_n. There is an increasing sequence (n_k) of indices such that $\lceil\!\lceil F'_n - F'_{n_k} \rceil\!\rceil^* \le 2^{-k}$ for $n \ge n_k$. Using them set

$$G \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \big| F'_{n_{k+1}} - F'_{n_k} \big| .$$

G is finite for the mean. Indeed, for $|r| \le 1$,

$$\lceil\!\lceil rG \rceil\!\rceil^* \le \sum_{k=1}^{K} \Big\lceil\!\!\Big\lceil r \cdot \big(F'_{n_{k+1}} - F'_{n_k}\big) \Big\rceil\!\!\Big\rceil^* + \sum_{k=K+1}^{\infty} \Big\lceil\!\!\Big\lceil F'_{n_{k+1}} - F'_{n_k} \Big\rceil\!\!\Big\rceil^* .$$

Given $\epsilon > 0$ we first choose K so large that the second summand is less than $\epsilon/2$ and then r so small that the first summand is also less than $\epsilon/2$. This shows that $\lim_{r\to 0} \lceil\!\lceil rG \rceil\!\rceil^* = 0$.

$$N \stackrel{\text{def}}{=} \bigcup_{n=1}^{\infty} N_n \cup [G = \infty]$$

is therefore a negligible set. If $\varpi \notin N$, then

$$F(\varpi) = F'_{n_1}(\varpi) + \sum_{k=1}^{\infty} \big(F'_{n_{k+1}}(\varpi) - F'_{n_k}(\varpi)\big) = \lim_{k\to\infty} F'_{n_k}(\varpi)$$

exists, since the infinite sum converges absolutely. Also,[1]

$$\begin{aligned}
\lceil\!\lceil F - F_{n_K} \rceil\!\rceil^* &= \Big\lceil\!\!\Big\lceil F - F'_{n_K} \Big\rceil\!\!\Big\rceil^* \\
&\le \Big\lceil\!\!\Big\lceil N \cdot (F - F'_{n_K}) \Big\rceil\!\!\Big\rceil^* + \Big\lceil\!\!\Big\lceil N^c \cdot (F - F'_{n_K}) \Big\rceil\!\!\Big\rceil^* \\
&\le \Big\lceil\!\!\Big\lceil \sum_{k=K+1}^{\infty} |F'_{n_{k+1}} - F'_{n_k}| \Big\rceil\!\!\Big\rceil^* \le 2^{-K} \xrightarrow[K\to\infty]{} 0 .
\end{aligned}$$

Thus $\left(F'_{n_k}\right)_{k=1}^{\infty}$ converges to F not only pointwise but also in mean. Given $\epsilon > 0$, let K be so large that both

$$\lceil\!\lceil F_m - F_n \rceil\!\rceil^* < \epsilon/2 \quad \text{for } m, n \geq n_K$$

and

$$\lceil\!\lceil F - F_{n_k} \rceil\!\rceil^* = \lceil\!\lceil F - F'_{n_k} \rceil\!\rceil^* < \epsilon/2 \quad \text{for } \ k \geq K \ .$$

For any $n \geq N \stackrel{\text{def}}{=} n_K$

$$\lceil\!\lceil F - F_n \rceil\!\rceil^* < \lceil\!\lceil F - F_{n_K} \rceil\!\rceil^* + \lceil\!\lceil F_{n_K} - F_n \rceil\!\rceil^* < \epsilon \ ,$$

showing that the original sequence $\left(F_n\right)$ converges to F in mean. Its subsequence $\left(F_{n_k}\right)$ clearly converges $\lceil\!\lceil \ \rceil\!\rceil^*$-almost everywhere to F. ∎

Henceforth we shall not be so excruciatingly punctilious. If we have to perform algebraic or limit arguments on a sequence of processes that are defined merely almost everywhere, we shall without mention replace every one of them with a process that is defined and finite everywhere, and perform the arguments on the resulting sequence; this affects neither the means of the processes nor their convergence in mean or almost everywhere.

Exercise 3.2.11 Define the linear combination, minimum, maximum, and product of two processes defined a.e., and prove the first two statements of theorem 3.2.10. Show that $\mathfrak{F}[\lceil\!\lceil \ \rceil\!\rceil^*]$ is not in general an algebra.

Exercise 3.2.12 (i) Let (F_n) be a mean-convergent sequence with limit F. Any process differing negligibly from F is also a mean limit of (F_n). Any two mean limits of (F_n) differ negligibly. (ii) Suppose that the processes F_n are finite for the mean $\lceil\!\lceil \ \rceil\!\rceil^*$ and $\sum_n \lceil\!\lceil F_n \rceil\!\rceil^*$ is finite. Then $\sum_n |F_n|$ is finite for the mean $\lceil\!\lceil \ \rceil\!\rceil^*$.

Integrable Processes and the Stochastic Integral

Definition 3.2.13 *An $\lceil\!\lceil \ \rceil\!\rceil^*$-almost everywhere defined process F is $\lceil\!\lceil \ \rceil\!\rceil^*$**-integrable** if there exists a sequence $\left(X_n\right)$ of elementary integrands converging in $\lceil\!\lceil \ \rceil\!\rceil^*$-mean to F: $\lceil\!\lceil F - X_n \rceil\!\rceil^* \xrightarrow[n\to\infty]{} 0$.*

*The collection of $\lceil\!\lceil \ \rceil\!\rceil^*$-integrable processes is denoted by $\mathfrak{L}^1[\lceil\!\lceil \ \rceil\!\rceil^*]$ or simply by $\mathfrak{L}^1$. In other words, $\mathfrak{L}^1$ is the $\lceil\!\lceil \ \rceil\!\rceil^*$-closure of $\mathcal{E}$ in $\mathfrak{F}$ (see exercise 3.2.15). If the mean is Daniell's mean $\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p}$ and we want to stress this point, then we shall also talk about **Z–p-integrable** processes and write $\mathfrak{L}^1[\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p}]$ or $\mathfrak{L}^1[Z\text{–}p]$. If the probability also must be exhibited, we write $\mathfrak{L}^1[Z\text{–}p; \mathbb{P}]$ or $\mathfrak{L}^1[\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p;\mathbb{P}}]$.*

Definition 3.2.14 *Suppose that the mean $\lceil\!\lceil \ \rceil\!\rceil^*$ is Daniell's mean $\lceil\!\lceil \ \rceil\!\rceil^*_{Z\text{–}p}$ or at least controls the elementary integral (definition 3.2.1), and suppose that F is an $\lceil\!\lceil \ \rceil\!\rceil^*$-integrable process. Let $\left(X_n\right)$ be a sequence of elementary integrands converging in $\lceil\!\lceil \ \rceil\!\rceil^*$-mean to F; the integral $\int F \, dZ$ is defined as the limit in p-mean of the sequence $\left(\int X_n \, dZ\right)$ in L^p. In other words, the extended integral is the extension by $\lceil\!\lceil \ \rceil\!\rceil^*$-continuity of the elementary integral. It is also called the **Itô stochastic integral**.*

This is unequivocal except perhaps for the definition of the integral. How do we know that the sequence $\left(\int X_n\, dZ\right)$ has a limit? Since $\lceil\!\lceil\ \rceil\!\rceil^*$ controls the elementary integral, we have

$$\left\lceil\!\!\left\lceil \int X_n\, dZ - \int X_m\, dZ \right\rceil\!\!\right\rceil_p$$

by equation (3.2.1):
$$\le C\cdot\lceil\!\lceil X_n - X_m \rceil\!\rceil^*$$
$$\le C\cdot\lceil\!\lceil F - X_n \rceil\!\rceil^* + C\cdot\lceil\!\lceil F - X_m \rceil\!\rceil^* \xrightarrow[n,m\to\infty]{} 0 .$$

The sequence $\left(\int X_n \;\; dZ\right)$ is therefore Cauchy in L^p and has a limit in p-mean (exercise A.8.1). How do we know that this limit does not depend on the particular sequence $\left(X_n\right)$ of elementary integrands chosen to approximate F in $\lceil\!\lceil\ \rceil\!\rceil^*$-mean? If $\left(X'_n\right)$ is a second such sequence, then clearly $\lceil\!\lceil X_n - X'_n \rceil\!\rceil^* \to 0$, and since the mean controls the elementary integral, $\lceil\!\lceil \int X_n\, dZ - \int X'_n\, dZ \rceil\!\rceil_p \to 0$: the limits are the same.

Let us be punctilious about this. The integrals $\int X_n\, dZ$ are by definition random variables. They form a Cauchy sequence in p-mean. There is not only one p-mean limit but many, all differing negligibly. The integral $\int F\, dZ$ above is by nature a ***class*** in $L^p(\mathbb{P})$! We won't be overly religious about this point; for instance, we won't hesitate to multiply a random variable f with the class $\int X \;dZ$ and understand $f\cdot\int X \;dZ$ to be the class $\dot f\cdot\int X \;dZ$. Yet there are some occasions where the distinction is important (see definition 3.7.6). Later on we shall pick from the class $\int F\, dZ$ a random variable in a nearly unique manner (see page 134).

Exercise 3.2.15 (i) A process F is $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable if and only if there exist integrable processes F_n with $F = \sum_n F_n$ and $\sum_n \lceil\!\lceil F_n \rceil\!\rceil^* < \infty$. (ii) An integrable process F is finite for the mean. (iii) The mean satisfies the all-important property (M) of definition 3.2.1 on sequences $\left(X_n\right)$ of positive integrable processes.

Exercise 3.2.16 (i) Assume that the mean controls the elementary integral $\int \cdot\, dZ : \mathcal{E} \to L^p$ (see definition 3.2.1 on page 94). Then the extended integral is a linear map $\int \cdot\, dZ : \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*] \to L^p$ again controlled by the mean:

$$\left\lceil\!\!\left\lceil \int F\, dZ \right\rceil\!\!\right\rceil_p \le C^{(3.2.1)} \cdot \lceil\!\lceil F \rceil\!\rceil^* , \qquad\qquad F \in \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*] .$$

(ii) Let $\lceil\!\lceil\ \rceil\!\rceil^* \le \lceil\!\lceil\ \rceil\!\rceil'^*$ be two means on $\mathcal{E}$. Then a $\lceil\!\lceil\ \rceil\!\rceil'^*$-integrable process is $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable. If both means control the elementary stochastic integral, then their integral extensions coincide on $\mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil'^*] \subset \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]$.

(iii) If Z is an L^q-integrator and $0 \le p < q < \infty$, then Z is an L^p-integrator; a Z–q-integrable process X is Z–p-integrable, and the integrals in either sense coincide.

Exercise 3.2.17 If the martingale M is an L^1-integrator, then $\mathbb{E}[\int X\, dM] = 0$ for any M–1-integrable process X with $X_0 = 0$.

Exercise 3.2.18 If $\mathcal{F}_\infty$ is countably generated, then the pseudometric space $\mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]$ is separable.

Exercise 3.2.19 Suppose that we start with a measured filtration $(\mathcal{F}_{.}, \mathbb{P})$ and an L^p-integrator Z in the sense of the original definition 2.1.7 on page 49. To obtain path regularity and simple truths like exercise 3.2.5, we replace $\mathcal{F}_{.}$ by its natural enlargement $\mathcal{F}^{\mathbb{P}}_{.+}$ and Z by a nice modification. $\mathfrak{L}^1$ is then the closure of $\mathcal{E}^{\mathbb{P}} = \mathcal{E}[\mathcal{F}^{\mathbb{P}}_{.+}]$ under $\lceil\!\lceil \ \rceil\!\rceil^*_{Z-p}$. Show that the original set $\mathcal{E}$ of elementary integrands is dense in $\mathfrak{L}^1$.

Permanence Properties of Integrable Functions

From now on we shall make use of all of the properties that make $\lceil\!\lceil \ \rceil\!\rceil^*$ a mean. We continue to write simply "integrable" and "negligible" instead of the more precise "$\lceil\!\lceil \ \rceil\!\rceil^*$-integrable" and "$\lceil\!\lceil \ \rceil\!\rceil^*$-negligible," etc. The next result is obvious:

Proposition 3.2.20 *Let $\big(F_n\big)$ be a sequence of integrable processes converging in $\lceil\!\lceil \ \rceil\!\rceil^*$-mean to F. Then F is integrable. If $\lceil\!\lceil \ \rceil\!\rceil^*$ controls the elementary integral in $\lceil\!\lceil \ \rceil\!\rceil_p$-mean, i.e., as a linear map to L^p, then*

$$\int F\, dZ = \lim_{n\to\infty} \int F_n\, dZ \quad \text{in } \lceil\!\lceil \ \rceil\!\rceil_p\text{-mean.}$$

Permanence Under Algebraic and Order Operations

Theorem 3.2.21 *Let $0 \le p < \infty$ and Z an L^p-integrator. Let F and F' be $\lceil\!\lceil \ \rceil\!\rceil^*$-integrable processes and $r \in \mathbb{R}$. Then the combinations $F + F'$, rF, $F \vee F'$, $F \wedge F'$, and $F \wedge 1$ are $\lceil\!\lceil \ \rceil\!\rceil^*$-integrable. So is the product $F \cdot F'$, provided that at least one of F, F' is bounded.*

Proof. We start with the sum. For any two elementary integrands X, X'

we have $\qquad |(F + F') - (X + X')| \le |F - X| + |F' - X'|\,,$

and so $\qquad \lceil\!\lceil (F + F') - (X + X') \rceil\!\rceil^* \le \lceil\!\lceil F - X \rceil\!\rceil^* + \lceil\!\lceil F' - X' \rceil\!\rceil^*\,.$

Since the right-hand side can be made as small as one pleases by the choice of X, X', so can the left-hand side. This says that $F + F'$ is integrable, inasmuch as $X + X'$ is an elementary integrand. The same argument applies to the other combinations:

$$\begin{aligned}
|(rF) - (rX)| &\le ([\lfloor r \rfloor] + 1) \cdot |F - X|\,;\\
|(F \vee F') - (X \vee X')| &\le |F - X| + |F' - X'|\,;\\
|(F \wedge F') - (X \wedge X')| &\le |F - X| + |F' - X'|\,;\\
||F| - |X|| &\le |F - X|\,;\ |F \wedge 1 - X \wedge 1| \le |F - X|\,;\\
|(F \cdot F') - (X \cdot X')| &\le |F| \cdot |F' - X'| + |X'| \cdot |F - X|\\
&\le \|F\|_\infty \cdot |F' - X'| + \|X'\|_\infty \cdot |F - X|\,.
\end{aligned}$$

We apply $\lceil\!\lceil \ \rceil\!\rceil^*$ to these inequalities and obtain

$$\lceil\!\lceil (rF) - (rX) \rceil\!\rceil^* \le ([|r|] + 1) \lceil\!\lceil F - X \rceil\!\rceil^* ;$$
$$\Big\lceil\!\Big\lceil (F \vee F') - (X \vee X') \Big\rceil\!\Big\rceil^* \le \lceil\!\lceil F - X \rceil\!\rceil^* + \Big\lceil\!\Big\lceil F' - X' \Big\rceil\!\Big\rceil^* ;$$
$$\Big\lceil\!\Big\lceil (F \wedge F') - (X \wedge X') \Big\rceil\!\Big\rceil^* \le \lceil\!\lceil F - X \rceil\!\rceil^* + \Big\lceil\!\Big\lceil F' - X' \Big\rceil\!\Big\rceil^* ;$$
$$\lceil\!\lceil |F| - |X| \rceil\!\rceil^* \le \lceil\!\lceil F - X \rceil\!\rceil^* ; \ \lceil\!\lceil F \wedge 1 - X \wedge 1 \rceil\!\rceil^* \le \lceil\!\lceil F - X \rceil\!\rceil^* ;$$
$$\Big\lceil\!\Big\lceil (F \cdot F') - (X \cdot X') \Big\rceil\!\Big\rceil^* \le \|F\|_\infty \cdot \Big\lceil\!\Big\lceil F' - X' \Big\rceil\!\Big\rceil^* + \|X'\|_\infty \cdot \lceil\!\lceil F - X \rceil\!\rceil^* .$$

Given an $\epsilon > 0$, we may choose elementary integrands X, X' so that the right-hand sides are less than ϵ. This is possible because the processes F, F' are integrable and shows that the processes $rF, F \vee F' \ldots$ are integrable as well, inasmuch as the processes $rX, X \vee X' \ldots$ appearing on the left are elementary.

The last case, that of the product, is marginally more complicated than the others. Given $\epsilon > 0$, we first choose X' elementary so that

$$\Big\lceil\!\Big\lceil F' - X' \Big\rceil\!\Big\rceil^* \le \frac{\epsilon}{2(1 + \|F\|_\infty)} ,$$

using the fact that the process F is bounded. Then we choose X elementary so that

$$\lceil\!\lceil F - X \rceil\!\rceil^* \le \frac{\epsilon}{2(1 + \|X'\|_\infty)} .$$

Then again $\lceil\!\lceil F \cdot F' - X \cdot X' \rceil\!\rceil^* \le \epsilon$, showing that $F \cdot F'$ is integrable, inasmuch as the product $X \cdot X'$ is an elementary integrand. ∎

Permanence Under Pointwise Limits of Sequences

The algebraic and order permanence properties of $\mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]$ are thus as good as one might hope for, to wit as good as in the case of the Lebesgue integral. Let us now turn to the permanence properties concerning limits. The first result is plain from theorem 3.2.10.

Theorem 3.2.22 *$\mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*]$ is complete in $\lceil\!\lceil\ \rceil\!\rceil^*$-mean. Every mean Cauchy sequence (F_n) has a subsequence that converges pointwise $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e. to a mean limit of (F_n).*

The existence of an a.e. convergent subsequence of a mean-convergent sequence (F_n) is frequently very helpful in identifying the limit, as we shall presently see. We know from ordinary integration theory that there is, in general, no hope that the sequence (F_n) itself converges almost everywhere.

Theorem 3.2.23 (The Monotone Convergence Theorem) *Let (F_n) be a monotone sequence of integrable processes with $\lim_{r \to 0} \sup_n \lceil\!\lceil rF_n \rceil\!\rceil^* = 0$. (For $p > 0$ and $\lceil\!\lceil\ \rceil\!\rceil^* = \lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$ this reads simply $\sup_n \lceil\!\lceil F_n \rceil\!\rceil^*_{Z\text{–}p} < \infty$.) Then (F_n) converges to its pointwise limit in mean.*

Proof. As $\big(F_n(\varpi)\big)$ is monotone it has a limit $F(\varpi)$ at all points ϖ of the base space, possibly $\pm\infty$. Let us assume first that the sequence $\big(F_n\big)$ is increasing. We start by showing that $\big(F_n\big)$ is mean-Cauchy. Indeed, assume it were not. There would then exist an $\epsilon > 0$ and a subsequence $\big(F_{n_k}\big)$ with $\lceil\!\lceil F_{n_{k+1}} - F_{n_k} \rceil\!\rceil^* > \epsilon$. There would further exist positive elementary integrands X_n with

$$\lceil\!\lceil (F_{n_{k+1}} - F_{n_k}) - X_k \rceil\!\rceil^* < 2^{-k}. \tag{$*$}$$

Let $|r| \le 1$ and $K < L \in \mathbb{N}$. Then

$$\begin{aligned}\Big\lceil\!\Big\lceil r\sum_{k=1}^{L} X_k \Big\rceil\!\Big\rceil^* &\le \Big\lceil\!\Big\lceil r\sum_{k=1}^{L}\big((F_{n_{k+1}} - F_{n_k}) - X_k\big)\Big\rceil\!\Big\rceil^* + \Big\lceil\!\Big\lceil r\sum_{k=1}^{L}(F_{n_{k+1}} - F_{n_k})\Big\rceil\!\Big\rceil^* \\ &\le \Big\lceil\!\Big\lceil r\sum_{k=1}^{K}\big((F_{n_{k+1}} - F_{n_k}) - X_k\big)\Big\rceil\!\Big\rceil^* + 2^{-K} + \Big\lceil\!\Big\lceil rF_{n_{L+1}}\Big\rceil\!\Big\rceil^* + \lceil\!\lceil rF_{n_1} \rceil\!\rceil^*.\end{aligned}$$

Given $\epsilon > 0$ we first fix $K \in \mathbb{N}$ so large that $2^{-K} < \epsilon/4$. Then we find r_ϵ so that the other three terms are smaller than $\epsilon/4$ each, for $|r| \le r_\epsilon$. By assumption, r_ϵ can be so chosen independently of L. That is to say,

$$\sup_L \Big\lceil\!\Big\lceil r\textstyle\sum_{k=1}^{L} X_k \Big\rceil\!\Big\rceil^* \xrightarrow[r\to 0]{} 0 .$$

Property (M) of the mean (see page 91) now implies that $\lceil\!\lceil X_k \rceil\!\rceil^* \to 0$. Thanks to $(*)$, $\lceil\!\lceil F_{n_{k+1}} - F_{n_k} \rceil\!\rceil^* \xrightarrow[k\to\infty]{} 0$, which is the desired contradiction. Now that we know that $\big(F_n\big)$ is Cauchy we employ theorem 3.2.10: there is a mean-limit F' and a subsequence[2] (F_{n_k}) so that $F_{n_k}(\varpi)$ converges to $F'(\varpi)$ as $k \to \infty$, for all ϖ outside some negligible set N. For all ϖ, though, $F_n(\varpi) \xrightarrow[n\to\infty]{} F(\varpi)$. Thus

$$F(\varpi) = \lim_{n\to\infty} F_n(\varpi) = \lim_{k\to\infty} F_{n_k}(\varpi) = F'(\varpi) \quad \text{for } \varpi \notin N \ :$$

F is equal almost surely to the mean-limit F' and thus is a mean-limit itself. If $\big(F_n\big)$ is decreasing rather than increasing, $\big(-F_n\big)$ increases pointwise – and by the above in mean – to $-F$: again $F_n \to F$ in mean. ∎

Theorem 3.2.24 (The Dominated Convergence Theorem or DCT) *Let* $\big(F_n\big)$ *be a sequence of integrable processes. Assume both*

(i) $\big(F_n\big)$ *converges pointwise* $\lceil\!\lceil\ \rceil\!\rceil^*$*-almost everywhere to a process* F*; and*

(ii) there exists a process $G \in \mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$ *with* $|F_n| \le G$ *for all indices* $n \in \mathbb{N}$*.*

Then $\big(F_n\big)$ *converges to* F *in* $\lceil\!\lceil\ \rceil\!\rceil^*$*-mean, and consequently* F *is integrable.*

The Dominated Convergence Theorem is central. Most other results in integration theory follow from it. It is false without some domination condition like (ii), as is well known from ordinary integration theory.

[2] Not the same as in the previous argument, which was, after all, shown not to exist.

Proof. As in the proof of the Monotone Convergence Theorem we begin by showing that the sequence $\big(F_n\big)$ is Cauchy. To this end consider the positive process

$$G_N = \sup\{|F_n - F_m| : m,n \geq N\} = \lim_{K\to\infty} \bigvee_{m,n=N}^{K} |F_n - F_m| \ \leq 2G \ .$$

Thanks to theorem 3.2.21 and the MCT, G_N is integrable. Moreover, $\big(G_N(\varpi)\big)$ converges decreasingly to zero at all points ϖ at which $\big(F_n(\varpi)\big)$ converges, that is to say, almost everywhere. Hence $[\![G_N]\!]^* \to 0$. Now $[\![F_n - F_m]\!]^* \leq [\![G_N]\!]^*$ for $m,n \geq N$, so $\big(F_n\big)$ is Cauchy in mean. Due to theorem 3.2.22 the sequence has a mean limit F' and a subsequence $\big(F_{n_k}\big)$ that converges pointwise a.e. to F'. Since $\big(F_{n_k}\big)$ also converges to F a.e., we have $F = F'$ a.e. Thus $[\![F_n - F]\!]^* = [\![F_n - F']\!]^* \xrightarrow[n\to\infty]{} 0$. Now apply proposition 3.2.20. ▮

Integrable Sets

Definition 3.2.25 ***A set is integrable*** *if its indicator function is integrable.*[1]

Proposition 3.2.26 *The union and relative complement of two integrable sets are integrable. The intersection of a countable family of integrable sets is integrable. The union of a countable family of integrable sets is integrable provided that it is contained in an integrable set C.*

Proof. For ease of reading we use the same symbol for a set and its indicator function.[1] For instance, $A_1 \cup A_2 = A_1 \vee A_2$ in the sense that the indicator function on the left is the pointwise maximum of the two indicator functions on the right.

Let $A_1, A_2, \ldots$ be a countable family of integrable sets. Then

$$A_1 \cup A_2 = A_1 \vee A_2 \ ,$$

$$A_1 \backslash A_2 = A_1 - (A_1 \wedge A_2) \ ,$$

$$\bigcap_{n=1}^{\infty} A_n = \bigwedge_{n=1}^{\infty} A_n = \lim_{N\to\infty} \bigwedge_{n=1}^{N} A_n \ ,$$

and

$$\bigcup_{n=1}^{\infty} A_n = C - \bigwedge_{n=1}^{\infty} (C - A_n) \ ,$$

in the sense that the set on the left has the indicator function on the right, which is integrable by theorem 3.2.24. ▮

A collection of subsets of a set that is closed under taking finite unions, relative differences, and countable intersections is called a ***δ-ring***. Proposition 3.2.26 can thus be read as saying that the integrable sets form a δ-ring.

Proposition 3.2.27 *Let F be an integrable process. (i) The sets $[F > r]$, $[F \ge r]$, $[F < -r]$, and $[F \le -r]$ are integrable, whenever $r \in \mathbb{R}$ is strictly positive. (ii) F is the limit a.e. and in mean of a sequence (F_n) of integrable step processes with $|F_n| \le |F|$.*

Proof. For the first claim, note that the set[1]

$$[F > 1] = \lim_{n\to\infty} 1 \wedge \big(n(F - F \wedge 1)\big)$$

is integrable. Namely, the processes $F_n = 1 \wedge \big(n(F - F \wedge 1)\big)$ are integrable and are dominated by $|F|$; by the Dominated Convergence Theorem, their limit is integrable. This limit is 0 at any point ϖ of the base space where $F(\varpi) \le 1$ and 1 at any point ϖ where $F(\varpi) > 1$; in other words, it is the (indicator function of the) set $[F > 1]$, which is therefore integrable. Note that here we use for the first (and only) time the fact that $\mathcal{E}$ is closed under chopping. The set $[F > r]$ equals $[F/r > 1]$ and is therefore integrable as well. Next, $[F \ge r] = \bigcap_{n>1/r}[F > r - 1/n]$, $[F < -r] = [-F > r]$, and $[F \le -r] = [-F \ge r]$.

For the next claim, let F_n be the step process over integrable sets[1]

$$F_n = \sum_{k=1}^{2^{2n}} k2^{-n} \cdot \big[k2^{-n} < F \le (k+1)2^{-n}\big]$$
$$+ \sum_{k=1}^{2^{2n}} -k2^{-n} \cdot \big[-k2^{-n} > F \ge -(k+1)2^{-n}\big] .$$

By (i), the sets

$$\big[k2^{-n} < F \le (k+1)2^{-n}\big] = \big[k2^{-n} < F\big] \setminus \big[(k+1)2^{-n} < F\big]$$

are integrable if $k \ne 0$. Thus F_n, being a linear combination of integrable processes, is integrable. Now (F_n) converges pointwise to F and is dominated by $|F|$, and the claim follows. ∎

Notation 3.2.28 *The integral of an integrable set A is written $\int A \; dZ$ or $\int_A dZ$. Let $F \in \mathfrak{L}^1[Z\text{–}p]$. With an integrable set A being a bounded (idempotent) process, the product $A \cdot F = 1_A \cdot F$ is integrable; its integral is variously written*

$$\int_A F \, dZ \quad or \quad \int A \cdot F \, dZ \, .$$

Exercise 3.2.29 Let (F_n) be a sequence of bounded integrable processes, all vanishing off the same integrable set A and converging uniformly to F. Then F is integrable.

Exercise 3.2.30 In the stochastic case there exists a countable collection of $[\![\ ]\!]^*$-integrable sets that covers the whole space, for example, $\{[\![0, k]\!] : k \in \mathbb{N}\}$. We say that the mean is ***σ-finite***. In consequence, any collection $\mathcal{M}$ of mutually disjoint non-negligible $[\![\ ]\!]^*$-integrable sets is at most countable.

3.3 Countable Additivity in p-Mean

The development so far rested on the assumption (IC-p) on page 89 that our L^p-integrator be continuous in L^p-mean along increasing sequences of $\mathcal{E}$, or σ-additive in p-mean. This assumption is, on the face of it, rather stronger than mere right-continuity in probability, and was needed to establish properties (iv) and (v) of Daniell's mean in theorem 3.1.6 on page 90. We show in this section that continuity in L^p-mean along increasing sequences is actually equivalent with right-continuity in probability, in the presence of the boundedness condition (B-p). First the case $p = 0$:

Lemma 3.3.1 *An L^0-integrator is σ-additive in probability.*

Proof. It is to be shown that for any decreasing sequence $\big(X^{(k)}\big)_{k=1}^{\infty}$ of elementary integrands with pointwise infimum zero, $\lim_k \int X^{(k)}\, dZ = 0$ in probability, under the assumptions (B-0) that Z is a bounded linear map from $\mathcal{E}$ to L^0 and (RC-0) that it is right-continuous in measure (exercise 3.1.5). As so often before, the argument is very nearly the same as in standard integration theory. Let us fix representations[1, 3]

$$X_s^{(k)}(\omega) = f_0^{(k)}(\omega) \cdot [0,0]_s + \sum_{n=1}^{N(k)} f_n^{(k)}(\omega) \cdot \big(t_n^{(k)}, t_{n+1}^{(k)}\big]_s$$

as in equation (2.1.1) on page 46. Clearly

$$\int f_0^{(k)} \cdot [\![0]\!]\, dZ = f_0^{(k)} \cdot Z_0 \xrightarrow[k\to\infty]{} 0 \ :$$

we may as well assume that $f_0^{(k)} = 0$. Scaling reduces the situation to the case that $\big|X^{(k)}\big| \le 1$ for all $k \in \mathbb{N}$. It eases the argument further to assume that the partitions $\{t_1^{(k)}, \ldots, t_{N(k)}^{(k)}\}$ become finer as k increases.

Let then $\epsilon > 0$ be given. Let U be an instant past which the $X^{(k)}$ all vanish. The continuity condition (B-0) provides a $\delta > 0$ such that[1]

$$\lceil\!\lceil \delta \cdot [\![0, U]\!] \rceil\!\rceil_{Z\text{–}0} < \epsilon/3 \ . \tag{$*$}$$

Next let us define instants $u_n^{(k)} < v_n^{(k)}$ as follows: for $k = 1$ we set $u_n^{(1)} = t_n^{(1)}$ and choose $v_n^{(1)} \in (t_n^{(1)}, t_{n+1}^{(1)})$ so that

$$\lceil\!\lceil Z_u - Z_t \rceil\!\rceil_0 < 3^{-1-n-1}\epsilon \ \text{ for } \ u_n^{(1)} \le t < u \le v_n^{(1)} \ ; \qquad 1 \le n \le N(1) \ .$$

The right-continuity of Z makes this possible. The intervals $[u_n^{(1)}, v_n^{(1)}]$ are clearly mutually disjoint. We continue by induction. Suppose that $u_n^{(j)}$ and $v_n^{(j)}$ have been found for $1 \le j < k$ and $1 \le n \le N(j)$, and let $t_n^{(k)}$ be one

[3] $[0,0]_s$ is the indicator function of $\{0\}$ evaluated at s, etc.

of the partition points for $X^{(k)}$. If $t_n^{(k)}$ lies in one of the intervals previously constructed or is a left endpoint of one of them, say $t_n^{(k)} \in [u_m^{(j)}, v_m^{(j)})$, then we set $u_n^{(k)} = u_m^{(j)}$ and $v_n^{(k)} = v_m^{(j)}$; in the opposite case we set $u_n^{(k)} = t_n^{(k)}$ and choose $v_n^{(k)} \in (t_n^{(k)}, t_{n+1}^{(k)})$ so that

$$\lceil\!\lceil Z_u - Z_t \rceil\!\rceil_0 < 3^{-k-n-1}\epsilon \quad \text{for} \quad u_n^{(k)} \le t < u \le v_n^{(k)} ; \qquad 1 \le n \le N(k) .$$

The right-continuity in probability of Z makes this possible. This being done we set

$$\mathring{N}^{(k)} \stackrel{\text{def}}{=} \bigcup_{n=1}^{N(k)} (u_n^{(k)}, v_n^{(k)}) \subset N^{(k)} \stackrel{\text{def}}{=} \bigcup_{n=1}^{N(k)} (u_n^{(k)}, v_n^{(k)}] \qquad k = 1, 2, \ldots .$$

Both $\mathring{N}^{(k)}$ and $N^{(k)}$ are finite unions of mutually disjoint intervals and increase with k. Furthermore

$$\sum_{n=1}^{N(k)} \left\lceil\!\!\left\lceil Z_{v_n'^{(k)}} - Z_{u_n'^{(k)}} \right\rceil\!\!\right\rceil_0 < \epsilon/3 , \quad \text{for any} \quad u_n^{(k)} \le u_n'^{(k)} \le v_n'^{(k)} \le v_n^{(k)} . \qquad (**)$$

We shall estimate separately the integrals of the elementary integrands in the sum

$$X^{(k)} = X^{(k)} \cdot \big(N^{(k)} \times \Omega\big) \;+\; X^{(k)} \cdot \Big(1 - \big(N^{(k)} \times \Omega\big)\Big) .$$

$$\begin{aligned}\int X^{(k)} \cdot N^{(k)} \, dZ &= \sum_{n=1}^{N(k)} f_n^{(k)} \cdot \int \bigcup_{m=1}^{N(k)} (\!(t_n^{(k)}, t_{n+1}^{(k)}]\!] \cap (\!(u_m^{(k)}, v_m^{(k)}]\!] \, dZ \\ &= \sum_{n=1}^{N(k)} f_n^{(k)} \cdot \int (\!(t_n^{(k)} \vee u_n^{(k)}, t_{n+1}^{(k)} \wedge v_n^{(k)}]\!] \, dZ \\ &= \sum_{n=1}^{N(k)} f_n^{(k)} \cdot \Big(Z_{t_{n+1}^{(k)} \wedge v_n^{(k)}} - Z_{t_n^{(k)} \vee u_n^{(k)}} \Big) .\end{aligned}$$

Since $\big|f_n^{(k)}\big| \le 1$, inequality $(**)$ yields

$$\left\lceil\!\!\left\lceil \int X^{(k)} \cdot N^{(k)} \, dZ \right\rceil\!\!\right\rceil_0 \le \epsilon/3 . \qquad (***)$$

Let us next estimate the remaining summand $X'^{(k)} = X^{(k)} \cdot \big((1 - N^{(k)}) \times \Omega\big)$. We start on this by estimating the process

$$X^{(k)} \cdot \big((1 - \mathring{N}^{(k)}) \times \Omega\big) ,$$

which evidently majorizes $X'^{(k)}$. Since every partition point of $X^{(k)}$ lies either inside one of the intervals $(u_n^{(k)}, v_n^{(k)})$ that make up $N^{(k)}$ or is a left endpoint of one of them, the paths of $X^{(k)} \cdot \big((1 - \mathring{N}^{(k)}) \times \Omega\big)$ are upper semicontinuous (see page 376). That is to say, for every $\omega \in \Omega$ and $\alpha > 0$, the set

$$C_\alpha(\omega) = \Big\{ s \in \mathbb{R}_+ : \ X_s^{(k)}(\omega) \cdot \big(1 - \mathring{N}^{(k)}\big) \ge \alpha \Big\}$$

is a finite union of closed intervals and is thus compact. These sets shrink as k increases and have void intersection. For every $\omega \in \Omega$ there is therefore an index $K(\omega)$ such that $C_\alpha(\omega) = \emptyset$ for all $k \ge K(\omega)$. We conclude that the maximal function

$$\big(X'^{(k)}\big)^\star_U = \sup_{0 \le s \le U} X'^{(k)}_s \le \sup_{0 \le s \le U} X^{(k)} \cdot \big((1 - \mathring{N}^{(k)}) \times \Omega\big)$$

decreases pointwise to zero, *a fortiori* in measure. Let then K be so large that for $k \ge K$ the set

$$B \stackrel{\text{def}}{=} \Big[\big(X'^{(k)}\big)^\star_U > \delta \Big] \ \text{ has } \ \mathbb{P}[B] < \epsilon/3 \ .$$

The dZ-integrals of $X'^{(k)}$ and $X'^{(k)} \wedge \delta$ agree pathwise outside B. Measured with $\lceil\!\lceil \ \rceil\!\rceil_0$ they differ thus by at most $\epsilon/3$. Since $X'^{(k)} \wedge \delta \le \delta \cdot [\![0, U]\!]$, inequality $(*)$ yields

$$\Big\lceil\!\Big\lceil \int X'^{(k)} \wedge \delta \, dZ \Big\rceil\!\Big\rceil_0 \le \epsilon/3 \ , \text{ and thus } \ \Big\lceil\!\Big\lceil \int X'^{(k)} \, dZ \Big\rceil\!\Big\rceil_0 \le 2\epsilon/3 \ , \quad k \ge K \ .$$

In view of $(***)$ we get $\lceil\!\lceil \int X^{(k)} \, dZ \rceil\!\rceil_0 \le \epsilon$ for $k \ge K$. ∎

Proposition 3.3.2 *An L^p-integrator is σ-additive in p-mean, $0 \le p < \infty$.*

Proof. For $p = 0$ this was done in lemma 3.3.1 above, so we need to consider only the case $p > 0$. Part (ii) of the Stone–Weierstraß theorem A.2.2 provides a locally compact Hausdorff space $\widehat{\boldsymbol{B}}$ and a map $j : \boldsymbol{B} \to \widehat{\boldsymbol{B}}$ with dense image such that every $X \in \mathcal{E}$ is of the form $\widehat{X} \circ j$ for some unique continuous function $\widehat{X}$ on $\widehat{\boldsymbol{B}}$. $\widehat{X}$ is called the Gelfand transform of X. The map $X \mapsto \widehat{X}$ is an algebraic and order isomorphism of $\mathcal{E}$ onto an algebra and vector lattice $\widehat{\mathcal{E}}$ closed under chopping of continuous bounded functions of compact support on $\widehat{\boldsymbol{B}}$ ($\widehat{X}$ has support in $\widehat{[X \ne 0]} \in \widehat{\mathcal{E}}$). The Gelfand transform $\widehat{\int}$, defined by

$$\widehat{\int} \widehat{X} \stackrel{\text{def}}{=} \int X \, dZ \ , \qquad X \in \mathcal{E} \ .$$

is plainly a vector measure on $\widehat{\mathcal{E}}$ with values in L^p that satisfies (B-p). (IC-p) is also satisfied, thanks to Dini's theorem A.2.1. For if the sequence $\big(\widehat{X}^{(n)}\big)$ in $\widehat{\mathcal{E}}$ increases pointwise to the continuous (!) function $\widehat{X} \in \widehat{\mathcal{E}}$, then the convergence is uniform and (B-p) implies that $\widehat{\int} \widehat{X}^{(n)} \to \widehat{\int} \widehat{X}$ in p-mean. Daniell's procedure of the preceding pages provides an integral extension of $\widehat{\int}$ for which the Dominated Convergence Theorem holds.

Let us now consider an increasing sequence $\big(X^{(n)}\big)$ in $\mathcal{E}_+$ that increases pointwise on $\boldsymbol{B}$ to $X \in \mathcal{E}$. The extensions $\widehat{X}^{(n)}$ will increase on $\widehat{\boldsymbol{B}}$ to some function $\widehat{H}$. While $\widehat{H}$ does not necessarily equal the extension $\widehat{X}$ (!), it is clearly less than or equal to it. By the Dominated Convergence Theorem for the integral extension of $\widehat{\int}$, $\int X^{(n)}\, dZ = \widehat{\int} \widehat{X}^{(n)}$ converges *in p-mean* to an element f of L^p. Now Z is certainly an L^0-integrator and thus $\int X^{(n)}\, dZ \to \int X\, dZ$ in measure (lemma 3.3.1). Thus $f = \int X\, dZ$, and $\int X^{(n)}\, dZ \to \int X\, dZ$ in p-mean. This very argument is repeated in slightly more generality in corollary A.2.7 on page 370. ▮

Exercise 3.3.3 Assume that for every $t \ge 0$, $\mathcal{A}_t$ is an algebra or vector lattice closed under chopping of $\mathcal{F}_t$-adapted bounded random variables that contains the constants and generates $\mathcal{F}_t$. Let $\mathcal{E}^0$ denote the collection of all elementary integrands X that have $X_t \in \mathcal{A}_t$ for all $t \ge 0$. Assume further that the right-continuous adapted process Z satisfies

$$\lceil Z^t \rceil^0_{\mathcal{I}^p} \stackrel{\text{def}}{=} \sup\left\{ \left\| \int X\, dZ^t \right\|_p : X \in \mathcal{E}^0,\ |X| \le 1 \right\} < \infty$$

for some $p > 0$ and all $t \ge 0$. Then Z is an L^p-integrator, and $\lceil Z^t \rceil^0_{\mathcal{I}^p} = \lceil Z^t \rceil_{\mathcal{I}^p}$ for all t.

Exercise 3.3.4 Let $0 < p < \infty$. An L^0-integrator Z is a local L^p-integrator iff $\big\{ \int [\![0,T]\!] \cdot X\, dZ : X \in \mathcal{E}\,,\ |X| \le 1 \big\}$ is bounded in L^p for arbitrarily large stopping times T.

The Integration Theory of Vectors of Integrators

We have mentioned before that often whole vectors $\boldsymbol{Z} = (Z^1, Z^2, \ldots, Z^d)$ of integrators drive a stochastic differential equation. It is time to consider their integration theory. An obvious way is to regard every component Z^η as an L^p-integrator, to declare $\boldsymbol{X} = (X_1, X_2, \ldots, X_d)$ $\boldsymbol{Z}$-integrable if X_η is Z^η–p-integrable for every $\eta \in \{1, \ldots, d\}$, and to define

$$\int \boldsymbol{X}\, d\boldsymbol{Z} \stackrel{\text{def}}{=} \int X_\eta\, dZ^\eta = \sum_{1 \le \eta \le d} \int X_\eta\, dZ^\eta\,, \tag{3.3.1}$$

simply extending the definition (2.2.2). Let us take another point of view, one that leads to better constants in estimates and provides a guide to the integration theory of random measures (section 3.10). Denote by $\boldsymbol{H}$ the discrete space $\{1, \ldots, d\}$ and by $\check{\boldsymbol{B}}$ the set $\boldsymbol{H} \times \boldsymbol{B}$ equipped with its elementary integrands $\check{\mathcal{E}} \stackrel{\text{def}}{=} C_{00}(\boldsymbol{H}) \otimes \mathcal{E}$. Now read a d-tuple $\boldsymbol{Z} = (Z^1, Z^2, \ldots, Z^d)$ of processes on $\boldsymbol{B}$ not as a vector-valued function on $\boldsymbol{B}$ but rather as a ***scalar*** function $(\eta, \varpi) \mapsto Z^\eta(\varpi)$ on the d-fold product $\check{\boldsymbol{B}}$. In this interpretation $\boldsymbol{X} \mapsto \int \boldsymbol{X}\, d\boldsymbol{Z}$ is a vector measure $\check{\mathcal{E}} \to L^p(\mathbb{P})$, and the extension theory of the previous sections applies. In particular, THE Daniell mean is defined as

$$[\![\boldsymbol{F}]\!]^*_{\boldsymbol{Z}-p} \stackrel{\text{def}}{=} \inf_{\substack{\boldsymbol{H} \in \check{\mathcal{E}}^\uparrow,\\ \boldsymbol{H} \ge |\boldsymbol{F}|}} \ \sup_{\substack{\boldsymbol{X} \in \check{\mathcal{E}},\\ |\boldsymbol{X}| \le \boldsymbol{H}}} \left[\!\!\left[\int \boldsymbol{X}\, d\boldsymbol{Z} \right]\!\!\right]_p \tag{3.3.2}$$

on functions $\boldsymbol{F} : \check{\boldsymbol{B}} \to \overline{\mathbb{R}}$. It is a fine exercise toward checking one's understanding of Daniell's procedure to show that $\int \cdot \, d\boldsymbol{Z}$ satisfies (IC-p), that therefore $\| \ \|^*_{\boldsymbol{Z}\text{–}p}$ is a mean satisfying

$$\left\lceil\!\left\lceil \int \boldsymbol{X}\, d\boldsymbol{Z} \right\rceil\!\right\rceil_p \le \| \boldsymbol{X} \|^*_{\boldsymbol{Z}\text{–}p} , \tag{3.3.3}$$

and that not only the integration theory developed so far but its continuation in the subsequent sections applies *mutatis perpauculis mutandis*. In particular, inequality (3.3.3) will imply that there is a unique extension

$$\int \cdot \, d\boldsymbol{Z} : \mathfrak{L}^1[\lceil\!\lceil \ \rceil\!\rceil^*_{\boldsymbol{Z}\text{–}p}] \to L^p$$

satisfying the same inequality. That extension is actually given by equation (3.3.1). For more along these lines see section 3.10.

3.4 Measurability

Measurability describes the local structure of the integrable processes. Lusin observed that Lebesgue integrable functions on the line are uniformly continuous on arbitrarily large sets. It is rather intuitive to use this behavior to *define* measurability. It turns out to be efficient as well.

As before, $\lceil\!\lceil \ \rceil\!\rceil^*$ is an arbitrary mean on the algebra and vector lattice closed under chopping $\mathcal{E}$ of bounded functions that live on the ambient set $\boldsymbol{B}$. In order to be able to speak about the uniform continuity of a function on a set $A \subset \boldsymbol{B}$, the ambient space $\boldsymbol{B}$ is equipped with the ***$\mathcal{E}$-uniformity***, the smallest uniformity with respect to which the functions of $\mathcal{E}$ are all uniformly continuous. The reader not yet conversant with uniformities may wish to read page 373 up to lemma A.2.16 on page 375 and to note the following: to say that a real-valued function on $A \subset \boldsymbol{B}$ is $\mathcal{E}$-uniformly continuous is the same as saying that it agrees with the restriction to A of a function in the uniform closure of $\mathcal{E} \oplus \mathbb{R}$ or that it is, on A, the uniform limit of functions in $\mathcal{E} \oplus \mathbb{R}$. To say that a numerical function on $A \subset \boldsymbol{B}$ is $\mathcal{E}$-uniformly continuous is the same as saying that it is, on A, the uniform limit of functions in $\mathcal{E} \oplus \mathbb{R}$, *with respect to the arctan metric.*

By way of motivation of definition 3.4.2 we make the following observation, whose proof is left to the reader:

Observation 3.4.1 *Let $F : \boldsymbol{B} \to \mathbb{R}$ be $(\mathcal{E}, \lceil\!\lceil \ \rceil\!\rceil^*)$-integrable and $\epsilon > 0$. There exists a set $U \in \mathcal{E}_+^\uparrow$ with $\lceil\!\lceil U \rceil\!\rceil^* \le \epsilon$ on whose complement F is the uniform limit of elementary integrands and thus is $\mathcal{E}$-uniformly continuous.*

Definition 3.4.2 *Let A be a $\lceil\!\lceil \ \rceil\!\rceil^*$-integrable set. A process*[4] *F almost everywhere defined on A is called* ***$\lceil\!\lceil \ \rceil\!\rceil^*$-measurable on*** *$A$ if for every $\epsilon > 0$*

[4] "Process" shall mean any $\lceil\!\lceil \ \rceil\!\rceil^*$-a.e. defined function on the ambient space that has values in some uniform space.

there is a $\llbracket\ \rrbracket^*$*-integrable subset* A_0 *of* A *with*[1] $\llbracket A\setminus A_0 \rrbracket^* < \epsilon$ *on which* F *is* $\mathcal{E}$*-uniformly continuous. A process* F *is called* $\llbracket\ \rrbracket^*$***-measurable*** *if it is measurable on every integrable set.*

Unless there is need to stress that this definition refers to the mean $\llbracket\ \rrbracket^*$*, we shall simply talk about measurability. If we want to make the point that* $\llbracket\ \rrbracket^*$ *is Daniell's mean* $\llbracket\ \rrbracket^*_{Z\text{–}p}$*, we shall talk about* ***Z–p-measurability*** *(this is actually independent of* p *– see corollary 3.6.11 on page 128).*

This definition is quite intuitive, describing as it does a considerable degree of smoothness. It says that F is measurable if it is on arbitrarily large sets as smooth as an elementary integrand, in other words, that it is "***largely as smooth as an elementary integrand***." It is also quite workable in that it admits fast proofs of the permanence properties. We start with a tiny result that will however facilitate the arguments greatly.

Lemma 3.4.3 *Let* A *be an integrable set and* (F_n) *a sequence of processes that are measurable on* A*. For every* $\epsilon > 0$ *there exists an integrable subset* A_0 *of* A *with* $\llbracket A\setminus A_0 \rrbracket^* \le \epsilon$ *such that every one of the* F_n *is uniformly continuous on* A_0*.*

Proof. Let $A_1 \subset A$ be integrable with $\llbracket A\setminus A_1 \rrbracket^* < \epsilon \cdot 2^{-1}$ and so that, on A_1, F_1 is uniformly continuous. Next let $A_2 \subset A_1$ be integrable with $\llbracket A_1\setminus A_2 \rrbracket^* < \epsilon{\cdot}2^{-2}$ and so that, on A_2, F_2 is uniformly continuous. Continue by induction, and set $A_0 = \bigcap_{n=1}^\infty A_n$. Then A_0 is integrable due to proposition 3.2.26,

$$\llbracket A\setminus A_0 \rrbracket^* = \Big\llbracket (A\setminus A_1) \cup \bigcup_{n>1} (A_n \setminus A_{n-1}) \Big\rrbracket^* \le \sum \epsilon \cdot 2^{-n} = \epsilon\,,$$

by the countable subadditivity of $\llbracket\ \rrbracket^*$, and every F_n is uniformly continuous on A_0, inasmuch as it is so on the larger set A_n. ∎

Permanence Under Limits of Sequences

Theorem 3.4.4 (Egoroff's Theorem) *Let* (F_n) *be a sequence of* $\llbracket\ \rrbracket^*$*-measurable processes with values in a metric space* (S, ρ)*, and assume that* (F_n) *converges* $\llbracket\ \rrbracket^*$*-almost everywhere to a process* F*. Then* F *is* $\llbracket\ \rrbracket^*$*-measurable.*

Moreover, for every integrable set A *and* $\epsilon > 0$ *there is an integrable subset* A_0 *of* A *with* $\llbracket A\setminus A_0 \rrbracket^* < \epsilon$ *on which* (F_n) *converges uniformly to* F *– we shall describe this behavior by saying "*(F_n) ***converges uniformly on arbitrarily large sets,****" or even simply by "*(F_n) ***converges largely uniformly.****"*

Proof. Let an integrable set A and an $\epsilon > 0$ be given. There is an integrable set $A_1 \subset A$ with $\llbracket A\setminus A_1 \rrbracket^* < \epsilon/2$ on which every one of the F_n is uniformly continuous. Then $\rho(F_m, F_n)$ is uniformly continuous on A_1, and therefore is, on A_1, the uniform limit of a sequence in $\mathcal{E}$, and thus $A_1 \cdot \rho(F_m, F_n)$ is integrable for every $m, n \in \mathbb{N}$ (exercise 3.2.29). Therefore

$$A_1 \cap \left[\rho(F_m, F_n) > \frac{1}{r}\right]$$

is an integrable set for $r = 1, 2, \ldots$, and then so is the set (see proposition 3.2.26)

$$B_p^r \stackrel{\text{def}}{=} A_1 \cap \bigcup_{m,n \geq p} \left[\rho(F_m, F_n) > \frac{1}{r}\right].$$

As p increases, B_p^r decreases, and the intersection $\bigcap_p B_p^r$ is contained in the negligible set of points where (F_n) does not converge. Thus $\lim_{p\to\infty} [\![B_p^r]\!]^* = 0$. There is a natural number $p(r)$ such that $[\![B_{p(r)}^r]\!]^* < 2^{-r-1}\epsilon$. Set

$$B \stackrel{\text{def}}{=} \bigcup_r B_{p(r)}^r \quad \text{and} \quad A_0 \stackrel{\text{def}}{=} A_1 \setminus B .$$

It is evident that $[\![A_1 \setminus A_0]\!]^* = [\![B]\!]^* < \epsilon/2$ and thus $[\![A \setminus A_0]\!]^* < \epsilon$. It is left to be shown that (F_n) converges uniformly on A_0. The limit F is then clearly also uniformly continuous there. To this end, let $\delta > 0$ be given. We let $N = p(r)$, where r is chosen so that $1/r < \delta$. Now if ϖ is any point in A_0 and $m, n \geq N$, then ϖ is not in the "bad set" $B_{p(r)}^r$; therefore $\rho\big(F_n(\varpi), F_m(\varpi)\big) \leq 1/r < \delta$, and thus $\rho\big(F(\varpi), F_n(\varpi)\big) \leq \delta$ for all $\varpi \in A_0$ and $n \geq N$. ∎

Corollary 3.4.5 *A numerical process*[4] *F is $[\![\ ]\!]^*$-measurable if and only if it is $[\![\ ]\!]^*$-almost everywhere the limit of a sequence of elementary integrands.*

Proof. The condition is sufficient by Egoroff's theorem. Toward its necessity we must assume that the mean is σ-finite, in the sense that there exists a countable collection of $[\![\ ]\!]^*$-integrable subsets B_n that exhaust the ambient set. The B_n can and will be chosen increasing with n. In the case of the stochastic integral take $B_n = [\![0, n]\!]$. Then find, for every integer n, a $[\![\ ]\!]^*$-integrable subset G_n of B_n with $[\![B_n \setminus G_n]\!]^* < 2^{-n}$ and an elementary integrand X_n that differs from F uniformly by less than 2^{-n} on G_n. The sequence (X_n) converges to F in every point of $G = \bigcup_N \bigcap_{n \geq N} G_n$, a set of $[\![\ ]\!]^*$-negligible complement. ∎

Permanence Under Algebraic and Order Operations

Theorem 3.4.6 *(i) Suppose that $F_1, \ldots, F_N$ are $[\![\ ]\!]^*$-measurable processes*[4] *with values in complete uniform spaces $(S_1, \mathfrak{u}_1), \cdots, (S_N, \mathfrak{u}_N)$, and ϕ is a continuous map from the product $S_1 \times \ldots \times S_N$ to another uniform space $(S, \mathfrak{u})$. Then the composition $\phi(F_1, \ldots, F_N)$ is $[\![\ ]\!]^*$-measurable. (ii) Algebraic and order combinations of measurable processes are measurable.*

Exercise 3.4.7 The conclusion (i) stays if ϕ is a Baire function.

Proof. (i) Let an integrable set A and an $\epsilon > 0$ be given. There is an integrable subset A_0 of A with $⟦A - A_0⟧^* < \epsilon$ on which every one of the F_n is uniformly continuous. By lemma A.2.16 (iv) the sets $F_n(A_0) \subset S_n$ are relatively compact, and by exercise A.2.15 ϕ is uniformly continuous on the compact product Π of their closures. Thus $\phi(F_1, \ldots, F_N) : A_0 \to \Pi \to S$ is uniformly continuous as the composition of uniformly continuous maps.

(ii) Let F_1, F_2 be measurable. Inasmuch as $+ : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $F_1 + F_2$ is measurable. The same argument applies with $+$ replaced by $\cdot, \wedge, \vee$, etc. ∎

Exercise 3.4.8 (Localization Principle) The notion of measurability is local: (i) A process F $⟦\ ⟧^*$-measurable on the $⟦\ ⟧^*$-integrable set A is $⟦\ ⟧^*$-measurable on every integrable subset of A. (ii) A process F $⟦\ ⟧^*$-measurable on the $⟦\ ⟧^*$-integrable sets A_1, A_2 is measurable on their union. (iii) If the process F is $⟦\ ⟧^*$-measurable on the $⟦\ ⟧^*$-integrable sets $A_1, A_2, \ldots$, then it is measurable on every $⟦\ ⟧^*$-integrable subset of their union $\bigcup_n A_n$.

Exercise 3.4.9 (i) Let $\mathcal{D}$ be any collection of bounded functions whose linear span is $⟦\ ⟧^*$-mean–dense in $\mathfrak{L}^1[⟦\ ⟧^*]$. Replacing the $\mathcal{E}$-uniformity on $\boldsymbol{B}$ by the $\mathcal{D}$-uniformity does not change the notion of measurability. In the case of the stochastic integral, therefore, a real-valued process is measurable if and only if it equals on arbitrarily large sets a continuous adapted process (take $\mathcal{D} = \mathfrak{C}$).

(ii) The notion of measurability of $F : \boldsymbol{B} \to S$ also does not change if the uniformity on S is replaced with another one that has the same topology, provided both uniformities are complete (apply theorem 3.4.6 to the identity map $S \to S$).

In particular, a process that is measurable as a numerical function and happens to take only real values is measurable as a real-valued function.

The Integrability Criterion

Let us now show that the notion of measurability captures exactly the "local smoothness" of the integrable processes:

Theorem 3.4.10 *A numerical process F is $⟦\ ⟧^*$-integrable if and only if it is $⟦\ ⟧^*$-measurable and finite in $⟦\ ⟧^*$-mean.*

Proof. An integrable process is finite for the mean (exercise 3.2.15) and, being the pointwise a.e. limit of a sequence of elementary integrands (theorem 3.2.22), is measurable (theorem 3.4.4). The two conditions are therefore necessary.

To establish the sufficiency let $\mathcal{C}$ be a maximal collection of mutually disjoint non-negligible integrable sets on which F is uniformly continuous. Due to the stipulated σ-finiteness of our mean there exists a countable collection $\{B_k\}$ of integrable sets that cover the base space, and $\mathcal{C}$ is countable: $\mathcal{C} = \{A_1, A_2, \ldots\}$ (see exercise 3.2.30). Now the complement of $C \stackrel{\text{def}}{=} \bigcup_n A_n$ is negligible; if it were not, then one of the integrable sets $B_k \setminus C$ would not be negligible and would contain a non-negligible integrable subset on which F is uniformly continuous – this would contradict the maximality of $\mathcal{C}$. The processes $F_n = F\cdot(\bigcup_{k\le n} A_k)$ are integrable, converge a.e. to F, and are dominated by $|F| \in \mathfrak{F}[\lceil\!\lceil\ \rceil\!\rceil^*]$. Thanks to the Dominated Convergence Theorem, F is integrable. ∎

Measurable Sets

Definition 3.4.11 ***A set is measurable*** *if its indicator function is measurable*[1] *– we write "measurable" instead of "$\lceil\!\lceil\ \rceil\!\rceil^*$-measurable," etc.*

Since sets are but idempotent functions, it is easy to see how their measurability interacts with that of arbitrary functions:

Theorem 3.4.12 *(i) A set M is measurable if and only if its intersection with every integrable set A is integrable. The measurable sets form a σ-algebra.*

(ii) If F is a measurable process, then the sets $[F > r]$, $[F \ge r]$, $[F < r]$, and $[F \le r]$ are measurable for any number r, and F is almost everywhere the pointwise limit of a sequence (F_n) of step processes with measurable steps.

(iii) A numerical process F is measurable if and only if the sets $[F > d]$ are measurable for every dyadic rational d.

Proof. These are standard arguments. (i) if $M \cap A$ is integrable, then it is measurable on A. The condition is thus sufficient. Conversely, if M is measurable and A integrable, then $M \cap A$ is measurable and has finite mean; so it is integrable (3.4.10). For the second claim let $A_1, A_2, \ldots$ be a countable family of measurable sets.[1]

Then
$$\big(A_1\big)^c = 1 - A_1 \ ,$$
$$\bigcap_{n=1}^{\infty} A_n = \bigwedge_{n=1}^{\infty} A_n = \lim_{N\to\infty} \bigwedge_{n=1}^{N} A_n,$$
and
$$\bigcup_{n=1}^{\infty} A_n = \bigvee_{n=1}^{\infty} A_n = \lim_{N\to\infty} \bigvee_{n=1}^{N} A_n,$$
in the sense that the set on the left has the indicator function on the right, which is measurable.

(ii) For the first claim, note that the process
$$\lim_{n\to\infty} 1 \wedge \big(n(F - F \wedge 1)\big)$$
is measurable, in view of the permanence properties. It vanishes at any point ϖ where $F(\varpi) \le 1$ and equals 1 at any point ϖ where $F(\varpi) > 1$; in other words, this limit is the (indicator function of the) set $[F > 1]$, which is therefore measurable. The set $[F > r]$ equals $[F/r > 1]$ when $r > 0$ and is thus measurable as well. $[F > 0] = \bigcup_{n=1}^{\infty}[F > 1/n]$ is measurable. Next, $[F \ge r] = \bigcap_{n>1/r}[F > r - 1/n]$, $[F < -r] = [-F > r]$, and $[F \le -r] = [-F \ge r]$. Finally, when $r \le 0$, then $[F > r]{=}[-F \ge -r]^c$, etc.

For the next claim, let F_n be the step process over measurable sets[1]

$$F_n = \sum_{k=-2^{2n}}^{2^{2n}} k2^{-n} \cdot \big[k2^{-n} < F \le (k+1)2^{-n}\big] \,. \tag{$*$}$$

The sets $[k2^{-n} < F \le (k+1)2^{-n}] = [k2^{-n} < F] \cap [(k+1)2^{-n} < F]^c$ are measurable, and the claim follows by inspection.

(iii) The necessity follows from the previous result. So does the sufficiency: The sets appearing in $(*)$ are then measurable, and F is as well, being the limit of linear combinations of measurable processes. ∎

3.5 Predictable and Previsible Processes

The Borel functions on the line are measurable for every measure. They form the *smallest* class that contains the elementary functions and has the usual permanence properties for measurability: closure under algebraic and order combinations, and under pointwise limits of sequences – and therein lies their virtue. Namely, they lend themselves to this argument: a property of functions that holds for the elementary ones and persists under limits of sequences, etc., holds for Borel functions. For instance, if two measures μ, ν satisfy $\mu(\phi) \le \nu(\phi)$ for step functions ϕ, then the same inequality is satisfied on Borel functions ϕ – observe that it makes no sense in general to state this inequality for integrable functions, inasmuch as a μ-integrable function may not even be ν-measurable. But the Borel functions also form a *large* class in the sense that every function measurable for some measure μ is μ-a.e. equal to a Borel function, and that takes the sting out of the previous observation: on that Borel function μ and ν can be compared.

It is the purpose of this section to identify and analyze the stochastic analog of the Borel functions.

Predictable Processes

The Borel functions on the line are the sequential closure[5] of the step functions or elementary integrands $\mathfrak{e}$. The analogous notion for processes is this:

Definition 3.5.1 *The sequential closure (in $\overline{\mathbb{R}}^{\boldsymbol{B}}$!) of the elementary integrands $\mathcal{E}$ is the collection of* ***predictable processes*** *and is denoted by $\mathcal{P}$. The σ-algebra of sets in $\mathcal{P}$ is also denoted by $\mathcal{P}$. If there is need to indicate the filtration, we write $\mathcal{P}[\mathcal{F}_.]$.*

An elementary integrand X is prototypically predictable in the sense that its value X_t at any time t is measurable on some strictly earlier σ-algebra $\mathcal{F}_s$: at time s the value X_t can be foretold. This explains the choice of the word "predictable."

[5] See pages 391–393.

$\mathcal{P}$ is of course also the name of the σ-algebra generated by the idempotents (sets[6]) in $\mathcal{E}$. These are the finite unions of elementary stochastic intervals of the form $(\!(S,T]\!]$. This again is the difference of $[\![0,T]\!]$ and $[\![0,S]\!]$. Thus $\mathcal{P}$ also agrees with the σ-algebra spanned by the family of stochastic intervals $\{[\![0,T]\!] : \text{T an elementary stopping time}\}$.

Egoroff's theorem 3.4.4 implies that a predictable process is measurable for any mean $\lceil\!\lceil\ \rceil\!\rceil^*$. Conversely, any $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable process F coincides $\lceil\!\lceil\ \rceil\!\rceil^*$-almost everywhere with some predictable process. Indeed, there is a sequence $(X^{(n)})$ of elementary integrands that converges $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e. to F (see corollary 3.4.5); the predictable process $\liminf X^{(n)}$ qualifies.

The next proposition provides a stock-in-trade of predictable processes.

Proposition 3.5.2 *(i) Any left-open right-closed stochastic interval $(\!(S,T]\!]$, $S \le T$, is predictable. In fact, whenever f is a random variable measurable on $\mathcal{F}_S$, then $f \cdot (\!(S,T]\!]$ is predictable*[6]*; if it is Z–p-integrable, then its integral is as expected – see exercise 2.1.14:*

$$f \cdot \big(Z_T - Z_S\big) \in \int f \cdot (\!(S,T]\!]\, dZ\ . \tag{3.5.1}$$

(ii) A left-continuous adapted process X is predictable. The continuous adapted processes generate $\mathcal{P}$.

Proof. (i) Let $T^{(n)}$ be the stopping times of exercise 1.3.20:

$$T^{(n)} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{k+1}{n} \cdot \Big[\frac{k}{n} < T \le \frac{k+1}{n}\Big] \ +\ \infty\cdot[T=\infty]\,, \qquad n \in \mathbb{N}.$$

Recall that $(T^{(n)})$ decreases to T. Next let $f^{(m)}$ be $\mathcal{F}_S$-measurable simple functions with $|f^{(m)}| \le |f|$ that converge pointwise to f. Set[6]

$$\begin{aligned} X^{(m,n)} &\stackrel{\text{def}}{=} f^{(m)} \cdot (\!(S^{(n)} \wedge m, T^{(n)} \wedge m]\!] \\ &= f^{(m)} \cdot [S^{(n)} < m] \cdot (\!(S^{(n)} \wedge m, T^{(n)} \wedge m]\!]\ . \end{aligned}$$

Since $f^{(m)} \in \mathcal{F}_S \subset \mathcal{F}_{S^{(n)}}$ (exercise 1.3.16),

$$f^{(m)} \cdot [S^{(n)} \le m] \cdot [S^{(n)} \wedge m \le t] = \begin{cases} f^{(m)} \cdot [S^{(n)} \le m] \in \mathcal{F}_m \subset \mathcal{F}_t\ , & t \ge m, \\ f^{(m)} \cdot [S^{(n)} \le t\,] \in \mathcal{F}_t\ , & t < m, \end{cases}$$

and $f^{(m)} \cdot [S^{(n)} \le m]$ is measurable on $\mathcal{F}_{S^{(n)} \wedge m}$. By exercise 2.1.5, $X^{(m,n)}$ is an elementary integrand. Therefore

$$f \cdot (\!(S,T]\!] = \lim_{m\to\infty} \lim_{n\to\infty} X^{(m,n)}$$

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[6] In accordance with convention A.1.5 on page 364, sets are identified with their (idempotent) indicator functions. A stochastic interval $(\!(S,T]\!]$, for instance, has at the instant s the value $(\!(S,T]\!]_s = [S < s \le T] = \begin{cases} 1 & \text{if } S(\omega) < s \le T(\omega), \\ 0 & \text{elsewhere.} \end{cases}$

is predictable. If Z is an L^p-integrator, then the integral of $X^{(m,n)}$ is $f^{(m)} \cdot \big(Z_{T^{(n)}\wedge m} - Z_{S^{(n)}\wedge m}\big)$ (ibidem). Now $|X^{(m,n)}| \le \|f^{(m)}\|_\infty \cdot [\![0, m]\!]$, so the Dominated Convergence Theorem in conjunction with the right-continuity of Z gives[6]

$$f^{(m)} \cdot \big(Z_{T\wedge m} - Z_{S\wedge m}\big) \in \int f^{(m)} \cdot (\!(S \wedge m, T \wedge m]\!]\, dZ$$

as $n \to \infty$. The integrands are dominated by $|f| \cdot (\!(S, T]\!]$; so if this process is Z–p-integrable, then a second application of the Dominated Convergence Theorem produces (3.5.1) as $m \to \infty$.

(ii) To start with assume that X is continuous. Thanks to corollary 1.3.12 the random times $T_0 = 0$,

$$T^n_{k+1} = \inf\{t : \big|X - X^{T^n_k}\big|^\star_t \ge 2^{-n}\}$$

are all stopping times. The process[6]

$$X_0 \cdot [\![0]\!] + \sum_{k=0}^{\infty} X_{T^n_k} \cdot (\!(T^n_k, T^n_{k+1}]\!]$$

is predictable and uniformly as close as 2^{-n} to X. Hence X is predictable. Now to the case that X is merely left-continuous and adapted. Replacing X by $-k \vee X \wedge k$, $k \in \mathbb{N}$, and taking the limit we may assume that X is bounded. Let $\phi^{(n)}$ be a positive continuous function with support in $[0, 1/n]$ and Lebesgue integral 1. Since the path of X is Lebesgue measurable and bounded, the convolution

$$X^{(n)}_t \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} X_s \cdot \phi^{(n)}(t-s)\, ds = \int_{t-1/n}^{t} X_s \cdot \phi^{(n)}(t-s)\, ds$$

exists (set $X_s = 0$ for $s < 0$). Every $X^{(n)}$ is an adapted process with continuous paths and is therefore predictable. The sequence $\big(X^{(n)}\big)$ converges pointwise to X, by the left-continuity of this process. Hence X is predictable.

Since in particular every elementary integrand can be approximated in this way by continuous adapted processes, the latter generate $\mathcal{P}$. ▮

Exercise 3.5.3 $\mathcal{F}_.$ and its right-continuous version $\mathcal{F}_{.+}$ have the same predictables.

Exercise 3.5.4 If Z is predictable and T a stopping time, then the stopped process Z^T is predictable. The variation process of a right-continuous predictable process of finite variation is predictable.

Exercise 3.5.5 Suppose G is Z–0-integrable; let S, T be two stopping times; and let $f \in L^0(\mathcal{F}_S)$. Then $f \cdot (\!(S, T]\!] \cdot G$ is Z–0-integrable and

$$\int f \cdot (\!(S, T]\!] \cdot G\, dZ \doteq \dot{f} \cdot \int (\!(S, T]\!] \cdot G\, dZ\,. \tag{3.5.2}$$

Previsible Processes

For the remainder of the chapter we resurrect our hitherto unused standing assumption 2.2.12 *that the measured filtration* $(\Omega, \mathcal{F}_., \mathbb{P})$ *satisfies the natural conditions.*

One should respect a process that tries to be predictable and *nearly* succeeds:

Definition 3.5.6 *A process X is called* ***previsible*** *with $\mathbb{P}$ if there exists a predictable process $X_{\mathbb{P}}$ that cannot be distinguished from X with $\mathbb{P}$. The collection of previsible processes is denoted by $\mathcal{P}^{\mathbb{P}}$, and so is the collection of sets in $\mathcal{P}^{\mathbb{P}}$.*

A previsible process is measurable for any of the Daniell means associated with an integrator. This follows from exercise 3.2.5 and uses the regularity of the filtration.

Exercise 3.5.7 (i) $\mathcal{P}^{\mathbb{P}}$ is sequentially closed and the idempotent functions (sets) in $\mathcal{P}^{\mathbb{P}}$ form a σ-algebra. (ii) In the presence of the natural conditions a measurable (in the sense of page 23) previsible process is predictable.

Exercise 3.5.8 Redo exercise 3.5.4 for previsible processes.

Predictable Stopping Times

On the half-line any singleton $\{t\}$ is integrable for any measure dz, since it is a compact Borel set. Its dz-measure is $\Delta z_t = z_t - z_{t-}$. The stochastic analog of a singleton is the graph of a random time, and the stochastic analog of the Borels are the predictable sets. It is natural to ask when the graph of a random time T is a predictable set and, if so, what the dZ-integral of its graph $[\![T]\!]$ is. The answer is given in theorem 3.5.13 in terms of the predictability of T:

Definition 3.5.9 *A random time T is called* ***predictable*** *if there exists a sequence of stopping times $T_n \le T$ that are strictly less than T on $[T > 0]$ and increase to T everywhere; the sequence $\big(T_n\big)$ is said to* ***predict*** *or to* ***announce*** *T.*

A predictable time is a stopping time (exercise 1.3.15). Before showing that it is precisely the predictable stopping times that answer our question it is expedient to develop their properties.

Exercise 3.5.10 (i) Instants are predictable. If T is any stopping time, then $T+\epsilon$ is predictable, as long as $\epsilon > 0$. The infimum of a finite number of predictable times and the supremum of a countable number of predictable times are predictable.

(ii) For any $A \in \mathcal{F}_0$ the reduced time 0_A is predictable; if S is predictable, then so is its reduction S_A, in particular $S_{[S>0]}$. If S, T are stopping times, S predictable, then the reduction $S_{[S \le T]}$ is predictable.

Exercise 3.5.11 Let S, T be predictable stopping times. Then all stochastic intervals that have $S, T, 0$, or ∞ as endpoints are predictable sets. In particular, $[\![0, T)\!)$, the graph $[\![T]\!]$, and $[\![T, \infty)\!)$ are predictable.

Lemma 3.5.12 *(i) A random time T nearly equal to a predictable stopping time S is itself a predictable stopping time; the σ-algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ agree.*
(ii) Let T be a stopping time, and assume that there exists a sequence (T_n) of stopping times that are almost surely less than T, almost surely strictly so on $[T>0]$, and that increase almost surely to T. Then T is predictable.
(iii) The limit S of a decreasing sequence $\big(S_n\big)$ of predictable stopping times is a predictable stopping time provided $\big(S_n\big)$ is almost surely ultimately constant.

Proof. We employ – for the first time in this context – the natural conditions.

(i) Suppose that S is announced by (S_n) and that $[S \neq T]$ is nearly empty. Then, due to the regularity of the filtration, the random variables[6]

$$T_n \stackrel{\text{def}}{=} S_n \cdot [S=T] + \big(0 \vee (T-1/n) \wedge n\big) \cdot [S \neq T]$$

are stopping times. The T_n evidently increase to T, strictly so on $[T>0]$. If $A \in \mathcal{F}_T$, then $A \cap [S \le t]$ nearly equals $A \cap [T \le t] \in \mathcal{F}_t$, so $A \cap [S \le t] \in \mathcal{F}_t$ by regularity. This says that A belongs to $\mathcal{F}_S$.

(ii) Replacing T_n by $\bigvee_{m \le n} T_m$ we may assume that (T_n) increases everywhere. $T_\infty \stackrel{\text{def}}{=} \sup T_n$ is a stopping time (exercise 1.3.15) nearly equal to T (exercise 1.3.27). It suffices therefore to show that T_∞ is predictable. In other words, we may assume that (T_n) increases everywhere to T, almost surely strictly so on $[T>0]$. The set $N \stackrel{\text{def}}{=} [T>0] \cap \bigcup_n [T=T_n]$ is nearly empty, and the chopped reductions $T_{nN^c} \wedge n$ increase strictly to T_{N^c} on $\big[T_{N^c} > 0\big]$: T_{N^c} is predictable. T, being nearly equal to T_{N^c}, is predictable as well.

(iii) To say that $\big(S_n(\omega)\big)$ is ***ultimately constant*** means of course that for every $\omega \in \Omega$ there is an $N(\omega)$ such that $S(\omega) = S_n(\omega)$ for all $n \ge N(\omega)$. To start with assume that S_1 is bounded, say $S_1 \le k$. For every n let S'_n be a stopping time less than or equal to S_n, strictly less than S_n where $S_n > 0$, and having

$$\mathbb{P}\big[S'_n < S_n - 2^{-n}\big] < 2^{-n-1} .$$

Such exist as S_n is predictable. Since $\mathcal{F}_{\boldsymbol{.}}$ is right-continuous, the random variables $S''_n \stackrel{\text{def}}{=} \inf_{\nu \ge n} S'_\nu$ are stopping times (exercise 1.3.30). Clearly $S''_n < S$ almost surely on $[S>0]$, namely, at all points $\omega \in [S>0]$ where $S_n(\omega)$ is ultimately constant. Since $\mathbb{P}\big[S''_n < S\big] \le 2^{-n}$, $\big(S''_n\big)$ increases almost surely to S. By (ii) S is predictable. In the general case we know now that $S \wedge k = \inf_n S_n \wedge k$ is predictable. Then so is the pointwise supremum $S = \bigvee_k S \wedge k$ (exercise 1.3.15). ∎

Theorem 3.5.13 *(i) Let $B \subset \boldsymbol{B}$ be previsible and $\epsilon > 0$. There is a predictable stopping time T whose graph is contained in B and such that $\mathbb{P}\big[\pi_\Omega[B]\big] < \mathbb{P}[T < \infty] + \epsilon$.*

(ii) A random time is predictable if and only if its graph is previsible.

Proof. (i) Let $B_{\mathbb{P}}$ be a predictable set that cannot be distinguished from B. Theorem A.5.14 on page 438 provides a predictable stopping time S whose graph lies inside $B_{\mathbb{P}}$ and satisfies $\mathbb{P}\big[\pi_\Omega[B_{\mathbb{P}}]\big] < \mathbb{P}[S < \infty] + \epsilon$ (see figure A.17 on page 436). The projection of $[\![S]\!] \setminus B$ is nearly empty and by regularity belongs to $\mathcal{F}_0$. The reduction of S to its complement is a predictable stopping time that meets the description.

(ii) The necessity of the condition was shown in exercise 3.5.11. Assume then that the graph $[\![T]\!]$ of the random time T is a previsible set. There are predictable stopping times S_k whose graphs are contained in that of T and so that $\mathbb{P}[T \neq S_k] \le 1/k$. Replacing S_k by $\inf_{\kappa \le k} S_\kappa$ we may assume the S_k to be decreasing. They are clearly ultimately constant. Thanks to lemma 3.5.12 their infimum S is predictable. The set $[S \neq T]$ is evidently nearly empty; so in view of lemma 3.5.12 (ii) T is predictable. ∎

The Strict Past of a Stopping Time The question at the beginning of the section is half resolved: the stochastic analog of a singleton $\{t\}$ *qua* integrand has been identified as the graph of a predictable time T. We have no analog yet of the fact that the measure of $\{t\}$ is $\Delta z_t = z_t - z_{t-}$. Of course in the stochastic case the right question to ask is this: for which random variables f is the process $f \cdot [\![T]\!]$ previsible, and what is its integral? Theorem 3.5.14 gives the answer in terms of the ***strict past*** of T. This is simply the σ-algebra $\mathcal{F}_{T-}$ generated by $\mathcal{F}_0$ and the collection

$$\big\{A \cap [t < T] : t \in \mathbb{R}_+, A \in \mathcal{F}_t\big\} .$$

A generator is "an event that occurs and is observable at some instant t strictly prior to T." A stopping time is evidently measurable on its strict past.

Theorem 3.5.14 *Let T be a stopping time, f a real-valued random variable, and Z an L^0-integrator. Then $f \cdot [\![T]\!]$ is a previsible process*[6] *if and only if both $f \cdot [T < \infty]$ is measurable on the strict past of T and the reduction $T_{[f \neq 0]}$ is predictable; and in this case*

$$f \cdot \Delta Z_T \in \int f \cdot [\![T]\!] \, dZ . \tag{3.5.3}$$

Before proving this theorem it is expedient to investigate the strict past of stopping times.

Lemma 3.5.15 *(i) If $S \le T$, then $\mathcal{F}_{S-} \subset \mathcal{F}_{T-} \subset \mathcal{F}_T$; and if in addition $S < T$ on $[T > 0]$, then $\mathcal{F}_S \subset \mathcal{F}_{T-}$.*

(ii) Let T_n be stopping times increasing to T. Then $\mathcal{F}_{T-} = \bigvee \mathcal{F}_{T_n-}$. If the T_n announce T, then $\mathcal{F}_{T-} = \bigvee \mathcal{F}_{T_n}$.

(iii) If X is a previsible process and T any stopping time, then X_T is measurable on $\mathcal{F}_{T-}$.

(iv) If T is a predictable stopping time and $A \in \mathcal{F}_{T-}$, then the reduction T_A is predictable.

Proof. (i) A generator $A \cap [t < S]$ of $\mathcal{F}_{S-}$ can be written as the intersection of $\big(A \cap [t < S]\big)$ with $[t < T]$ and belongs to $\mathcal{F}_{T-}$ inasmuch as $[t < S] \in \mathcal{F}_t$. A generator $A \cap [t < T]$ of $\mathcal{F}_{T-}$ belongs to $\mathcal{F}_T$ since

$$(A \cap [t < T]) \cap [T \le u] = \begin{cases} \emptyset \in \mathcal{F}_u & \text{for } u \le t \\ A \cap [T \le u] \cap [T \le t]^c \in \mathcal{F}_u & \text{for } u > t. \end{cases}$$

Assume that $S < T$ on $[T > 0]$, and let $A \in \mathcal{F}_S$. Then $A \cap [T > 0] = A \cap \bigcup_{q \in \mathbb{Q}_+} [S < q] \cap [q < T]$ belongs to $\mathcal{F}_{T-}$, and so does $A \cap [T = 0] \in \mathcal{F}_0$. This proves the second claim of (i).

(ii) A generator $A \cap [t < T] = \bigcup_n A \cap [t < T_n]$ clearly lies in $\bigvee_n \mathcal{F}_{T_n -}$. If the T_n announce T, then by (i) $\mathcal{F}_{T-} \subset \bigvee_n \mathcal{F}_{T_n -} \subset \bigvee_n \mathcal{F}_{T_n} \subset \mathcal{F}_{T-}$.

(iii) Assume first that X is of the form $X = A \times (s, t]$ with $A \in \mathcal{F}_s$. Then $X_T = A \cap [s < T \le t] = \big(A \cap [s < T]\big) \setminus \big(\Omega \cap [t < T]\big) \in \mathcal{F}_{T-}$. By linearity, $X_T \in \mathcal{F}_{T-}$ for all $X \in \mathcal{E}$. The processes X with $X_T \in \mathcal{F}_{T-}$ evidently form a sequentially closed family, so every predictable process has this property. An evanescent process clearly has it as well, so every previsible process has it.

(iv) Let $\big(T^n\big)$ be a sequence announcing T. Since $A \in \bigvee \mathcal{F}_{T^n}$, there are sets $A^n \in \bigcup \mathcal{F}_{T^n}$ with $\mathbb{P}\big[|A - A^n|\big] < 2^{-n-1}$. Taking a subsequence, we may assume that $A^n \in \mathcal{F}_{T^n}$. Then $A_N \overset{\text{def}}{=} \bigcap_{n > N} A^n \in \mathcal{F}_T$, and $T^n_{A^n} \wedge n$ announces T_{A_N}. This sequence of predictable stopping times is ultimately constant and decreases almost surely to T_A, so T_A is predictable. ∎

Proof of Theorem 3.5.14. If $X \overset{\text{def}}{=} f \cdot [\![T]\!]$ is previsible,[6] then $X_T = f \cdot [T < \infty]$ is measurable on $\mathcal{F}_{T-}$ (lemma 3.5.15 (iii)), and $T_{[f \ne 0]}$ is predictable since it has previsible graph $[X \ne 0]$ (theorem 3.5.13). The conditions listed are thus necessary.

To show their sufficiency we replace first of all f by $f \cdot [T < \infty]$, which does not change X. We may thus assume that f is measurable on $\mathcal{F}_{T-}$, and that $T = T_{[f \ne 0]}$ is predictable. If f is a set in $\mathcal{F}_{T-}$, then X is the graph of a predictable stopping time (ibidem) and thus is predictable (exercise 3.5.11). If f is a step function over $\mathcal{F}_{T-}$, a linear combination of sets, then X is predictable as a linear combination of predictable processes. The usual sequential closure argument shows that X is predictable in general.

It is left to be shown that equation (3.5.3) holds. We fix a sequence $\big(T^n\big)$ announcing T and an L^0-integrator Z. Since f is measurable on the span of the $\mathcal{F}_{T^n}$, there are $\mathcal{F}_{T^n}$-measurable step functions f^n that converge in probability to f. Taking a subsequence we can arrange things so that $f^n \to f$ almost surely. The processes $X^n \overset{\text{def}}{=} f^n \cdot (\!(T^n, T]\!]$, previsible by proposition 3.5.2, converge to $X = f \cdot [\![T]\!]$ except possibly on the evanescent set $\mathbb{R}_+ \times [f_n \not\to f]$, so the limit is previsible. To establish equation (3.5.3) we note that $f^m \cdot (\!(T^n, T]\!]$ is Z–0-integrable for $m \le n$ (exercise 3.5.5) with

$$f^m \cdot (Z_T - Z_{T^n}) \in \int f^m \cdot (\!(T^n, T]\!]\, dZ\ .$$

We take $n \to \infty$ and get $f^m \cdot \Delta Z_T \in \int f^m \cdot [\![T]\!] \, dZ$. Now, as $m \to \infty$, the left-hand side converges almost surely to $f \cdot \Delta Z_T$. If the $|f^m|$ are uniformly bounded, say by M, then $f^m \cdot [\![T]\!]$ converges to $f \cdot [\![T]\!]$ Z–0-a.e., being dominated by $M \cdot [\![T]\!]$. Then $f^m \cdot [\![T]\!]$ converges to $f \cdot [\![T]\!]$ in Z–0-mean, thanks to the Dominated Convergence Theorem, and (3.5.3) holds. We leave to the reader the task of extending this argument to the case that f is almost surely finite (replace M by $\sup |f^m|$ and use corollary 3.6.10 to show that $\sup |f^m| \cdot [\![T]\!]$ is finite in Z–0-mean). ∎

Corollary 3.5.16 *A right-continuous previsible process X with finite maximal process is locally bounded.*

Proof. Let $t < \infty$ and $\epsilon > 0$ be given. By the choice of $\lambda > 0$ we can arrange things so that $T^\lambda = \inf\{t : |X_t| \geq \lambda\}$ has $\mathbb{P}[T^\lambda < t] < \epsilon/2$. The graph of T^λ is the intersection of the previsible sets $[|X| \geq \lambda]$ and $[\![0, T^\lambda]\!]$. Due to theorem 3.5.13, T^λ is predictable: there is a stopping time $S < T^\lambda$ with $\mathbb{P}[S < T^\lambda \wedge t] < \epsilon/2$. Then $\mathbb{P}[S < t] < \epsilon$ and $|X^S|$ is bounded by λ. ∎

Accessible Stopping Times

For an application in section 4.4 let us introduce stopping times that are "partly predictable" and those that are "nowhere predictable:"

Definition 3.5.17 *A stopping time T is* ***accessible on a set $A \in \mathcal{F}_T$*** *of strictly positive measure if there exists a predictable stopping time S that agrees with T on A – clearly T is then accessible on the larger set $[S = T]$ in $\mathcal{F}_T \cap \mathcal{F}_S$. If there is a countable cover of Ω by sets on which T is accessible, then T is simply called* ***accessible****. On the other hand, if T agrees with no predictable stopping time on any set of strictly positive probability, then T is called* ***totally inaccessible****.*

For example, in a realistic model for atomic decay, the first time T a Geiger counter detects a decay should be totally inaccessible: there is no circumstance in which the decay is foreseeable.

Given a stopping time T, let $\mathfrak{A}$ be a maximal collection of mutually disjoint sets on which T is accessible. Since the sets in $\mathfrak{A}$ have strictly positive measure, there are at most countably many of them, say $\mathfrak{A} = \{A_1, A_2, \ldots\}$. Set $A \stackrel{\text{def}}{=} \bigcup A_n$ and $I \stackrel{\text{def}}{=} A^c$. Then clearly the reduction T_A is accessible and T_I is totally inaccessible:

Proposition 3.5.18 *Any stopping time T is the infimum of two stopping times T_A, T_I having disjoint graphs, with T_A accessible – wherefore $[\![T_A]\!]$ is contained in the union of countably many previsible graphs – and T_I totally inaccessible.*

Exercise 3.5.19 Let $V \in \mathfrak{D}$ be previsible, and let $\lambda \geq 0$. (i) Then

$$T_V^\lambda = \inf\{t : |V_t| \geq \lambda\} \quad \text{and} \quad T_{\Delta V}^\lambda = \inf\{t : \Delta V_t \geq \lambda\}$$

are predictable stopping times. (ii) There exists a sequence $\{T_n\}$ of predictable stopping times with disjoint graphs such that $[\Delta V \neq 0] \subset \bigcup_n [\![T_n]\!]$.

Exercise 3.5.20 If M is a uniformly integrable martingale and T a predictable stopping time, then $M_{T-} \in \mathbb{E}[M_\infty|\mathcal{F}_{T-}]$ and thus $\mathbb{E}[\Delta M_T|\mathcal{F}_{T-}] \doteq 0$.

Exercise 3.5.21 For deterministic instants t, $\mathcal{F}_{t-}$ is the σ-algebra generated by $\{\mathcal{F}_s : s < t\}$. The σ-algebras $\mathcal{F}_{t-}$ make up the ***left-continuous version*** $\mathcal{F}_{\cdot-}$ of $\mathcal{F}_{\cdot}$. Its predictables and previsibles coincide with those of $\mathcal{F}_{\cdot}$.

3.6 Special Properties of Daniell's Mean

In this section a probability $\mathbb{P}$, an exponent $p \geq 0$, and an $L^p(\mathbb{P})$-integrator Z are fixed. The mean is Daniell's mean $[\![\ ]\!]^*_{Z-p}$, computed with respect to $\mathbb{P}$. As usual, mention of $\mathbb{P}$ is suppressed in the notation. Recall that we often use the words *Z–p-integrable*, *Z–p-a.e.*, *Z–p-measurable*, etc., instead of $[\![\ ]\!]^*_{Z-p}$-integrable, $[\![\ ]\!]^*_{Z-p}$-a.e., $[\![\ ]\!]^*_{Z-p}$-measurable, etc.

Maximality

Proposition 3.6.1 *$[\![\ ]\!]^*_{Z-p}$ is **maximal**. That is to say, if $[\![\ ]\!]^*$ is any mean less than or equal to $[\![\ ]\!]^*_{Z-p}$ on positive elementary integrands, then the inequality $[\![F]\!]^* \leq [\![F]\!]^*_{Z-p}$ holds for all processes F.*

Proof. Suppose that $[\![F]\!]^*_{Z-p} < a$. There exists an $H \in \mathcal{E}_+^\uparrow$, limit of an increasing sequence of positive elementary integrands $X^{(n)}$, with $|F| \leq H$ and $[\![H]\!]^*_{Z-p} < a$. Then

$$[\![F]\!]^* \leq [\![H]\!]^* = \sup_n \left[\!\!\left[X^{(n)} \right]\!\!\right]^* \leq \sup_n \left[\!\!\left[X^{(n)} \right]\!\!\right]^*_{Z-p} = [\![H]\!]^*_{Z-p} < a . \qquad \blacksquare$$

Exercise 3.6.2 $\|\ \|^*_{Z-p}$ and $\|\ \|^*_{Z-[\alpha]}$ are maximal as well.

Exercise 3.6.3 Suppose that Z is an L^p-integrator, $p \geq 1$, and $X \mapsto \int X\, dZ$ has been extended in some way to a vector lattice $\mathfrak{L}$ of processes such that the Dominated Convergence Theorem holds. Then there exists a mean $[\![\ ]\!]^*$ such that the integral is the extension by $[\![\ ]\!]^*$-continuity of the elementary integral, at least on the $[\![\ ]\!]^*$-closure of $\mathcal{E}$ in $\mathfrak{L}$.

Exercise 3.6.4 If $[\![\ ]\!]^*$ is any mean, then

$$[\![F]\!]^{**} \stackrel{\text{def}}{=} \sup\{[\![F]\!]' : \ [\![\ ]\!]' \text{ a mean with } [\![\ ]\!]' \leq [\![\ ]\!]^* \text{ on } \mathcal{E}_+ \}$$

defines a mean $[\![\ ]\!]^{**}$, evidently a maximal one. It is given by Daniell's up-and-down procedure:

$$[\![F]\!]^{**} = \begin{cases} \sup\{[\![X]\!]^* : \ X \in \mathcal{E}_+ \,,\ X \leq F\} & \text{if } F \in \mathcal{E}_+^\uparrow \\ \inf\{[\![H]\!]^{**} : \ |F| \leq H \in \mathcal{E}_+^\uparrow\} & \text{for arbitrary } F. \end{cases} \tag{3.6.1}$$

Exercise 3.6.3 says that an integral extension featuring the Dominated Convergence Theorem can be had essentially only by using a mean that controls the elementary integral. Other examples can be found in definition (4.2.9) and exercise 4.5.18: Daniell's procedure is not so ad hoc as it may seem at first. Exercise 3.6.4 implies that we might have also defined Daniell's mean as the maximal mean that agrees with the semivariation on $\mathcal{E}_+$. That would have left us, of course, with the onus to show that there exists at least one such mean.

It seems at this point, though, that Daniell's mean is the worst one to employ, whichever way it is constructed. Namely, the larger the mean, the smaller evidently the collection of integrable functions. In order to integrate as large a collection as possible of processes we should try to find as small a mean as possible that still controls the elementary integral. This can be done in various non-canonical and uninteresting ways. We prefer to develop some nice and useful properties that are direct consequences of the maximality of Daniell's mean.

Continuity Along Increasing Sequences

It is well known that the outer measure μ^* associated with a measure μ satisfies $0 \le A_n \uparrow A \implies \mu^*(A_n) \uparrow \mu^*(A)$, making it a capacity. The Daniell mean has the same property:

Proposition 3.6.5 *Let $\lceil\!\lceil \ \rceil\!\rceil^*$ be a maximal mean on $\mathcal{E}$. For any increasing sequence $\big(F^{(n)}\big)$ of positive numerical processes,*

$$\left\lceil\!\!\left\lceil \sup F^{(n)} \right\rceil\!\!\right\rceil^* = \sup_n \left\lceil\!\!\left\lceil F^{(n)} \right\rceil\!\!\right\rceil^* .$$

Proof. We start with an observation, which might be called *upper regularity: for every positive integrable process F and every $\epsilon > 0$ there exists a process $H \in \mathcal{E}_+^{\uparrow}$ with $H > F$ and $\lceil\!\lceil H - F \rceil\!\rceil^* \le \epsilon$.* Indeed, there exists an $X \in \mathcal{E}_+$ with $\lceil\!\lceil F - X \rceil\!\rceil^* < \epsilon/2$; equation (D.1) provides an $H^\epsilon \in \mathcal{E}_+^{\uparrow}$ with $|F - X| \le H^\epsilon$ and $\lceil\!\lceil H^\epsilon \rceil\!\rceil^* < \epsilon/2$; and evidently $H \stackrel{\text{def}}{=} X + H^\epsilon$ meets the description.

Now to the proof proper. Only the inequality

$$\left\lceil\!\!\left\lceil \sup F^{(n)} \right\rceil\!\!\right\rceil^* \le \sup_n \left\lceil\!\!\left\lceil F^{(n)} \right\rceil\!\!\right\rceil^* \tag{?}$$

needs to be shown, the reverse inequality being obvious from the solidity of $\lceil\!\lceil \ \rceil\!\rceil^*$. To start with, assume that the $F^{(n)}$ are $\lceil\!\lceil \ \rceil\!\rceil^*$-integrable. Let $\epsilon > 0$. Using the upper regularity choose for every n an $H^{(n)} \in \mathcal{E}_+^{\uparrow}$ with $F^{(n)} \le H^{(n)}$ and $\lceil\!\lceil H^{(n)} - F^{(n)} \rceil\!\rceil^* < \epsilon/2^n$, and set $F = \sup F^{(n)}$ and $\overline{H}^{(N)} = \sup_{n \le N} H^{(n)}$. Then $F \le H \stackrel{\text{def}}{=} \sup_N \overline{H}^{(N)} \in \mathcal{E}_+^{\uparrow}$.

Now
$$\begin{aligned} \overline{H}^{(N)} &= \sup\nolimits_{n \le N} \big(F^{(n)} + (H^{(n)} - F^{(n)})\big) \\ &\le F^{(N)} + \textstyle\sum_{n \le N} H^{(n)} - F^{(n)} \end{aligned}$$

and so
$$\left\lceil\!\!\left\lceil \overline{H}^{(N)} \right\rceil\!\!\right\rceil^* \le \sup\nolimits_N \left\lceil\!\!\left\lceil F^{(N)} \right\rceil\!\!\right\rceil^* + \epsilon .$$

Now $\lceil\!\lceil\ \rceil\!\rceil^*$ is continuous along the increasing sequence $\big(\overline{H}^{(N)}\big)$ of $\mathcal{E}_+^\uparrow$, so

$$\lceil\!\lceil F \rceil\!\rceil^* \le \lceil\!\lceil H \rceil\!\rceil^* = \sup_N \left\lceil\!\!\left\lceil \overline{H}^{(N)} \right\rceil\!\!\right\rceil^* \le \sup_N \left\lceil\!\!\left\lceil F^{(N)} \right\rceil\!\!\right\rceil^* + \epsilon \ ,$$

which in view of the arbitraryness of ϵ implies that $\lceil\!\lceil F \rceil\!\rceil^* \le \sup_n \lceil\!\lceil F^{(n)} \rceil\!\rceil^*$. Next assume that the $F^{(n)}$ are merely $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable. Then

$$\underline{F}^{(n)} \stackrel{\text{def}}{=} \big(F^{(n)} \wedge n\big)\cdot[\![0, n]\!]$$

is $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable (theorem 3.4.10). Since $\sup \underline{F}^{(n)} = \sup F^{(n)}$, the first part of the proof gives $\lceil\!\lceil \sup F^{(n)} \rceil\!\rceil^* = \lceil\!\lceil \sup \underline{F}^{(n)} \rceil\!\rceil^* \le \sup \lceil\!\lceil F^{(n)} \rceil\!\rceil^*$.

Now if the $F^{(n)}$ are arbitrary positive $\overline{\mathbb{R}}$-valued processes, choose for every $n, k \in \mathbb{N}$ a process $H^{(n,k)} \in \mathcal{E}_+^\uparrow$ with $F^{(n)} \le H^{(n,k)}$ and

$$\left\lceil\!\!\left\lceil H^{(n,k)} \right\rceil\!\!\right\rceil^* \le \left\lceil\!\!\left\lceil F^{(n)} \right\rceil\!\!\right\rceil^* + 1/k \ .$$

This is possible by the very definition of the Daniell mean; if $\lceil\!\lceil F^{(n)} \rceil\!\rceil^* = \infty$, then $H^{(n,k)} \stackrel{\text{def}}{=} \infty$ qualifies. Set

$$\overline{F}^{(N)} = \inf_{n \ge N} \inf_k H^{(n,k)} \ , \qquad N \in \mathbb{N} \ .$$

The $\overline{F}^{(n)}$ are $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable, satisfy $\lceil\!\lceil F^{(n)} \rceil\!\rceil^* = \lceil\!\lceil \overline{F}^{(n)} \rceil\!\rceil^*$, and increase with n, whence the desired inequality (?):

$$\left\lceil\!\!\left\lceil \sup F^{(n)} \right\rceil\!\!\right\rceil^* \le \left\lceil\!\!\left\lceil \sup \overline{F}^{(n)} \right\rceil\!\!\right\rceil^* = \sup \left\lceil\!\!\left\lceil \overline{F}^{(n)} \right\rceil\!\!\right\rceil^* = \sup \left\lceil\!\!\left\lceil F^{(n)} \right\rceil\!\!\right\rceil^* . \qquad \blacksquare$$

Predictable Envelopes

A subset A of the line is contained in a Borel set $\widetilde{A}$ whose measure equals the outer measure of A. A similar statement holds for the Daniell mean:

Proposition 3.6.6 *Let $\lceil\!\lceil\ \rceil\!\rceil^*$ be a maximal mean on $\mathcal{E}$.*

(i) If F is a $\lceil\!\lceil\ \rceil\!\rceil^*$-negligible process, then there is a predictable process $\widetilde{F} \ge |F|$ that is also $\lceil\!\lceil\ \rceil\!\rceil^*$-negligible.

(ii) If F is a $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable process, then there exist predictable processes $\underset{\sim}{F}$ and $\widetilde{F}$ that differ $\lceil\!\lceil\ \rceil\!\rceil^*$-negligibly and sandwich F: $\underset{\sim}{F} \le F \le \widetilde{F}$.

*(iii) Let F be a non-negative process. There exists a predictable process $\widetilde{F} \ge F$ such that $\lceil\!\lceil r\widetilde{F} \rceil\!\rceil^* = \lceil\!\lceil rF \rceil\!\rceil^*$ for all $r \in \mathbb{R}$ and such that every $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable process bigger than or equal to F is $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e. bigger than or equal to $\widetilde{F}$. If F is a set,[7] then $\widetilde{F}$ can be chosen to be a set as well. If F is finite for $\lceil\!\lceil\ \rceil\!\rceil^*$, then $\widetilde{F}$ is $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable. $\widetilde{F}$ is called a **predictable $\lceil\!\lceil\ \rceil\!\rceil^*$-envelope** of F.*

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[7] In accordance with convention A.1.5 on page 364, sets are identified with their (idempotent) indicator functions.

Proof. (i) For every $n \in \mathbb{N}$ there is an $H^{(n)} \in \mathcal{E}_+^\uparrow$ satisfying both $|F| \le H^{(n)}$ and $\lceil\!\lceil H^{(n)} \rceil\!\rceil^* \le 1/n$ (see equation (D.1)). $\widetilde{F} \stackrel{\text{def}}{=} \inf_n H^{(n)}$ meets the description.

(ii) To start with, assume that F is $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable. Let $(X^{(n)})$ be a sequence of elementary integrands converging $\lceil\!\lceil\ \rceil\!\rceil^*$-almost everywhere to F. The process $Y = (\liminf X^{(n)} - F) \vee 0$ is $\lceil\!\lceil\ \rceil\!\rceil^*$-negligible. $\underset{\sim}{F} \stackrel{\text{def}}{=} \liminf X^{(n)} - \widetilde{Y}$ is less than or equal to F and differs $\lceil\!\lceil\ \rceil\!\rceil^*$-negligibly from F. $\widetilde{F}$ is constructed similarly. Next assume that F is positive and let $F^{(n)} = (n \wedge F) \cdot [\![0, n]\!]$. Then $\underset{\sim}{F} \stackrel{\text{def}}{=} \limsup \underset{\sim}{F^{(n)}}$ and $\widetilde{F} \stackrel{\text{def}}{=} \liminf \widetilde{F^{(n)}}$ qualify. Finally, if F is arbitrary, write it as the difference of two positive measurable processes: $F = F^+ - F^-$, and set $\underset{\sim}{F} = \underset{\sim}{F^-} - \widetilde{F^+}$ and $\widetilde{F} = \widetilde{F^+} - \underset{\sim}{F^-}$.

(iii) To start with, assume that F is finite for $\lceil\!\lceil\ \rceil\!\rceil^*$. For every $q \in \mathbb{Q}_+$ and $k \in \mathbb{N}$ there is an $H^{(q,k)} \in \mathcal{E}_+^\uparrow$ with $H^{(q,k)} \ge F$ and

$$\left\lceil\!\!\left\lceil q \cdot H^{(q,k)} \right\rceil\!\!\right\rceil^* \le \lceil\!\lceil q \cdot F \rceil\!\rceil^* + 2^{-k} .$$

(If $\lceil\!\lceil q \cdot F \rceil\!\rceil^* = \infty$, then $H^{(q,k)} = \infty$ clearly qualifies.) The predictable process $\widehat{F} \stackrel{\text{def}}{=} \bigwedge_{q,k} H^{(q,k)}$ is greater than or equal to F and has $\lceil\!\lceil q\widehat{F} \rceil\!\rceil^* = \lceil\!\lceil qF \rceil\!\rceil^*$ for all positive rationals q; since $\widehat{F}$ is evidently finite for $\lceil\!\lceil\ \rceil\!\rceil^*$, it is $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable, and the previous equality extends by continuity to all positive reals. Next let $\{X_\alpha\}$ be a maximal collection of non-negative predictable and $\lceil\!\lceil\ \rceil\!\rceil^*$-non-negligible processes with the property that

$$F + \textstyle\sum_\alpha X_\alpha \le \widehat{F} .$$

Such a collection is necessarily countable (theorem 3.2.23). It is easy to see that

$$\widetilde{F} \stackrel{\text{def}}{=} \widehat{F} - \textstyle\sum_\alpha X_\alpha$$

meets the description. For if $H \ge F$ is a $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable process, then $H \wedge \widetilde{F}$ is integrable; the envelope $\widetilde{H \wedge \widetilde{F}}$ of part (ii) can be chosen to be smaller than $\widetilde{F}$; the positive process $\widetilde{F} - \widetilde{H \wedge \widetilde{F}}$ is both predictable and $\lceil\!\lceil\ \rceil\!\rceil^*$-integrable; if it were not $\lceil\!\lceil\ \rceil\!\rceil^*$-negligible, it could be adjoined to $\{X_\alpha\}$, which would contradict the maximality of this family; thus $\widetilde{F} - \widetilde{H \wedge \widetilde{F}}$ and $\widetilde{F} - H \wedge \widetilde{F}$ are $\lceil\!\lceil\ \rceil\!\rceil^*$-negligible, or, in other words, $H \ge \widetilde{F}$ $\lceil\!\lceil\ \rceil\!\rceil^*$-almost everywhere.

If F is not finite for $\lceil\!\lceil\ \rceil\!\rceil^*$, then let $\widetilde{F}^{(n)}$ be an envelope for $F \wedge (n \cdot [\![0, n]\!])$. This can evidently be arranged so that $\widetilde{F}^{(n)}$ increases with n. Set $\widetilde{F} = \sup_n \widetilde{F}^{(n)}$. If $H \ge F$ is $\lceil\!\lceil\ \rceil\!\rceil^*$-measurable, then $H \ge \widetilde{F}^{(n)}$ $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e., and consequently $H \ge \widetilde{F}$ $\lceil\!\lceil\ \rceil\!\rceil^*$-a.e. It follows from equation (D.1) that $\lceil\!\lceil F \rceil\!\rceil^* = \lceil\!\lceil \widetilde{F} \rceil\!\rceil^*$.

To see the homogeneity let $r > 0$ and let $\widetilde{rF}$ be an envelope for rF. Since $r^{-1} \cdot \widetilde{rF} \geq F$, we have $\widetilde{rF} \geq r\widetilde{F}$ $\lceil\!\lceil \ \rceil\!\rceil^*$-a.e. and

$$\lceil\!\lceil rF \rceil\!\rceil^* = \left\lceil\!\left\lceil \widetilde{rF} \right\rceil\!\right\rceil^* \geq \left\lceil\!\left\lceil r\widetilde{F} \right\rceil\!\right\rceil^* \geq \lceil\!\lceil rF \rceil\!\rceil^* ,$$

whence equality throughout. Finally, if F is a set with envelope $\widetilde{F}$, then $[\widetilde{F} \geq 1]$ is a smaller envelope and a set. ▮

We apply this now in particular to the Daniell mean of an L^0-integrator Z.

Corollary 3.6.7 *Let A be a subset of Ω, not necessarily measurable. If $B \stackrel{\text{def}}{=} [0,\infty) \times A$ is Z–0-negligible, then the whole path of Z nearly vanishes on A.*

Proof. Let $\widetilde{B}$ be a predictable Z–0-envelope of B and C its complement. Since the natural conditions are in force, the debut T of C is a stopping time (corollary A.5.12). Replace $\widetilde{B}$ by $\widetilde{B} \setminus (\!(T,\infty)\!)$. This does not disturb T, but has the effect that now the graph of T separates $\widetilde{B}$ from its complement C. Now fix an instant $t < \infty$ and an $\epsilon > 0$ and set $T^\epsilon = \inf\{s : |Z_s| \geq \epsilon\}$.

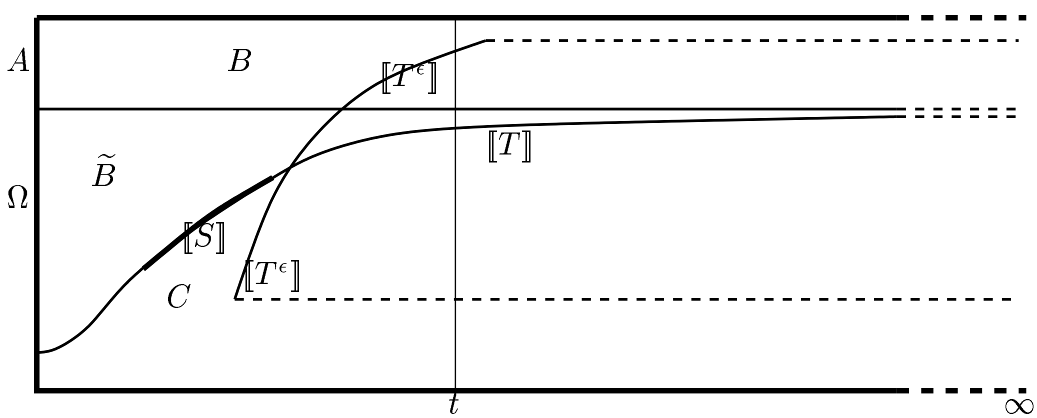

Figure 3.10

The stochastic interval $[\![0, T \wedge T^\epsilon \wedge t]\!]$ intersects C in a predictable subset of the graph of T, which is therefore the graph of a predictable stopping time S (theorem 3.5.13). The rest, $[\![0, T \wedge T^\epsilon \wedge t]\!] \setminus C \subset \widetilde{B}$, is Z–0-negligible. The random variable $Z_{T \wedge T^\epsilon \wedge t}$ is a member of the class $\int [\![0, T \wedge T^\epsilon \wedge t]\!] \, dZ = \int [\![S]\!] \, dZ$ (exercise 3.5.5), which also contains ΔZ_S (theorem 3.5.14). Now $\Delta Z_S = 0$ on A, so we conclude that, on A, $Z_{T^\epsilon \wedge t} = Z_{T \wedge T^\epsilon \wedge t} \doteq 0$. Since $|Z_{T^\epsilon}| \geq \epsilon$ on $[T^\epsilon \leq t]$ (proposition 1.3.11), we must conclude that $A \cap [T^\epsilon \leq t]$ is negligible. This holds for all $\epsilon > 0$ and $t < \infty$, so $A \cap [Z^\star_\infty > 0]$ is negligible. As $[Z^\star_\infty > 0] = \bigcup_n [Z^\star_n > 0] \in \mathcal{A}_{\infty\sigma}$, $A \cap [Z^\star_\infty > 0]$ is actually nearly empty. ▮

Exercise 3.6.8 Let $X \geq 0$ be predictable. Then $\widetilde{XF} = X\widetilde{F}$ Z–p-a.e.

Exercise 3.6.9 Let $\widetilde{B}$ be a predictable Z–0-envelope of $B \subset \boldsymbol{B}$. Any two Z–0-measurable processes X, X' that agree Z–0-almost everywhere on B agree Z–0-almost everywhere on $\widetilde{B}$.

Regularity

Here is an analog of the well-known fact that the measure of a Lebesgue integrable set is the supremum of the Lebesgue measures of the compact sets contained in it (exercise A.3.14). The role of the compact sets is taken by the collection $\mathcal{P}_{00}$ of predictable processes that are bounded and vanish after some instant.

Corollary 3.6.10 *For any Z–p-measurable process F,*

$$\lceil\!\lceil F \rceil\!\rceil^*_{Z-p} = \sup\left\{ \left\lceil\!\!\left\lceil \int Y\, dZ \right\rceil\!\!\right\rceil_p : Y \in \mathcal{P}_{00},\ |Y| \le |F| \right\} .$$

Proof. Since $\lceil\!\lceil \int Y\, dZ \rceil\!\rceil_p \le \lceil\!\lceil Y \rceil\!\rceil^*_{Z-p} \le \lceil\!\lceil F \rceil\!\rceil^*_{Z-p}$, one inequality is obvious. For the other, the solidity of $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-p}$ and proposition 3.6.6 allow us to assume that F is positive and predictable: if necessary, we replace F by $\widetilde{|F|}$. To start with, assume that F is Z–p-integrable and let $\epsilon > 0$. There are an $X \in \mathcal{E}_+$ with $\lceil\!\lceil F - X \rceil\!\rceil^*_{Z-p} < \epsilon/3$ and a $Y' \in \mathcal{E}$ with $|Y'| \le X$ such that

$$\left\lceil\!\!\left\lceil \int Y'\, dZ \right\rceil\!\!\right\rceil_p > \lceil\!\lceil X \rceil\!\rceil^*_{Z-p} - \epsilon/3 .$$

The process $Y \stackrel{\text{def}}{=} (-F) \vee Y' \wedge F$ belongs to $\mathcal{P}_{00}$, $|Y' - Y| \le \big|F - X\big|$, and

$$\left\lceil\!\!\left\lceil \int Y\, dZ \right\rceil\!\!\right\rceil_{L^p} \ge \left\lceil\!\!\left\lceil \int Y'\, dZ \right\rceil\!\!\right\rceil_{L^p} - \epsilon/3 \ge \lceil\!\lceil X \rceil\!\rceil^*_{Z-p} - 2\epsilon/3 \ge \lceil\!\lceil F \rceil\!\rceil^*_{Z-p} - \epsilon .$$

Since $|Y| \le F$ and $\epsilon > 0$ was arbitrary, the claim is proved in the case that F is Z–p-integrable. If F is merely Z–p-measurable, we apply proposition 3.6.5. The $F^{(n)} \stackrel{\text{def}}{=} \big(|F| \wedge n\big) \cdot [\![0, n]\!]$ increase to $|F|$. If $\lceil\!\lceil F \rceil\!\rceil^*_{Z-p} > a$, then $\lceil\!\lceil F^{(n)} \rceil\!\rceil^*_{Z-p} > a$ for large n, and the argument above produces a $Y \in \mathcal{P}_{00}$ with $|Y| \le F^{(n)} \le F$ and $\lceil\!\lceil \int Y\, dZ \rceil\!\rceil_p > a$. ∎

Corollary 3.6.11 *(i) A process F is Z–p-negligible if and only if it is Z–0-negligible, and is Z–p-measurable if and only if it is Z–0-measurable.*

(ii) Let $F \ge 0$ be any process and $\widetilde{F}$ predictable. Then $\widetilde{F}$ is a predictable Z–p-envelope of F if and only if it is a predictable Z–0-envelope.

Proof. (i) By the countable subadditivity of $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-p}$ and $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-0}$ it suffices to prove the first claim under the additional assumption that $|F|$ is majorized by an elementary integrand, say $|F| \le n \cdot [\![0, n]\!]$. The infimum of a predictable Z–p-envelope and a predictable Z–0-envelope is a predictable envelope in the sense both of $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-p}$ and $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-0}$ and is integrable in both senses, with the same integral. So if $\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}0} = 0$, then $\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}p} = 0$. In view of corollary 3.4.5, Z–p-measurability is determined entirely by the Z–p-negligible sets: the Z–p-measurable and Z–0-measurable real-valued processes are the same. We leave part (ii) to the reader. ∎

Definition 3.6.12 *In view of corollary 3.6.11 we shall talk henceforth about* ***Z-negligible*** *and* ***Z-measurable processes****, and about* ***predictable Z-envelopes****.*

Exercise 3.6.13 Let Z be an L^p-integrator, T a stopping time, and G a process.[6] Then

$$\lceil\!\lceil G \cdot [\![0,T]\!] \rceil\!\rceil^*_{Z\text{–}p} = \lceil\!\lceil G \rceil\!\rceil^*_{Z^T\text{–}p} .$$

Consequently, G is Z^T–p-integrable if and only if $G \cdot [\![0,T]\!]$ is Z–p-integrable, and in that case

$$\int G\, dZ^T = \int G \cdot [\![0,T]\!]\, dZ .$$

Exercise 3.6.14 Let Z, Z' be L^0-integrators. If F is both Z–0-integrable (Z–0-negligible, Z–0-measurable) and Z'–0-integrable (Z'–0-negligible, Z'–0-measurable), then it is $(Z{+}Z')$–0-integrable ($(Z{+}Z')$–0-negligible, $(Z{+}Z')$–0-measurable).

Exercise 3.6.15 Suppose Z is a local L^p-integrator. According to proposition 2.1.9, Z is an L^0-integrator, and the notions of negligibility and measurability for Z have been defined in section 3.2. On the other hand, given the definition of a local L^p-integrator one might want to define negligibility and measurability locally. No matter:

Let (T_n) be a sequence of stopping times that increase without bound and reduce Z to L^p-integrators. A process is Z-negligible or Z-measurable if and only if it is Z^{T_n}-negligible or Z^{T_n}-measurable, respectively, for every $n \in \mathbb{N}$.

Exercise 3.6.16 The Daniell mean is also ***minimal*** in this sense: if $\lceil\!\lceil\ \rceil\!\rceil^*$ is a mean such that $\lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}p} \le \lceil\!\lceil X \rceil\!\rceil^*$ for all elementary integrands X, then $\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}p} \le \lceil\!\lceil F \rceil\!\rceil^*$ for all *predictable* F.

Exercise 3.6.17 A process X is Z–0-integrable if and only if for every $\epsilon > 0$ and α there is an $X' \in \mathcal{E}$ with $\| X - X' \|^*_{Z\text{–}[\alpha]} < \epsilon$.

Exercise 3.6.18 Let Z be an L^p-integrator, $0 \le p < \infty$. There exists a positive σ-additive measure μ on $\mathcal{P}$ that has the same negligible sets as $\lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}p}$. If $p \ge 1$, then μ can be chosen so that $|\mu(X)| \le \lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}p}$. Such a measure is called a ***control measure*** for Z.

Exercise 3.6.19 Everything said so far in this chapter remains true *mutatis mutandis* if $L^p(\mathbb{P})$ is replaced by the closure $L^1(\|\ \|^*)$ of the step functions over $\mathcal{F}_\infty$ under a mean $\|\ \|^*$ that has the same negligible sets as $\mathbb{P}$.

Stability Under Change of Measure

Let Z be an $L^0(\mathbb{P})$-integrator and $\mathbb{P}'$ a measure on $\mathcal{F}_\infty$ absolutely continuous with respect to $\mathbb{P}$. Since the injection of $L^0(\mathbb{P})$ into $L^0(\mathbb{P}')$ is bounded, Z is an $L^0(\mathbb{P}')$-integrator (proposition 2.1.9). How do the integrals compare?

Proposition 3.6.20 *A Z–0;$\mathbb{P}$-negligible (-measurable, -integrable) process is Z–0;$\mathbb{P}'$-negligible (-measurable, -integrable). The stochastic integral of a Z–0;$\mathbb{P}$-integrable process does not depend on the choice of the probability $\mathbb{P}$ within its equivalence class.*

Proof. For simplicity of reading let us write $\lceil\!\lceil\ \rceil\!\rceil$ for $\lceil\!\lceil\ \rceil\!\rceil_{Z\text{–}0;\mathbb{P}}$, $\lceil\!\lceil\ \rceil\!\rceil'$ for $\lceil\!\lceil\ \rceil\!\rceil_{Z\text{–}0;\mathbb{P}'}$, and $\lceil\!\lceil\ \rceil\!\rceil'^{*}_{Z\text{–}0}$ for the Daniell mean formed with $\lceil\!\lceil\ \rceil\!\rceil'$. Exercise A.8.12 on page 450 furnishes an increasing right-continuous function $\Phi : (0,1] \to (0,1]$ with $\Phi(r) \xrightarrow[r\to 0]{} 0$ such that

$$\lceil\!\lceil f \rceil\!\rceil' \le \Phi\big(\lceil\!\lceil f \rceil\!\rceil\big)\ , \qquad f \in L^0(\mathbb{P})\ .$$

The monotonicity of Φ causes the same inequality to hold on $\mathcal{E}^{\uparrow}_{+}$:

$$\begin{aligned}\lceil\!\lceil H \rceil\!\rceil'^{*}_{Z\text{–}0} &= \sup\Big\{\Big\lceil\!\!\Big\lceil \int X\,dZ \Big\rceil\!\!\Big\rceil' :\ X \in \mathcal{E}\,,\ |X| \le H\Big\}\\ &\le \sup\Big\{\Phi\Big(\Big\lceil\!\!\Big\lceil \int X\,dZ \Big\rceil\!\!\Big\rceil\Big) :\ X \in \mathcal{E}\,,\ |X| \le H\Big\} \le \Phi\Big(\lceil\!\lceil H \rceil\!\rceil^{*}_{Z\text{–}0}\Big)\end{aligned}$$

for $H \in \mathcal{E}^{\uparrow}_{+}$; the right-continuity of Φ allows its extension to all processes F:

$$\begin{aligned}\lceil\!\lceil F \rceil\!\rceil'^{*}_{Z\text{–}0} &= \inf\Big\{\lceil\!\lceil H \rceil\!\rceil'_{Z\text{–}0} :\ H \in \mathcal{E}^{\uparrow}_{+}\ ,\ H \ge |F|\Big\}\\ &\le \inf\Big\{\Phi\Big(\lceil\!\lceil H \rceil\!\rceil^{*}_{Z\text{–}0}\Big) :\ H \in \mathcal{E}^{\uparrow}_{+}\ ,\ H \ge |F|\Big\} = \Phi\Big(\lceil\!\lceil F \rceil\!\rceil^{*}_{Z\text{–}0}\Big)\ .\end{aligned}$$

Since $\Phi(r) \xrightarrow[r\to 0]{} 0$, a $\lceil\!\lceil\ \rceil\!\rceil^{*}_{Z\text{–}0}$-negligible process is $\lceil\!\lceil\ \rceil\!\rceil'^{*}_{Z\text{–}0}$-negligible, and a $\lceil\!\lceil\ \rceil\!\rceil^{*}_{Z\text{–}0}$-Cauchy sequence is $\lceil\!\lceil\ \rceil\!\rceil'^{*}_{Z\text{–}0}$-Cauchy. A process that is negligible, integrable, or measurable in the sense $\lceil\!\lceil\ \rceil\!\rceil^{*}_{Z\text{–}0}$ is thus negligible, integrable, or measurable, respectively, in the sense $\lceil\!\lceil\ \rceil\!\rceil'^{*}_{Z\text{–}0}$. ∎

Exercise 3.6.21 For the conclusion that Z is an $L^0(\mathbb{P}')$-integrator and that a Z–0;$\mathbb{P}$-negligible (-measurable) process is Z–0;$\mathbb{P}'$-negligible (-measurable) it suffices to know that $\mathbb{P}'$ is *locally* absolutely continuous with respect to $\mathbb{P}$.

Exercise 3.6.22 Modify the proof of proposition 3.6.20 to show in conjunction with exercise 3.2.16 that, whichever gauge on L^p is used to do Daniell's extension with – even if it is not subadditive – , the resulting stochastic integral will be the same.

3.7 The Indefinite Integral

Again a probability $\mathbb{P}$, an exponent $p \ge 0$, and an $L^p(\mathbb{P})$-integrator Z are fixed, and the filtration satisfies the natural conditions.

For motivation consider a measure dz on $\mathbb{R}_+$. The indefinite integral of a function g against dz is commonly defined as the function $t \mapsto \int_0^t g_s\,dz_s$. For this to make sense it suffices that g be locally integrable, i.e., dz-integrable on every bounded set. For instance, the exponential function is locally Lebesgue integrable but not integrable, and yet is of tremendous use. We seek the stochastic equivalent of the notions of local integrability and of the indefinite integral.

The stochastic analog of a bounded interval $[0,t] \subset \mathbb{R}_+$ is a finite stochastic interval $[\![0,T]\!]$. What should it mean to say "G is $Z{-}p$-integrable on the stochastic interval $[\![0,T]\!]$"? It is tempting to answer "the process $G \cdot [\![0,T]\!]$ is $Z{-}p$-integrable.[6]" This would not be adequate, though. Namely, if Z is not an L^p-integrator, merely a local one, then $\lceil\!\lceil\ \rceil\!\rceil^*_{Z-p}$ may fail to be finite on elementary integrands and so may be no mean; it may make no sense to talk about $Z{-}p$-integrable processes. Yet in some suitable sense, we feel, there ought to be many. We take our clue from the classical formula

$$\int_0^t g\,dz \stackrel{\text{def}}{=} \int g \cdot 1_{[0,t]}\,dz = \int g\,dz^t\ ,$$

where z^t is the stopped distribution function $s \mapsto z^t_s \stackrel{\text{def}}{=} z_{t\wedge s}$. This observation leads directly to the following definition:

Definition 3.7.1 *Let Z be a local L^p-integrator, $0 \le p < \infty$. The process G is **$Z{-}p$-integrable on the stochastic interval** $[\![0,T]\!]$ if T reduces Z to an L^p-integrator and G is $Z^T{-}p$-integrable. In this case we write*

$$\int_0^T G\,dZ = \int_{[\![0}^{T]\!]} G\,dZ \stackrel{\text{def}}{=} \int G\,dZ^T\ .$$

If S is another stopping time, then

$$\int_{S+}^T G\,dZ = \int_{(\!(S}^{T]\!]} G\,dZ \stackrel{\text{def}}{=} \int G \cdot (\!(S,\infty)\!)\,dZ^T\ . \tag{3.7.1}$$

The expressions in the middle are designed to indicate that the endpoint $[\![T]\!]$ is included in the interval of integration and $[\![S]\!]$ is not, just as it should be when one integrates on the line against a measure that charges points. We will however usually employ the notation on the left with the understanding that the endpoints are always included in the domain of integration, unless the contrary is explicitly indicated, as in (3.7.1). An exception is the point ∞, which is never included in the domain of integration, so that $\int_S^\infty$ and $\int_S^{\infty-}$ mean the same thing. Below we also consider cases where the left endpoint $[\![S]\!]$ is included in the domain of integration and the right endpoint $[\![T]\!]$ is not. *For (3.7.1) to make sense we must assume of course that Z^T is an L^p-integrator.*

Exercise 3.7.2 If G is $Z{-}p$-integrable on $(\!(S^{(i)},T^{(i)}]\!]$, $i = 1,2$, then it is $Z{-}p$-integrable on the union $(\!(S^{(1)} \wedge S^{(2)}, T^{(1)} \vee T^{(2)}]\!]$.

Definition 3.7.3 *Let Z be a local L^p-integrator, $0 \le p < \infty$. The process G is **locally $Z{-}p$-integrable** if it is $Z{-}p$-integrable on arbitrarily large stochastic intervals, that is to say, if for every $\epsilon > 0$ and $t < \infty$ there is a stopping time T with $\mathbb{P}[T < t] < \epsilon$ that reduces Z to an L^p-integrator such that G is $Z^T{-}p$-integrable.*

Here is yet another indication of the flexibility of L^0-integrators:

Proposition 3.7.4 *Let Z be a local L^0-integrator. A locally Z–0-integrable process is Z–0-integrable on every almost surely finite stochastic interval.*

Proof. The stochastic interval $[\![0, U]\!]$ is called almost surely finite, of course, if $\mathbb{P}[U = \infty] = 0$. We know from proposition 2.1.9 that Z is an L^0-integrator. Thanks to exercise 3.6.13 it suffices to show that $G' \stackrel{\text{def}}{=} G \cdot [\![0, U]\!]$ is Z–0-integrable. Let $\epsilon > 0$. There exists a stopping time T with $\mathbb{P}[T < U] < \epsilon$ so that G and then G' are Z^T–0-integrable.[6] Then $G'' \stackrel{\text{def}}{=} G' \cdot [\![0, T]\!] = G \cdot [\![0, U \wedge T]\!]$ is Z–0-integrable (ibidem). The difference $G''' = G' - G''$ is Z-measurable and vanishes off the stochastic interval $I \stackrel{\text{def}}{=} (\!(T, U]\!]$, whose projection on Ω has measure less than ϵ, and so $\lceil\!\lceil G''' \rceil\!\rceil^*_{Z\text{–}0} \le \epsilon$ (exercise 3.1.2). In other words, G' differs arbitrarily little (by less than ϵ) in $\lceil\!\lceil\ \rceil\!\rceil^*_{Z\text{–}0}$-mean from a Z–0-integrable process (G''). It is thus Z–0-integrable itself (proposition 3.2.20). ∎

The Indefinite Integral

Let Z be a local L^p-integrator, $0 \le p < \infty$, and G a locally Z–p-integrable process. Then Z is an L^0-integrator and G is Z–0-integrable on every finite deterministic interval $[\![0, t]\!]$ (proposition 3.7.4). It is tempting to define the indefinite integral as the function $t \mapsto \int G\, dZ^t$. This is for every t a *class* in L^0 (definition 3.2.14). We can be a little more precise: since $\int X\, dZ^t \in \mathcal{F}_t$ when $X \in \mathcal{E}$, the limit $\int G\, dZ^t$ of such elementary integrals can be viewed as an equivalence class of $\mathcal{F}_t$-measurable random variables. It is desirable to have for the indefinite integral a process rather than a mere slew of classes. This is possible of course by the simple expedient of selecting from every class $\int G\, dZ^t \subset L^0(\mathcal{F}_t, \mathbb{P})$ a random variable measurable on $\mathcal{F}_t$. Let us do that and *temporarily* call the process so obtained $G{*}Z$:

$$(G{*}Z)_t \in \int G\, dZ^t \quad \text{and} \quad (G{*}Z)_t \in \mathcal{F}_t \quad \forall\, t\,. \tag{3.7.2}$$

This is not really satisfactory, though, since two different people will in general come up with wildly differing modifications $G{*}Z$. Fortunately, this deficiency is easily repaired using the following observation:

Lemma 3.7.5 *Suppose that Z is an L^0-integrator and that G is locally Z–0-integrable. Then any process $G{*}Z$ satisfying (3.7.2) is an L^0-integrator and consequently has an adapted modification that is right-continuous with left limits. If $G{*}Z$ is such a version, then*

$$(G{*}Z)_T \in \int_0^T G\, dZ \tag{3.7.3}$$

for any stopping time T for which the integral on the right exists – in particular for all almost surely finite stopping times T.

Proof. It is immediate from the Dominated Convergence Theorem that $G{*}Z$ is right-continuous in probability. For if $t_n \downarrow t$, then $(G{*}Z)_{t_n} - (G{*}Z)_t \in \int G \cdot (\!(t, t_n]\!]\, dZ \to 0$ $\lceil\!\lceil\ \rceil\!\rceil_{L^0}$-mean. To see that the L^0-boundedness condition (B-0) of definition 2.1.7 is satisfied as well, take an elementary integrand X as in (2.1.1), to wit,[6]

$$X = f_0 \cdot [\![0]\!] + \sum_{n=1}^{N} f_n \cdot (\!(t_n, t_{n+1}]\!] \ , \qquad f_n \in \mathcal{F}_{t_n} \ \text{ simple.}$$

Then
$$\begin{aligned} \int X \, d(G{*}Z) &= f_0 \cdot (G{*}Z)_0 + \sum_n f_n \cdot \big((G{*}Z)_{t_{n+1}} - (G{*}Z)_{t_n}\big) \\ &\in \dot{f}_0 \cdot \dot{G}_0 \dot{Z}_0 + \sum_n \dot{f}_n \cdot \int_{t_n+}^{t_{n+1}} G \, dZ \end{aligned}$$

by exercise 3.5.5:
$$\begin{aligned} &= \dot{f}_0 \dot{G}_0 \cdot \dot{Z}_0 + \int \sum_n f_n \cdot (\!(t_n, t_{n+1}]\!] \cdot G \, dZ \\ &= \int X \cdot G \, dZ \ . \end{aligned} \tag{3.7.4}$$

Multiply with $\lambda > 0$ and measure both sides with $\lceil\!\lceil\ \rceil\!\rceil_{L^0}$ to obtain

$$\begin{aligned} \left\lceil\!\!\left\lceil \int \lambda X \, d(G{*}Z) \right\rceil\!\!\right\rceil_{L^0} &= \left\lceil\!\!\left\lceil \int \lambda X \cdot G \, dZ \right\rceil\!\!\right\rceil_{L^0} \\ &\le \lceil\!\lceil \lambda X \cdot G \rceil\!\rceil^*_{Z\text{–}0} \le \lceil\!\lceil \lambda \cdot G \rceil\!\rceil^*_{Z^t\text{–}0} \end{aligned}$$

for all $X \in \mathcal{E}$ with $|X| \le 1$. The right-hand side tends to zero as $\lambda \to 0$: (B-0) is satisfied, and $G{*}Z$ indeed is an L^0-integrator.

Theorem 2.3.4 in conjunction with the natural conditions now furnishes the desired right-continuous modification with left limits. Henceforth $G{*}Z$ denotes such a version.

To prove equation (3.7.3) we start with the case that T is an elementary stopping time; it is then nothing but (3.7.4) applied to $X = [\![0, T]\!]$. For a general stopping time T we employ once again the stopping times $T^{(n)}$ of exercise 1.3.20. For any k they take only finitely many values less than k and decrease to T. In taking the limit as $n \to \infty$ in

$$(G{*}Z)_{T^{(n)} \wedge k} \in \int_0^{T^{(n)} \wedge k} G \, dZ \ ,$$

the left-hand side converges to $(G{*}Z)_{T \wedge k}$ by right-continuity, the right-hand side to[6] $\int_0^{T \wedge k} G \, dZ = \int G \cdot [\![0, T \wedge k]\!] \, dZ$ by the Dominated Convergence Theorem. Now take $k \to \infty$ and use the domination $\big|G \cdot [\![0, T \wedge k]\!]\big| \le \big|G \cdot [\![0, T]\!]\big|$ to arrive at $(G{*}Z)_T \doteq \int G \cdot [\![0, T]\!] \, dZ$. In view of exercise 3.6.13, this is equation (3.7.3). ∎

Any two modifications produced by lemma 3.7.5 are of course indistinguishable. This observation leads directly to the following:

Definition 3.7.6 *Let Z be an L^0-integrator and G a locally Z–0-integrable process. The* ***indefinite integral*** *is a process $G{*}Z$ that is right-continuous with left limits and adapted to $\mathcal{F}^{\mathfrak{P}}_{.+}$ and that satisfies*

$$(G{*}Z)_t \in \int_0^t G\,dZ \stackrel{\text{def}}{=} \int G\,dZ^t \quad \forall\, t \in [0,\infty)\,.$$

It is unique up to indistinguishability. If G is Z–0-integrable, it is understood that $G{*}Z$ is chosen so as to have almost surely a finite limit at infinity as well.

So far it was necessary to distinguish between random variables and their classes when talking about the stochastic integral, because the latter is by its very definition an equivalence class modulo negligible functions. Henceforth we shall do this: when we meet an L^0-integrator Z and a locally Z–0-integrable process G we shall pick once and for all an indefinite integral $G{*}Z$; then $\int_{S+}^T G\,dZ$ will denote the specific random variable $(G{*}Z)_T-(G{*}Z)_S$, etc. Two people doing this will not come up with precisely the same random variables $\int_{S+}^T G\,dZ$, but with nearly the same ones, since in fact the whole paths of their versions of $G{*}Z$ nearly agree. If G happens to be Z–0-integrable, then $\int G\,dZ$ is the almost surely defined random variable $(G{*}Z)_\infty$.

Vectors of integrators $\boldsymbol{Z} = (Z^1, Z^2, \ldots, Z^d)$ appear naturally as drivers of stochastic differentiable equations (pages 8 and 56). The gentle reader recalls from page 109 that the integral extension

$$\mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil^*_{\boldsymbol{Z}\text{–}p}] \ni \boldsymbol{X} \mapsto \int \boldsymbol{X}\,d\boldsymbol{Z}$$

of the elementary integral $\qquad \check{\mathcal{E}} \ni \boldsymbol{X} \mapsto \int \boldsymbol{X}\,d\boldsymbol{Z}$

is given by $\qquad (X_1,\ldots,X_d) = \boldsymbol{X} \mapsto \sum_{\eta=1}^d \int X_\eta\,dZ^\eta\,.$

Therefore $\qquad \boldsymbol{X}{*}\boldsymbol{Z} \stackrel{\text{def}}{=} \sum_{\eta=1}^d X_\eta{*}Z^\eta$

is reasonable notation for the indefinite integral of $\boldsymbol{X}$ against $d\boldsymbol{Z}$; the right-hand side is a càdlàg process unique up to indistinguishability and satisfies $(\boldsymbol{X}{*}\boldsymbol{Z})_T \in \int_0^T \boldsymbol{X}\ d\boldsymbol{Z} \quad \forall\, T \in \mathfrak{T}$. Henceforth $\int_0^T \boldsymbol{X}d\boldsymbol{Z}$ means the random variable $(\boldsymbol{X}{*}\boldsymbol{Z})_T$.

Exercise 3.7.7 Define $\lceil \boldsymbol{Z} \rceil_{\mathcal{I}^p}$ and show that $\lceil \boldsymbol{Z} \rceil_{\mathcal{I}^p} = \sup\{ \lceil \boldsymbol{X}{*}\boldsymbol{Z} \rceil_{\mathcal{I}^p} : \boldsymbol{X} \in \mathcal{E}_1^d \}$.

Exercise 3.7.8 Suppose we are faced with a whole collection $\mathfrak{P}$ of probabilities, the filtration $\mathcal{F}_.$ is right-continuous, and Z is an $L^0(\mathbb{P})$-integrator for every $\mathbb{P} \in \mathfrak{P}$. Let G be a predictable process that is locally Z–0;$\mathbb{P}$-integrable for every $\mathbb{P} \in \mathfrak{P}$. There is a right-continuous process $G{*}Z$ with left limits, adapted to the $\mathfrak{P}$***-regularization*** $\mathcal{F}_{.}^{\mathfrak{P}} \stackrel{\text{def}}{=} \bigcap_{\mathbb{P}\in\mathfrak{P}} \mathcal{F}_{.}^{\mathbb{P}}$, that is an indefinite integral in the sense of $L^0(\mathbb{P})$ for every $\mathbb{P} \in \mathfrak{P}$.

Exercise 3.7.9 If M is a right-continuous local martingale and G is locally M–1-integrable (see corollary 2.5.29), then $G{*}M$ is a local martingale.

Integration Theory of the Indefinite Integral

If a measure dy on $[0,\infty)$ has a density with respect to the measure dz, say $dy_t = g_t\,dz_t$, then a function f is dy-negligible (-integrable, -measurable) if and only if the product fg is dz-negligible (-integrable, -measurable). The corresponding statements are true in the stochastic case:

Theorem 3.7.10 *Let Z be an L^p-integrator, $p \in [0,\infty)$, and G a Z–p-integrable process. Then for all processes F*

$$\lceil\!\lceil F \rceil\!\rceil^*_{(G*Z)\text{–}p} = \lceil\!\lceil F \cdot G \rceil\!\rceil^*_{Z\text{–}p} \quad \textit{and} \quad \lceil G{*}Z \rceil_{\mathcal{I}^p} = \lceil\!\lceil G \rceil\!\rceil^*_{Z\text{–}p} \,. \tag{3.7.5}$$

Therefore a process F is $(G{*}Z)$–p-negligible (-integrable, -measurable) if and only if $F{\cdot}G$ is Z–p-negligible (-integrable, -measurable). If F is locally $(G{*}Z)$–p-integrable, then

$$F{*}(G{*}Z) = (FG){*}Z \;,$$

in particular

$$\int F\, d(G{*}Z) \doteq \int F \cdot G\, dZ$$

when F is $(G{*}Z)$–p-integrable.

Proof. Let $Y = G{*}Z$ denote the indefinite integral. The family of bounded processes X with $\int X\ dY = \int XG\ dZ$ contains $\mathcal{E}$ (equation (3.7.4) on page 133) and is closed under pointwise limits of bounded sequences. It contains therefore the family $\mathcal{P}_b$ of all bounded predictable processes. The assignment $F \mapsto \lceil\!\lceil FG \rceil\!\rceil^*_{Z\text{–}p}$ is easily seen to be a mean: properties (i) and (iii) of definition 3.2.1 on page 94 are trivially satisfied, (ii) follows from proposition 3.6.5, (iv) from the Dominated Convergence Theorem, and (v) from exercise 3.2.15. If F is predictable, then, due to corollary 3.6.10,

$$\begin{aligned}
\lceil\!\lceil F \rceil\!\rceil^*_{Y\text{–}p} &= \sup\Big\{ \Big\lceil\!\Big\lceil \int X\ dY \Big\rceil\!\Big\rceil_p :\ X \in \mathcal{P}_b\,,\ |X| \le |F| \Big\} \\
&= \sup\Big\{ \Big\lceil\!\Big\lceil \int XG\ dZ \Big\rceil\!\Big\rceil_p :\ X \in \mathcal{P}_b\,,\ |X| \le |F| \Big\} \\
&= \sup\Big\{ \Big\lceil\!\Big\lceil \int X'\ dZ \Big\rceil\!\Big\rceil_p :\ X' \in \mathcal{P}_b\,,\ |X'| \le |FG| \Big\} \\
&= \lceil\!\lceil FG \rceil\!\rceil^*_{Z\text{–}p} \,.
\end{aligned}$$

The maximality of Daniell's mean (proposition 3.6.1 on page 123) gives $\lceil\!\lceil FG \rceil\!\rceil^*_{Z\text{–}p} \le \lceil\!\lceil F \rceil\!\rceil^*_{Y\text{–}p}$ for all F. For the converse inequality let $\widetilde{F}$ be a predictable Y–p-envelope of $F \ge 0$ (proposition 3.6.6). Then

$$\lceil\!\lceil F \rceil\!\rceil^*_{Y\text{–}p} = \Big\lceil\!\Big\lceil \widetilde{F} \Big\rceil\!\Big\rceil^*_{Y\text{–}p} = \Big\lceil\!\Big\lceil \widetilde{F}G \Big\rceil\!\Big\rceil^*_{Z\text{–}p} \ge \lceil\!\lceil FG \rceil\!\rceil^*_{Z\text{–}p} \,.$$

This proves equation (3.7.5). The second claim is evident from this identity. The equality of the integrals in the last line holds for elementary integrands (equation (3.7.4)) and extends to Y–p-integrable processes by approximation in mean: if $\mathcal{E} \ni X^{(n)} \to F$ in $[\![\ ]\!]^*_{Y-p}$-mean, then $\mathcal{E} \ni X^{(n)} \cdot G \to F \cdot G$ in $[\![\ ]\!]^*_{Z-p}$-mean and so

$$\int F \, dY \doteq \lim \int X^{(n)} \, dY \doteq \lim \int X^{(n)} \cdot G \, dZ \doteq \int F \cdot G \, dZ$$

in the topology of L^p. We apply this to the processes[6] $F \cdot [\![0, t]\!]$ and find that $F*(G*Z)$ and $(FG)*Z$ are modifications of each other. Being right-continuous and adapted, they are indistinguishable (exercise 1.3.28). ∎

Corollary 3.7.11 *Let $G^{(n)}$ and G be locally Z–0-integrable processes, T_k finite stopping times increasing to ∞, and assume that*

$$\left[\!\!\left[G - G^{(n)} \right]\!\!\right]^*_{Z^{T_k-0}} \xrightarrow[n\to\infty]{} 0$$

for every $k \in \mathbb{N}$. Then the paths of $G^{(n)}*Z$ converge to the paths of $G*Z$ uniformly on bounded intervals, in probability. There are a subsequence $\left(G^{(n_k)}\right)$ and a nearly empty set N outside which the path of $G^{(n_k)}*Z$ converges uniformly on bounded intervals to the path of $G*Z$.

Proof. By lemma 2.3.2 and equation (3.7.5),

$$\begin{aligned} \delta_k^{(n)}(\lambda) &\stackrel{\text{def}}{=} \mathbb{P}\Big[\big(G*Z - G^{(n)}*Z\big)^\star_{T_k} > \lambda\Big] = \mathbb{P}\Big[\big((G - G^{(n)})*Z\big)^\star_{T_k} > \lambda\Big] \\ &\le \left\lceil \lambda^{-1}\big((G - G^{(n)})*Z\big)^{T_k} \right\rceil_{\mathcal{I}^0} = \left[\!\!\left[\lambda^{-1}\big(G - G^{(n)}\big) \right]\!\!\right]^*_{Z^{T_k-0}} \xrightarrow[n\to\infty]{} 0 \,. \end{aligned}$$

We take a subsequence $\left(G^{(n_k)}\right)$ so that $\delta_k^{(n_k)}(2^{-k}) \le 2^{-k}$ and set

$$N \stackrel{\text{def}}{=} \limsup_k \Big[\big(G*Z - G^{(n_k)}*Z\big)^\star_{T_k} > 2^{-k} \Big] \,.$$

This set belongs to $\mathcal{A}_{\infty\sigma}$ and by the Borel–Cantelli lemma is negligible: it is nearly empty. If $\omega \notin N$, then the path $\left(G^{(n_k)}*Z\right)_{\bullet}(\omega)$ converges evidently to the path $(G*Z)_{\bullet}(\omega)$ uniformly on every one of the intervals $[0, T_k(\omega)]$. ∎

If $X \in \mathcal{E}$, then $X*Z$ can jump only where Z jumps. Therefore:

Corollary 3.7.12 *If Z has continuous paths, then every indefinite integral $G*Z$ has a modification with continuous paths – which will then of course be chosen.*

Corollary 3.7.13 *Let A be a subset of Ω, not necessarily measurable, and assume the paths of the locally Z–0-integrable process G vanish almost surely on A. Then the paths of $G*Z$ also vanish almost surely, in fact nearly, on A.*

Proof. The set $[0,\infty) \times A \subset \boldsymbol{B}$ is by equation (3.7.5) $(G{*}Z)$-negligible. Corollary 3.6.7 says that the paths of $G{*}Z$ nearly vanish on A. ∎

Exercise 3.7.14 If G is Z–0-integrable, then $\lceil G{*}Z \rceil_{[\alpha]} = \|G\|^*_{Z\text{–}[\alpha]}$ for $\alpha > 0$.

Exercise 3.7.15 For any locally Z–0-integrable G and any almost surely finite stopping time T the processes $G{*}Z^T$ and $(G{*}Z)^T$ are indistinguishable.

Exercise 3.7.16 (P.–A. Meyer) Let Z, Z' be L^0-integrators and X, X' processes that are integrable for both. Let Ω_0 be a subset of Ω and $T : \Omega \to \mathbb{R}_+$ a time, neither of them necessarily measurable. If $X = X'$ and $Z = Z'$ up to and including (excluding) time T on Ω_0, then $X{*}Z = X'{*}Z'$ up to and including (excluding) time T on Ω_0, except possibly on an evanescent set.

A General Integrability Criterion

Theorem 3.7.17 *Let Z be an L^0-integrator, T an almost surely finite stopping time, and X a Z-measurable process. If $X^\star_T$ is almost surely finite, then X is Z–0-integrable on $[\![0,T]\!]$.*

This says – to put it plainly if a bit too strongly – that any reasonable process is Z–0-integrable. The assumptions concerning the integrand are often easy to check: X is usually given as a construct using algebraic and order combinations and limits of processes known to be Z-measurable, so the splendid permanence properties of measurability will make it obvious that X is Z-measurable; frequently it is also evident from inspection that the maximal process $X^\star$ is almost surely finite at any instant, and thus at any almost surely finite stopping time. In cases where the checks are that easy we shall not carry them out in detail but simply write down the integral without fear. That is the point of this theorem.

Proof. Let $\epsilon > 0$. Since $\left[X^\star_T \le K\right] \uparrow \Omega$ almost surely as $K \uparrow \infty$, and since outer measure $\mathbb{P}^*$ is continuous along increasing sequences, there is a number K with $\mathbb{P}^*\left[X^\star_T \le K\right] > 1 - \epsilon$. Write $X' = X \cdot [\![0,T]\!]$ and[1]

$$X' = X' \cdot \left[|X| \le K\right] + X' \cdot \left[|X| > K\right] = X^{(1)} + X^{(2)} .$$

Now Z^T is a global L^0-integrator, and so $X^{(1)}$ is Z^T–0-integrable, even Z–0-integrable (exercise 3.6.13). As to $X^{(2)}$, it is Z-measurable and its entire path vanishes on the set $\left[X^\star_T \le K\right]$. If Y is a process in $\mathcal{P}_{00}$ with $|Y| \le |X^{(2)}|$, then its entire path also vanishes on this set, and thanks to corollary 3.7.13 so does the path of $Y{*}Z$, at least almost surely. In particular, $\int Y\, dZ = 0$ almost surely on $\left[X^\star_T \le K\right]$. Thus $B \overset{\text{def}}{=} \left[\int Y\, dZ \ne 0\right]$ is a measurable set almost surely disjoint from $\left[X^\star_T \le K\right]$. Hence $\mathbb{P}[B] \le \epsilon$ and $\left\lceil\!\left\lceil \int Y\, dZ \right\rceil\!\right\rceil_0 \le \epsilon$. Corollary 3.6.10 shows that $\left\lceil\!\left\lceil X^{(2)} \right\rceil\!\right\rceil^*_{Z\text{–}0} \le \epsilon$. That is, X' differs from the Z–0-integrable process $X^{(1)}$ arbitrarily little in Z–0-mean and therefore is Z–0-integrable itself. That is to say, X is indeed Z–0-integrable on $[\![0,T]\!]$. ∎

Exercise 3.7.18 Suppose that F is a process whose paths all vanish outside a set $\Omega_0 \subset \Omega$ with $\mathbb{P}^*(\Omega_0) < \epsilon$. Then $\lceil\!\lceil F \rceil\!\rceil^*_{Z\text{–}0} < \epsilon$.

Exercise 3.7.19 If Z is previsible and $T \in \mathfrak{T}$, then the stopped process Z^T is previsible. If Z is a previsible integrator and X a Z–0-integrable process, then $X{*}Z$ is previsible.

Exercise 3.7.20 Let Z be an L^0-integrator and S, T two stopping times. (i) If G is a process Z–0-integrable on $(\!(S,T]\!]$ and $f \in L^0(\mathcal{F}_S, \mathbb{P})$, then the process $f \cdot G$ is Z–0-integrable on $(\!(S,T]\!]$

and
$$\int_{S+}^{T} f \cdot G\, dZ = f \cdot \int_{S+}^{T} G\, dZ \ \in \mathcal{F}_T \quad \text{a.s.}$$

Also, for $f \in L^0(\mathcal{F}_0, \mathbb{P})$, $\displaystyle\int_0^0 f\, dZ = \int_{[\![0}^{0]\!]} f\, dZ = f \cdot Z_0$.

(ii) If G is Z–0-integrable on $[\![0,T]\!]$, S is predictable, and f is measurable on the strict past of S and almost surely finite, then $f \cdot G$ is Z–0-integrable on $[\![S,T]\!]$, and

$$\int_S^T f \cdot G\, dZ = f \cdot \int_S^T G\, dZ . \tag{3.7.6}$$

(iii) Let (S_k) be a sequence of finite stopping times that increases to ∞ and f_k almost surely finite random variables measurable on $\mathcal{F}_{S_k}$. Then $G \stackrel{\text{def}}{=} \sum_k f_k \cdot (\!(S_k, S_{k+1}]\!]$ is locally Z–0-integrable, and its indefinite integral is given by

$$(G{*}Z)_t \doteq \textstyle\sum_k f_k \cdot \Big(Z^t_{S_{k+1}} - Z^t_{S_k} \Big) = \sum_k f_k \cdot \Big(Z_t^{S_{k+1}} - Z_t^{S_k} \Big) .$$

Exercise 3.7.21 Suppose Z is an L^p-integrator for some $p \in [0, \infty)$, and $X, X^{(n)}$ are previsible processes Z–p-integrable on $[\![0,T]\!]$ and such that $|X^{(n)} - X|^\star_T \xrightarrow[n\to\infty]{} 0$ in probability. Then X is Z–p-integrable on $[\![0,T]\!]$, and $|X^{(n)}{*}Z - X{*}Z|^\star_T \xrightarrow[n\to\infty]{} 0$ in L^p-mean (cf. [92]).

Approximation of the Integral via Partitions

The Lebesgue integral of a càglàd integrand, being a Riemann integral, can be approximated via partitions. So can the stochastic integral:

Definition 3.7.22 *A* ***stochastic partition*** *or* ***random partition*** *of the stochastic interval* $[\![0, \infty)\!)$ *is a finite or countable collection*

$$\mathcal{S} = \{0 = S_0 \le S_1 \le S_2 \le \ldots \le S_\infty \le \infty\}$$

of stopping times. $\mathcal{S}$ *is assumed to contain the stopping time* $S_\infty \stackrel{\text{def}}{=} \sup_k S_k$ *– which is no assumption at all when* $\mathcal{S}$ *is finite or* $S_\infty = \infty$*. It simplifies the notation in some formulas to set* $S_{\infty+1} \stackrel{\text{def}}{=} \infty$*. We say that the random partition* $\mathcal{T} = \{0 = T_0 \le T_1 \le T_2 \le \ldots \le T_\infty \le \infty\}$ ***refines*** $\mathcal{S}$ *if*

$$\bigcup\{[\![S]\!] : S \in \mathcal{S}\} \subseteq \bigcup\{[\![T]\!] : T \in \mathcal{T}\} .$$

The ***mesh*** *of* $\mathcal{S}$ *is the (non-adapted) process* mesh$[\mathcal{S}]$ *that at* $\varpi = (\omega, s) \in \boldsymbol{B}$ *has the value*

$$\inf\big\{\overline{\rho}\big(S(\omega), S'(\omega)\big) : S \le S' \text{ in } \mathcal{S} ,\ S(\omega) < s \le S'(\omega)\big\} .$$

*Here $\overline{\rho}$ is the arctan metric on $\overline{\mathbb{R}}_+$ (see item A.1.2 on page 363). With the random partition $\mathcal{S}$ and the process $Z \in \mathfrak{D}$ goes the **$\mathcal{S}$-scalæfication***

$$Z^{\mathcal{S}} \stackrel{\text{def}}{=} \sum_{0\le k\le\infty} Z_{S_k}\cdot[\![S_k, S_{k+1})\!) \stackrel{\text{def}}{=} \sum_{0\le k<\infty} Z_{S_k}\cdot[\![S_k, S_{k+1})\!) + Z_{S_\infty}\cdot[\![S_\infty,\infty)\!) ,$$

defined so as to produce again a process $Z^{\mathcal{S}} \in \mathfrak{D}$. ∎

The left-continuous version of the scalæfication of $X \in \mathfrak{D}$ evidently is

$$X_{.-}^{\mathcal{S}} = \sum\nolimits_{0\le k\le\infty} X_{S_k}\cdot(\!(S_k, S_{k+1}]\!]$$

with indefinite dZ-integral

$$\big(X_{.-}^{\mathcal{S}} {*} Z\big)_t = \sum\nolimits_{0\le k\le\infty} X_{S_k}\cdot\big(Z_t^{S_{k+1}} - Z_t^{S_k}\big) = \int_0^t X_{.-}^{\mathcal{S}}\, dZ . \tag{3.7.7}$$

Theorem 3.7.23 *Let $\mathcal{S}^n = \{0 = S_0^n \le S_1^n \le S_2^n \le \ldots \le \infty\}$ be a sequence of random partitions of $\boldsymbol{B}$ such that* mesh$[\mathcal{S}^n] \xrightarrow[n\to\infty]{} 0$ *except possibly on an evanescent set.*[8] *Assume that Z is an L^p-integrator for some $p \in [0,\infty)$, that $X \in \mathfrak{D}$, and that the maximal process of $X_{.-} \in \mathfrak{L}$ is Z–p-integrable on every interval $[\![0,u]\!]$ – recall from theorem 3.7.17 that this is automatic when $p = 0$. The indefinite integrals $X_{.-}^{\mathcal{S}^n} {*} Z$ of (3.7.7) then approximate the indefinite integral $X_{.-}{*}Z$ uniformly in p-mean, in the sense that for every instant $u < \infty$*

$$\left[\!\!\left[\, |X_{.-}{*}Z - X_{.-}^{\mathcal{S}^n}{*}Z|_u^\star \right]\!\!\right]_p \xrightarrow[n\to\infty]{} 0 . \tag{3.7.8}$$

Proof. Due to the left-continuity of $X_{.-}$, we evidently have

$$X_{.-}^{\mathcal{S}^n} \xrightarrow[n\to\infty]{} X_{.-} \quad \text{pointwise on } \boldsymbol{B}. \tag{3.7.9}$$

Also $\qquad |X_{.-}^{\mathcal{S}^n} - X_{.-}|\cdot[\![0,u]\!] \le 2|X_{.-}|_u^\star\cdot[\![0,u]\!] \in \mathfrak{L}^1[\lceil\!\lceil\ \rceil\!\rceil_{Z\text{–}p}^*]$,

so the Dominated Convergence Theorem gives $\lceil\!\lceil\, |X_{.-}^{\mathcal{S}^n} - X_{.-}|\cdot[\![0,u]\!] \,\rceil\!\rceil_{Z\text{–}p}^* \xrightarrow[n\to\infty]{} 0$. An application of the maximal theorem 2.3.6 on page 63 leads to

$$\left\| \, |X_{.-}^{\mathcal{S}^n}{*}Z - X_{.-}{*}Z|_u^\star \right\|_p \le C_p^{\star(2.3.5)} \cdot \left\lceil (X_{.-}^{\mathcal{S}^n} - X_{.-}){*}Z^u \right\rceil_{\mathcal{I}^p}$$

by theorem 3.7.10:
$$\le C_p^\star \cdot \left\| \, |X_{.-}^{\mathcal{S}^n} - X_{.-}|\cdot[\![0,u]\!] \right\|_{Z\text{–}p}^* \xrightarrow[n\to\infty]{} 0 .$$

This proves the claim for $p > 0$. The case $p = 0$ is similar. ∎

Note that equation (3.7.8) does not permit us to conclude that the approximants $X_{.-}^{\mathcal{S}^n}{*}Z$ converge to $X_{.-}{*}Z$ almost surely; for that, one has to choose the partitions $\mathcal{S}^n$ so that the convergence in equation (3.7.9) becomes uniform (theorem 3.7.26); even $\sup_{\varpi\in\boldsymbol{B}} \text{mesh}[\mathcal{S}^n](\varpi) \xrightarrow[n\to\infty]{} 0$ does not guarantee that.

[8] A partition $\mathcal{S}$ is assumed to contain $S_\infty \stackrel{\text{def}}{=} \sup_{k<\infty} S_k$, and defining $S_{\infty+1} \stackrel{\text{def}}{=} \infty$ simplifies some formulas.

Exercise 3.7.24 For every $X \in \mathfrak{D}$ and $(\mathcal{S}^n)$ there exists a subsequence $(\mathcal{S}^{n_k})$ that depends on Z and is impossible to find in practice, so that $X^{\mathcal{S}^{n_k}}_{.-}{*}Z \to X_{.-}{*}Z$ almost surely uniformly on bounded intervals.

Exercise 3.7.25 Any two stochastic partitions have a common refinement.

Pathwise Computation of the Indefinite Integral

The discussion of chapter 1, in particular theorem 1.2.8 on page 17, seems to destroy any hope that $\int G\,dZ$ can be understood pathwise, not even when both the integrand G and the integrator Z are nice and continuous, say. On the other hand, there are indications that the path of the indefinite integral $G{*}Z$ is determined to a large extent by the paths of G and Z alone: this is certainly true if G is elementary, and corollary 3.7.11 seems to say that this is still so almost surely if G is any integrable process; and exercise 3.7.16 seems to say the same thing in a different way.

There is, in fact, an *algorithm* implementable on an (ideal) computer that takes the paths $s \mapsto X_s(\omega)$ and $s \mapsto Z_s(\omega)$ and computes from them an approximate path $s \mapsto Y^{(\delta)}_s(\omega)$ of the indefinite integral $X_{.-}{*}Z$. If the parameter $\delta > 0$ is taken through a sequence (δ_n) that converges to zero sufficiently fast, the approximate paths $Y^{(\delta_n)}_{.}(\omega)$ converge uniformly on every finite stochastic interval to the path $(X_{.-}{*}Z)_{.}(\omega)$ of the indefinite integral. Moreover, the rate of convergence can be estimated.

This applies only to certain integrands, so let us be precise about the data. The filtration is assumed right-continuous. $\mathbb{P}$ is a fixed probability on $\mathcal{F}_\infty$, and Z is a right-continuous $L^0(\mathbb{P})$-integrator. As to the integrand, it equals the left-continuous version $X_{.-}$ of some real-valued càdlàg adapted process X; its value at time 0 is 0. The integrand might be the left-continuous version of a continuous function of some integrator, for a typical example. Such a process is adapted and left-continuous, hence predictable (proposition 3.5.2). Since its maximal function is finite at any instant, it is locally Z–0-integrable (theorem 3.7.17). Here is the typical approximate $Y^{(\delta)}$ to the indefinite integral $Y = X_{.-}{*}Z$: fix a threshold $\delta > 0$. Set $S_0 \stackrel{\text{def}}{=} 0$ and $Y^{(\delta)}_0 \stackrel{\text{def}}{=} 0$; then proceed recursively with

$$S_{k+1} \stackrel{\text{def}}{=} \inf\left\{t > S_k : \left|X_t - X_{S_k}\right| > \delta\right\} \tag{3.7.10}$$

and
$$Y^{(\delta)}_t \stackrel{\text{def}}{=} Y_{S_k} + X_{S_k}\cdot(Z_t - Z_{S_k}) \qquad \text{for} \quad S_k < t \le S_{k+1}$$

by induction:
$$= \sum_{\kappa=1}^{k} X_{S_\kappa}\cdot\Big(Z^t_{S_{\kappa+1}} - Z^t_{S_\kappa}\Big)\ . \tag{3.7.11}$$

In other words, the prescription is this: wait until the *change in the integrand* warrants a new computation, then do a linear approximation – the scheme above is an ***adaptive Riemann-sum scheme***.[9] Another way of looking at it is to note that (3.7.10) defines a stochastic partition $\mathcal{S} = \mathcal{S}^\delta$ and that by equation (3.7.7) the process $Y^{(\delta)}_{\bullet}$ is but the indefinite dZ-integral of $X^{\mathcal{S}}_{\bullet-}$.

The algorithm (3.7.10)–(3.7.11) converges pathwise provided δ is taken through a sequence (δ_n) that converges sufficiently quickly to zero:

Theorem 3.7.26 *Choose numbers $\delta_n > 0$ so that*

$$\sum_{n=1}^{\infty} \lceil n\delta_n \cdot Z^n \rceil_{\mathcal{I}^0} < \infty \tag{3.7.12}$$

(Z^n is Z stopped at the instant n). If X is any adapted càdlàg process, then $X_{\bullet-}$ is locally Z–0-integrable; and for nearly all $\omega \in \Omega$ the approximates $Y^{(\delta_n)}_{\bullet}(\omega)$ of (3.7.11) converge to the indefinite integral $(X_{\bullet-}{*}Z)_{\bullet}(\omega)$ uniformly on any finite interval, as $n \to \infty$.

Remarks 3.7.27 (i) The sequence (δ_n) depends only on the integrator sizes of the stopped processes Z^n. For instance, if $\lceil Z \rceil_{\mathcal{I}^p} < \infty$ for some $p > 0$, then the choice $\delta_n \stackrel{\text{def}}{=} n^{-q}$ will do as long as $q > 1 + 1 \vee 1/p$.

The algorithm (3.7.10)–(3.7.11) can be viewed as a black box – not hard to write as a program on a computer once the numbers δ_n are fixed – that takes two inputs and yields one output. One of the inputs is a path $Z_{\bullet}(\omega)$ of any integrator Z satisfying inequality (3.7.12); the other is a path $X_{\bullet}(\omega)$ of any $X \in \mathfrak{D}$. Its output is the path $(X_{\bullet-}{*}Z)_{\bullet}(\omega)$ of the indefinite integral – where the algorithm does not converge have the box produce the zero path.

(ii) Suppose we are not sure which probability $\mathbb{P}$ is pertinent and are faced with a whole collection $\mathfrak{P}$ of them. If the size of the integrator Z is bounded independently of $\mathbb{P} \in \mathfrak{P}$ in the sense that

$$f(\lambda) \stackrel{\text{def}}{=} \sup_{\mathbb{P} \in \mathfrak{P}} \lceil \lambda \cdot Z \rceil_{\mathcal{I}^p[\mathbb{P}]} \xrightarrow[\lambda \to 0]{} 0 , \tag{3.7.13}$$

then we choose δ_n so that $\sum_n f(n\delta_n) < \infty$. The proof of theorem 3.7.26 shows that the set where the algorithm (3.7.11) does not converge belongs to $\mathcal{A}_{\infty\sigma}$ and is negligible for all $\mathbb{P} \in \mathfrak{P}$ simultaneously, and that the limit is $X_{\bullet-}{*}Z$, understood as an indefinite integral in the sense $L^0(\mathbb{P})$ for all $\mathbb{P} \in \mathfrak{P}$.

(iii) Assume (3.7.13). By representing (X, Z) on canonical path space $\mathscr{D}^2$ (item 2.3.11), we can produce a ***universal integral***. This is a bilinear map $\mathscr{D} \times \mathscr{D} \to \mathscr{D}$, adapted to the canonical filtrations and written as a binary operation $_{\bullet-}\circledast$, such that $X_{\bullet-}{*}Z$ is but the composition $X_{\bullet-}\circledast Z$ of (X, Z) with this operation. We leave the details as an exercise.

[9] This is of course what one should do when computing the Riemann integral $\int_\eta^\theta f(x)\,dx$ for a continuous integrand f that for lack of smoothness does not lend itself to a simplex method or any other method whose error control involves derivatives: chopping the x-axis into lots of little pieces as one is ordinarily taught in calculus merely incurs round-off errors when f is constant or varies slowly over long stretches.

(iv) The theorem also shows that the problem about the "meaning in mean" of the stochastic differential equation (1.1.5) raised on page 6 is really no problem at all: the stochastic integral appearing in (1.1.5) can be read as a ***pathwise***[10] integral, provided we do not insist on understanding it as a Lebesgue–Stieltjes integral but rather as defined by the limit of the algorithm (3.7.11) – which surely meets everyone's intuitive needs for an integral[11] – and provided the integrand $b(X)$ belongs to $\mathfrak{L}$.

(v) Another way of putting this point is this. Suppose we are, as is the case in the context of stochastic differential equations, only interested in stochastic integrals of integrands in $\mathfrak{L}$; such are on $(\!(0,\infty)\!)$ the left-continuous versions $X_{.-}$ of càdlàg processes $X \in \mathfrak{D}$. Then the limit of the algorithm (3.7.11) serves as a perfectly intuitive *definition* of the integral. From this point of view one might say that the definition 2.1.7 on page 49 of an integrator serves merely to identify the conditions[12] under which this limit exists and defines an integral with decent limit properties. It would be interesting to have a proof of this that does not invoke the whole machinery developed so far.

Proof of Theorem 3.7.26. Since the filtration is right-continuous, one sees recursively that the S_k are stopping times (exercise 1.3.30). They increase strictly with k and their limit is ∞. For on the set $\left[\sup_k S_k < \infty\right]$ X must have an oscillatory discontinuity or be unbounded, which is ruled out by the assumption that $X \in \mathfrak{D}$: this set is void. The key to all further arguments is the observation that $Y^{(\delta)}$ is nothing but the indefinite integral of[6]

$$X_{.-}^{\mathcal{S}} = \sum_{k=0}^{\infty} X_{S_k} \cdot (\!(S_k, S_{k+1}]\!] ,$$

with $\mathcal{S}$ denoting the partition $\{0 = S_0 \le S_1 \le \cdots\}$. This is a predictable process (see proposition 3.5.2), and in view of exercise 3.7.20

$$Y^{(\delta)} = X_{.-}^{\mathcal{S}} * Z .$$

The very construction of the stopping times S_k is such that $X_{.-}$ and $X_{.-}^{\mathcal{S}}$ differ uniformly by less than δ. The indefinite integral $X_{.-}*Z$ may not exist in the sense Z–p if $p > 0$, but it does exist in the sense Z–0, since the maximal process of an $X_{.-} \in \mathfrak{L}$ is finite at any finite instant t (use theorem 3.7.17). There is an immediate estimate of the difference $X_{.-}*Z - Y^{(\delta)}$. Namely, let U be any finite stopping time. If $X_{.-} \cdot [\![0, U]\!]$ is Z–p-integrable for some $p \in [0, \infty)$, then the maximal difference of the indefinite integral $X_{.-}*Z$ from $Y^{(\delta)}$ can be estimated as follows:

$$\mathbb{P}\left[\left|X_{.-}*Z - Y^{(\delta)}\right|_U^\star > \lambda\right] = \mathbb{P}\left[\left|(X_{.-} - X_{.-}^{\mathcal{S}})*Z\right|_U^\star > \lambda\right]$$

[10] That is, computed separately for every single path $t \mapsto \big(X_t(\omega), Z_t(\omega)\big)$, $\omega \in \Omega$.
[11] See, however, pages 168–171 and 310 for further discussion of this point.
[12] They are (RC-0) and (B-p), ibidem.

by lemma 2.3.2: $$\leq \left\lceil \frac{1}{\lambda} \cdot (X_{.-} - X^{\mathcal{S}}_{.-}) * Z^U \right\rceil_{\mathcal{I}^p}$$

using (3.7.5) twice: $$\leq \left\lceil \frac{\delta}{\lambda} \cdot Z^U \right\rceil_{\mathcal{I}^p} . \tag{3.7.14}$$

At $p = 0$ inequality (3.7.14) has the consequence

$$\mathbb{P}\left[\left| X_{.-} * Z - Y^{(\delta_n)} \right|^\star_u > 1/n \right] \leq \lceil n\delta_n \cdot Z^n \rceil_{\mathcal{I}^0} , \qquad n \geq u .$$

Since the right-hand side is summable over n by virtue of the choice (3.7.12) of the δ_n, the Borel-Cantelli lemma yields at any instant u

$$\mathbb{P}\Big[\limsup_{n \to \infty} \left| X_{.-} * Z - Y^{(\delta_n)} \right|^\star_u > 0 \Big] = 0 . \qquad \blacksquare$$

Remark 3.7.28 The proof shows that the particular definition of the S_k in equation (3.7.10) is not important. What is needed is that $X_{.-}$ differ from X_{S_k} by less than δ on $(\!(S_k, S_{k+1}]\!]$ and of course that $\lim_k S_k = \infty$; and (3.7.10) is one way to obtain such S_k. We might, for instance, be confronted with several L^0-integrators $Z^1, \ldots, Z^d$ and left-continuous integrands $X_{1.-}, \ldots, X_{d.-}$. In that case we set $S_0 = 0$ and continue recursively by

$$S_{k+1} = \inf \Big\{ t > S_k : \sup_{1 \leq \eta \leq d} \left| X_{\eta t} - X_{\eta S_k} \right| > \delta \Big\}$$

and choose the δ_n so that $\sup_\eta \sum_{n=1}^\infty \lceil n\delta_n \cdot (Z^\eta)^n \rceil_{\mathcal{I}^0} < \infty$. Equation (3.7.10) then defines a black box that computes the integrals $X^\eta_{.-} * Z^\eta$ pathwise simultaneously for all $\eta \in \{1, \ldots, d\}$, and thus computes $\boldsymbol{X}_{.-} * \boldsymbol{Z} = \sum_\eta X_{\eta .-} * Z^\eta$.

Exercise 3.7.29 Suppose Z is a global L^0-integrator and the δ_n are chosen so that $\sum_n \lceil n\delta_n \cdot Z \rceil_{\mathcal{I}^0}$ is finite. If $X_{.-} \in \mathfrak{L}$ is Z–0-integrable, then the approximate path $Y^{(\delta_n)}_{.}(\omega)$ of (3.7.11) converges to the path of the indefinite integral $(X_{.-} * Z)_{.}(\omega)$ uniformly on $[0, \infty)$, for almost all $\omega \in \Omega$.

Exercise 3.7.30 We know from theorem 2.3.6 that $\| Z^\star_\infty \|_p \leq C_p^{\star(2.3.5)} \lceil Z \rceil_{\mathcal{I}^p}$. This can be used to establish the following strong version of the weak-type inequality (3.7.14), which is useful only when Z is $\mathcal{I}^p$-bounded on $[\![0, U]\!]$:

$$\Big\| \left| X_{.-} * Z - Y^{(\delta)} \right|^\star_U \Big\|_p \leq \delta C_p^\star \cdot \lceil Z^U \rceil_{\mathcal{I}^p} , \qquad 0 < p < \infty.$$

Exercise 3.7.31 The rate at which the algorithm (3.7.11) converges as $\delta \to 0$ does not depend on the integrand $X_{.-}$ and depends on the integrator Z only through the function $\lambda \mapsto \lceil \lambda Z \rceil_{\mathcal{I}^p}$. Suppose Z is an L^p-integrator for some $p > 0$, let U be a stopping time, and suppose $X_{.-}$ is a priori known to be Z–p-integrable on $[\![0, U]\!]$. (i) With δ as in (3.7.11) derive the confidence estimate

$$\mathbb{P}\left[\sup_{0 \leq s \leq U} |X_{.-} * Z - Y^{(\delta)}|_s > \lambda \right] \leq \left[\frac{\delta}{\lambda} \right]^{p \wedge 1} \cdot \lceil Z^U \rceil_{\mathcal{I}^p} .$$

How must δ be chosen, if with probability 0.9 the error is to be less than 0.05 units?

(ii) If the integrand $X_{.-}$ varies furiously, then the loop (3.7.10)–(3.7.11) inside our black box is run through very often, even when δ is moderately large, and round-off errors accumulate. It may even occur that the stopping times S_k follow each other so fast that the physical implementation of the loop cannot keep up. It is desirable to have an estimate of the number $N(U)$ of calculations needed before a given ultimate time U of interest is reached. Now, rather frequently the integrand $X_{.-}$ comes as follows: there are an L^q-integrator X' and a Lipschitz function[13] Φ such that $X_{.-}$ is the left-continuous version of $\Phi(X')$. In that case there is a simple estimate for $N(U)$: with $c_q = 1$ for $q \geq 2$ and $c_q \leq 2.00075$ for $0 < q < 2$,

$$\mathbb{P}[N(U) > K] \leq c_q \left(\frac{L \,¦ X'^U ¦_{\mathcal{I}^q}}{\delta \sqrt{K}} \right)^q .$$

Exercise 3.7.32 Let Z be an L^0-integrator. (i) If Z has continuous paths and X is Z–0-integrable, then $X{*}Z$ has continuous paths. (ii) If X is the uniform limit of elementary integrands, then $\Delta(X{*}Z) = X \cdot \Delta Z$. (iii) If $X \in \mathfrak{L}$, then $\Delta(X{*}Z) = X \cdot \Delta Z$. (See proposition 3.8.21 for a more general statement.)

Integrators of Finite Variation

Suppose our L^0-integrator Z is a process V of finite variation. Surely our faith in the merit of the stochastic integral would increase if in this case it were the same as the ordinary Lebesgue–Stieltjes integral computed path-by-path. In other words, we hope that for all instants t

$$\big(X{*}V\big)_t(\omega) = LS\!-\!\!\int_0^t X_s(\omega)\, dV_s(\omega) , \tag{3.7.15}$$

at least almost surely. Since both sides of the equation are right-continuous and adapted, $X{*}Z$ would then in fact be indistinguishable from the indefinite Lebesgue–Stieltjes integral.

There is of course no hope that equation (3.7.15) will be true for all integrands X. The left-hand side is only defined if X is locally V–0-integrable and thus "somewhat non-anticipating." And it also may happen that the left-hand side is defined but the right-hand side is not. The obstacle is that for the Lebesgue–Stieltjes integral on the right to exist in the usual sense it is necessary that the upper integral[6] $\int^* \big|X_s(\omega)\big| \cdot [0,t]_s \big|dV_s(\omega)\big|$ be finite; and for the equality itself the random variable $\omega \mapsto LS\!-\!\!\int_0^t X_s(\omega)\, dV_s(\omega)$ must be measurable on $\mathcal{F}_t$. The best we can hope for is that the class of integrands X for which equation (3.7.15) holds be rather large. Indeed it is:

Proposition 3.7.33 *Both sides of equation (3.7.15) are defined and agree almost surely in the following cases:*

(i) X is previsible and the right-hand side exists a.s.

(ii) V is increasing and X is locally V–0-integrable.

---

[13] $|\Phi(x) - \Phi(x')| \leq L|x - x'|$ for $x, x' \in \mathbb{R}$. The smallest such L is the Lipschitz constant of Φ.

Proof. (i) Equation (3.7.15) is true by definition if X is an elementary integrand. The class of processes X such that $LS\text{-}\int_0^t X_s\, dV_s$ belongs to the class of the stochastic integral $\int_0^t X\, dV$ and is thus almost surely equal to $(X{*}V)_t$ is evidently a vector space closed under limits of bounded monotone sequences. So, thanks to the monotone class theorem A.3.4, equation (3.7.15) holds for all bounded predictable X, and then evidently for all bounded previsible X. To say that $LS\text{-}\int_0^t X_s(\omega)\, dV_s(\omega)$ exists almost surely implies that $LS\text{-}\int_0^t |X|_s(\omega)\, |dV_s|(\omega)$ is finite almost surely. Then evidently $|X|$ is finite for the mean $⟦\ ⟧^*_{V\text{–}0}$, so by the Dominated Convergence Theorem $-n \vee X \wedge n$ converges in $⟦\ ⟧^*_{V\text{–}0}$-mean to X, and the dV-integrals of this sequence converge to both the right-hand side and the left-hand side of equation (3.7.15), which thus agree.

(ii) We split X into its positive and negative parts and prove the claim for them separately. In other words, we may assume $X \ge 0$. We sandwich X between two predictable processes $\underset{\sim}{X} \le X \le \widetilde{X}$ with $⟦\widetilde{X} - \underset{\sim}{X}⟧^*_{V\text{–}0} = 0$, as in proposition 3.6.6. Part (i) implies that $\int_{[0}^{\infty} (\widetilde{X}_s(\omega) - \underset{\sim}{X}_s(\omega)) \wedge n\, dV_s(\omega) = 0$ $\forall n$ and then $\int_{[0}^{\infty} \widetilde{X}_s(\omega) - \underset{\sim}{X}_s(\omega)\, dV_s(\omega) = 0$ for almost all $\omega \in \Omega$. Neither $\big(X{*}V\big)_t$ nor $LS\text{-}\int_0^t X_s\, dV_s$ change but negligibly if X is replaced by $\widetilde{X}$: we may assume that $X \ge 0$ is predictable. Equation (3.7.15) holds then for $X \wedge n$, and by the Monotone Convergence Theorem for X. ∎

Exercise 3.7.34 The conclusion continues to hold if X is $(\mathcal{E}, ⟦\ ⟧^*)$-integrable for the mean

$$F \mapsto ⟦F⟧^* \overset{\text{def}}{=} \left⟦ \int^* |F_s| \cdot [0,t]\, |dV_s| \right⟧^*_{L^0(\mathbb{P})} .$$

Exercise 3.7.35 Let V be an adapted process of integrable variation $\lceil V \rceil$. Let μ denote the σ-additive measure $X \mapsto \mathbb{E}[\int X\, d\lceil V \rceil]$ on $\mathcal{E}$. Its usual Daniell upper integral (page 396)

$$F \mapsto \int^* F\, d\mu = \inf\Big\{ \sum \mu(X^{(n)}) : X^{(n)} \in \mathcal{E}\,, \sum X^{(n)} \ge F \Big\}$$

gives rise to the usual Daniell mean $F \mapsto \|F\|^*_\mu \overset{\text{def}}{=} \int^* |F|\, d\mu$, which majorizes $⟦\ ⟧^*_{V\text{–}1}$ and so gives rise to fewer integrable processes.

If X is integrable for the mean $\|\ \|^*_\mu$, then X is V–1-integrable (but not necessarily vice versa); its path $t \mapsto X_t(\omega)$ is a.s. integrable for the scalar measure $dV(\omega)$ on the line; the pathwise integral LS–$\int X\, dV$ is integrable and is a member of the class $\int X\, dV$.

3.8 Functions of Integrators

Consider the classical formula

$$f(t) - f(s) = \int_s^t f'(\sigma)\, d\sigma\ . \tag{3.8.1}$$

The equation

$$\Phi(Z_T) - \Phi(Z_S) = \int_{S+}^T \Phi'(Z)\, dZ$$

suggests itself as an appealing analog when Z is a stochastic integrator and Φ a differentiable function. Alas, life is not that easy. Equation (3.8.1) remains true if $d\sigma$ is replaced by an arbitrary measure μ on the line provided that provisions for jumps are made; yet the assumption that the distribution function of μ have finite variation is crucial to the usual argument. This is not at our disposal in the stochastic case, as the example of theorem 1.2.8 shows. What can be said? We take our clue from the following consideration: if we want a representation of $\Phi(Z_t)$ in a "Generalized Fundamental Theorem of Stochastic Calculus" similar to equation (3.8.1), then $\Phi(Z)$ must be an integrator (cf. lemma 3.7.5). So we ask for which Φ this is the case. It turns out that $\Phi(Z)$ is rather easily seen to be an L^0-integrator if Φ is *convex*. We show this next.

For the applications later results in higher dimension are needed. Accordingly, let D be a convex open subset of $\mathbb{R}^d$ and let

$$\boldsymbol{Z} = (Z^1, \ldots, Z^d)$$

be a vector of L^0-integrators. We follow the custom of denoting partial derivatives by subscripts that follow a semicolon:

$$\Phi_{;\eta} \stackrel{\text{def}}{=} \frac{\partial \Phi}{\partial x^\eta} \ , \quad \Phi_{;\eta\theta} \stackrel{\text{def}}{=} \frac{\partial^2 \Phi}{\partial x^\eta \partial x^\theta} \ , \text{ etc.,}$$

and use the ***Einstein convention***: if an index appears twice in a formula, once as a subscript and once as a superscript, then summation over this index is implied. For instance, $\Phi_{;\eta} G^\eta$ stands for the sum $\sum_\eta \Phi_{;\eta} G^\eta$. Recall the convention that $X_{0-} = 0$ for $X \in \mathfrak{D}$.

Theorem 3.8.1 *Assume that $\Phi : D \to \mathbb{R}$ is continuously differentiable and convex, and that the paths both of the L^0-integrator $\boldsymbol{Z}_{\boldsymbol{.}}$ and of its left-continuous version $\boldsymbol{Z}_{.-}$ stay in D at all times. Then $\Phi(\boldsymbol{Z})$ is an L^0-integrator. There exists an adapted right-continuous increasing process $A = A[\Phi; \boldsymbol{Z}]$ with $A_0 = 0$ such that nearly*

$$\Phi(\boldsymbol{Z}) = \Phi(\boldsymbol{Z}_0) + \Phi_{;\eta}(\boldsymbol{Z})_{.-} {*} Z^\eta + A[\Phi; \boldsymbol{Z}] \ , \tag{3.8.2}$$

i.e.,

$$\Phi(\boldsymbol{Z}_t) = \Phi(\boldsymbol{Z}_0) + \sum_{1 \le \eta \le d} \int_{0+}^{t} \Phi_{;\eta}(\boldsymbol{Z}_{.-}) \, dZ^\eta + A_t \qquad \forall\, t \ge 0 \ .$$

Like every increasing process, A is the sum of a continuous increasing process $C = C[\Phi; \boldsymbol{Z}]$ that vanishes at $t = 0$ and an increasing pure jump process $J = J[\Phi; \boldsymbol{Z}]$, both adapted (see theorem 2.4.4). J is given at $t \ge 0$ by

$$J_t = \sum_{0 < s \le t} \Big(\Phi(\boldsymbol{Z}_s) - \Phi(\boldsymbol{Z}_{s-}) - \Phi_{;\eta}(\boldsymbol{Z}_{s-}) \cdot \Delta Z^\eta_s \Big) \ , \tag{3.8.3}$$

the sum on the right being a sum of positive terms.

Proof. In view of theorem 3.7.17, the processes $\Phi_{;\eta}(\boldsymbol{Z})_{.-}$ are $\boldsymbol{Z}$–0-integrable. Since Φ is convex,

$$\Phi(\boldsymbol{z}_2) - \Phi(\boldsymbol{z}_1) - \Phi_{;\eta}(\boldsymbol{z}_1)(\boldsymbol{z}_2^\eta - \boldsymbol{z}_1^\eta)$$

is non-negative for any two points $\boldsymbol{z}_1, \boldsymbol{z}_2 \in D$. Consider now a stochastic partition[8] $\mathcal{S} = \{0 = S_0 \le S_1 \le S_2 \le \ldots \le \infty\}$ and let $0 \le t < \infty$. Set

$$\begin{aligned} A_t^{\mathcal{S}} &\stackrel{\text{def}}{=} \Phi(\boldsymbol{Z}_t) - \Phi(\boldsymbol{Z}_0) - \sum_{0 \le k \le \infty} \Phi_{;\eta}(\boldsymbol{Z}_{S_k \wedge t}) \cdot (\boldsymbol{Z}^\eta_{S_{k+1}\wedge t} - \boldsymbol{Z}^\eta_{S_k \wedge t}) \qquad (3.8.4) \\ &= \sum_{0 \le k \le \infty} \Big(\Phi(\boldsymbol{Z}_{S_{k+1}\wedge t}) - \Phi(\boldsymbol{Z}_{S_k\wedge t}) - \Phi_{;\eta}(\boldsymbol{Z}_{S_k\wedge t}) \cdot (\boldsymbol{Z}^\eta_{S_{k+1}\wedge t} - \boldsymbol{Z}^\eta_{S_k\wedge t}) \Big) . \end{aligned}$$

On the interval of integration, which does not contain the point $t = 0$, $\Phi_{;\eta}(\boldsymbol{Z}) = \Phi_{;\eta}(\boldsymbol{Z}_{.-})_{.+}$. Thus the sum on the top line of this equation converges to the stochastic integral $\int_{0+}^t \Phi_{;\eta}(\boldsymbol{Z})_{.-}\, dZ^\eta$ as the partition $\mathcal{S}$ is taken through a sequence whose mesh goes to zero (theorem 3.7.23), and so $A_t^{\mathcal{S}}$ converges to

$$A_t \stackrel{\text{def}}{=} \Phi(\boldsymbol{Z}_t) - \Phi(\boldsymbol{Z}_0) - \int_{0+}^t \Phi_{;\eta}(\boldsymbol{Z}_{.-})\, dZ^\eta .$$

In fact, the convergence is uniform in t on bounded intervals, in measure. The second line of (3.8.4) shows that A_t increases with t. We can and shall choose a modification of A that has right-continuous and increasing paths. Exercise 3.7.32 on page 144 identifies the jump part of A as $\Delta A_s = \Delta\Phi(\boldsymbol{Z})_s - \Phi_{;\eta}(\boldsymbol{Z})_{s-} \cdot \Delta Z_s^\eta$, $s > 0$. The terms on the right are positive and have a finite sum over $s \le t$ since $A_t < \infty$; this observation identifies the jump part of A as stated. ∎

Remarks 3.8.2 (i) If Φ is, instead, the difference of two convex functions of class C^1, then the theorem remains valid, except that the processes A, C, J are now of finite variation, with the expression for J converging absolutely.

(ii) It would be incorrect to write

$$J_t = \sum_{0<s\le t} \Big(\Phi(\boldsymbol{Z}_s) - \Phi(\boldsymbol{Z}_{s-}) \Big) - \sum_{0<s\le t} \Phi_{;\eta}(\boldsymbol{Z}_{s-}) \cdot \Delta Z_s^\eta ,$$

on the grounds that neither sum on the right converges in general by itself.

Exercise 3.8.3 Suppose $\boldsymbol{Z}$ is continuous. Set $\nabla_\eta|\boldsymbol{z}| \stackrel{\text{def}}{=} \partial|\boldsymbol{z}|/\partial z^\eta = z^\eta/|\boldsymbol{z}|$ for $\boldsymbol{z} \ne 0$, with $\nabla|\boldsymbol{z}| \stackrel{\text{def}}{=} 0$ at $\boldsymbol{z} = 0$. There exists an increasing process L so that

$$|\boldsymbol{Z}|_t = |\boldsymbol{Z}_0| + \int_0^t \nabla_\eta|\boldsymbol{Z}_s|\, d\boldsymbol{Z}^\eta + L_t .$$

If $d = 1$, then L is known as the ***local time of Z at zero***.

Square Bracket and Square Function of an Integrator

A most important process arises by taking $d = 1$ and $\Phi(z) = z^2$. In this case the process $\Phi(Z_0) + A[\Phi; Z]$ of equation (3.8.2) is denoted by $[Z, Z]$ and is called the ***square bracket*** or ***square variation*** of Z. It is thus defined by

$$Z^2 = 2Z_{.-}{*}Z + [Z, Z] \quad \text{or} \quad Z_t^2 = 2\int_{0+}^{t} Z_{.-}\, dZ + [Z, Z]_t \ , \qquad t \geq 0 \ .$$

Note that the jump Z_0^2 is subsumed in $[Z, Z]$. For the simple function $\Phi(z) = z^2$, equation (3.8.4) reduces to $A_t^{\mathcal{S}} = \sum_{0 \leq k \leq \infty} \big(Z^t_{S_{k+1}} - Z^t_{S_k}\big)^2$, so that $[Z, Z]_t$ is the limit in measure of

$$Z_0^2 + \sum\nolimits_{0 \leq k \leq \infty} \big(Z^t_{S_{k+1}} - Z^t_{S_k}\big)^2 \ ,$$

taken as the random partition $\mathcal{S} = \{0 = S_0 \leq S_1 \leq S_2 \leq \ldots \leq \infty\}$ of $[\![0, \infty)\!)$ runs through a sequence whose mesh tends to zero. By equation (3.8.3), the jump part of $[Z, Z]$ is simply

$${}^{j}[Z, Z]_t = Z_0^2 + \sum\nolimits_{0 < s \leq t} (\Delta Z_s)^2 = \sum\nolimits_{0 \leq s \leq t} (\Delta Z_s)^2 \ .$$

Its continuous part is ${}^{c}[Z, Z]$, the ***continuous bracket***. Note that we make the convention $[Z, Z]_0 = {}^{j}[Z, Z]_0 = Z_0^2$ and ${}^{c}[Z, Z]_0 = 0$. Homogeneity mandates considering the square roots of these quantities. We set

$$S[Z] = \sqrt{[Z, Z]} \qquad \text{and} \qquad \sigma[Z] = \sqrt{{}^{c}[Z, Z]}$$

and call these processes the ***square function*** and the ***continuous square function*** of Z, respectively. Evidently

$$\sigma[Z] \leq S[Z] \quad \text{and} \quad \sqrt{{}^{j}[Z, Z]} \leq S[Z] \ .$$

The proof of theorem 3.8.1 exhibits $S_T[Z]$ as the limit in measure of the square roots

$$\sqrt{Z_0^2 + A_T^{\mathcal{S}}} = \Big(Z_0^2 + \sum\nolimits_{0 \leq k \leq \infty} (Z^T_{S_{k+1}} - Z^T_{S_k})^2\Big)^{1/2} \ , \tag{3.8.5}$$

taken as the random partition $\mathcal{S} = \{0 = S_0 \leq S_1 \leq S_2 \leq \ldots \leq \infty\}$ runs through a sequence whose mesh tends to zero. For an estimate of the speed of the convergence see exercise 3.8.14.

Theorem 3.8.4 (The Size of the Square Function) *At all stopping times T*

for all exponents $p > 0$ $\qquad \|S_T[Z]\|_{L^p} \leq K_p \cdot \lceil\!\lceil Z^T \rceil\!\rceil_{\mathcal{I}^p} \ .$ (3.8.6)

Also, for $p = 0$, $\qquad \|S_T[Z]\|_{[\alpha]} \leq K_0 \cdot \lceil\!\lceil Z^T \rceil\!\rceil_{[\alpha\kappa_0]} \ .$ (3.8.7)

The universal constants K_p, κ_0 are bounded by the Khintchine constants of theorem A.8.26 on page 455: $K_p^{(3.8.6)} \le K_p^{(A.8.9)}$ when p is strictly positive; and for $p = 0$, $K_0^{(3.8.7)} \le K_0^{(A.8.9)}$ and $\kappa_0^{(3.8.7)} \ge \kappa_0^{(A.8.9)}$.

Proof. In equation (3.8.5) set $X^{(0)} \stackrel{\text{def}}{=} [\![0]\!] = [\![S_0]\!]$ and $X^{(k+1)} \stackrel{\text{def}}{=} (\!(S_k, S_{k+1}]\!]$ for $k = 0, 1, \ldots \infty$. Then

$$Z_0^2 + A_T^{\mathcal{S}} = \sum_{0 \le k \le \infty} \Big(\int X^{(k)} \, dZ^T \Big)^2 ;$$

and since $\sum_k |X^{(k)}| \le 1$, corollary 3.1.7 on page 94 results in

$$\Big\| \sqrt{Z_0^2 + A_T^{\mathcal{S}}} \Big\|_{L^p} \le K_p^{(A.8.5)} \cdot \lceil\!\lceil Z^T \rceil\!\rceil_{\mathcal{I}^p} , \qquad p > 0,$$

and
$$\Big\| \sqrt{Z_0^2 + A_T^{\mathcal{S}}} \Big\|_{[\alpha]} \le K_0^{(A.8.6)} \cdot \lceil\!\lceil Z^T \rceil\!\rceil_{[\alpha\kappa_0]} , \qquad p = 0.$$

As the partition is taken through a sequence whose mesh tends to zero, Fatou's lemma produces the inequalities of the statement and exercise A.8.29 the estimates of the constants. True, corollary 3.1.7 was proved for *elementary* integrands $X^{(k)}$ only – for lack of others at the time – but the reader will have no problem seeing that it holds for integrable $X^{(k)}$ as well. ∎

Remark 3.8.5 Consider a standard Wiener process W. The variation

$$\lim_{\mathcal{S}} \sum_k \big| W_{S_{k+1}} - W_{S_k} \big| ,$$

taken over partitions $\mathcal{S} = \{S_k\}$ of any interval, however small, is infinite (theorem 1.2.8). The proof above shows that the limit exists and is finite, provided the absolute value of the differences is replaced with their squares. This explains the name square variation. Proposition 3.8.16 on page 153 makes the identification $[W, W]_t = \lim_{\mathcal{S}} \sum_k |W_{S_{k+1}} - W_{S_k}|^2 = t$.

Exercise 3.8.6 The square function of a previsible integrator is previsible.

Exercise 3.8.7 For $p > 0$ and Z–p-integrable X,

$$\lceil\!\lceil \sigma_\infty[X{*}Z] \rceil\!\rceil_{L^p} \le \lceil\!\lceil S_\infty[X{*}Z] \rceil\!\rceil_{L^p} \le K_p^{p\wedge 1} \cdot \lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}p}$$

and
$$\Big\lceil\!\Big\lceil \sqrt{{}^j[X{*}Z, X{*}Z]_\infty} \Big\rceil\!\Big\rceil_{L^p} \le K_p^{p\wedge 1} \cdot \lceil\!\lceil X \rceil\!\rceil^*_{Z\text{–}p} .$$

Exercise 3.8.8 Let $\boldsymbol{Z} = (Z^1, \ldots, Z^d)$ be L^0-integrators and $T \in \mathfrak{T}$. Then

for $p \in (0, \infty)$
$$\Big\| \Big(\sum_{\eta=1}^d [Z^\eta, Z^\eta]_T \Big)^{1/2} \Big\|_{L^p} \le K_p^{(3.8.6)} \cdot \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil_{\mathcal{I}^p} ;$$

and for $p = 0$
$$\Big\| \Big(\sum_{\eta=1}^d [Z^\eta, Z^\eta]_T \Big)^{1/2} \Big\|_{[\alpha]} \le K_0^{(3.8.7)} \cdot \lceil\!\lceil \boldsymbol{Z}^T \rceil\!\rceil_{[\alpha\kappa_0^{(3.8.7)}]} .$$

The Square Bracket of Two Integrators

This process associated with two integrators Y, Z is obtained by taking in theorem 3.8.1 the function $\Phi(y,z) = y \cdot z$ of two variables, which is the difference of two convex smooth functions:

$$y \cdot z = \frac{1}{2}\Big((y+z)^2 - (y^2+z^2)\Big) ,$$

and thus remark 3.8.2 (i) applies. The process $Y_0 Z_0 + A[\Phi;(Y,Z)]$ of finite variation that arises in this case is denoted by $[Y,Z]$ and is called the ***square bracket*** of Y and Z. It is thus defined by

$$YZ = Y_{.-}{*}Z + Z_{.-}{*}Y + [Y,Z] \tag{3.8.8}$$

or, equivalently, by

$$Y_t{\cdot}Z_t = \int_{0+}^t Y_{.-}\, dZ + \int_{0+}^t Z_{.-}\, dY + [Y,Z]_t , \qquad t \geq 0 .$$

For an algorithm computing $[Y,Z]$ see exercise 3.8.14. By equation (3.8.3) the jump part of $[Y,Z]$ is simply

$$^j[Y,Z]_t = Y_0 Z_0 + \sum_{0<s\leq t} (\Delta Y_s \cdot \Delta Z_s) = \sum_{0\leq s\leq t} (\Delta Y_s \cdot \Delta Z_s) .$$

Its continuous part is denoted by $^c[Y,Z]$ and vanishes at $t=0$. Both $[Y,Z]$ and $^c[Y,Z]$ are evidently linear in either argument, and so is their difference $^j[Y,Z]$. All three brackets have the structure of positive semidefinite inner products, so the usual Cauchy–Schwarz inequality holds. In fact, there is a slight generalization:

Theorem 3.8.9 (Kunita–Watanabe) *For any two L^0-integrators Y, Z there exists a nearly empty set N such that for all $\omega \in \Omega_0 \stackrel{\text{def}}{=} \Omega \setminus N$ and any two processes U, V with Borel measurable paths*

$$\int_0^\infty |UV|\, d\big|[Y,Z]\big| \leq \Big(\int_0^\infty U^2\, d[Y,Y]\Big)^{1/2} \cdot \Big(\int_0^\infty V^2\, d[Z,Z]\Big)^{1/2} ,$$

$$\int_0^\infty |UV|\, d\big|{}^c[Y,Z]\big| \leq \Big(\int_0^\infty U^2\, d{}^c[Y,Y]\Big)^{1/2} \cdot \Big(\int_0^\infty V^2\, d{}^c[Z,Z]\Big)^{1/2} ,$$

$$\int_0^\infty |UV|\, d\big|{}^j[Y,Z]\big| \leq \Big(\int_0^\infty U^2\, d{}^j[Y,Y]\Big)^{1/2} \cdot \Big(\int_0^\infty V^2\, d{}^j[Z,Z]\Big)^{1/2} .$$

Proof. Consider the polynomial

$$\begin{aligned} p(\lambda) &\stackrel{\text{def}}{=} A + 2B\lambda + C\lambda^2 \\ &\stackrel{\text{def}}{=} ([Y,Y]_t - [Y,Y]_s) + 2([Y,Z]_t - [Y,Z]_s)\lambda + ([Z,Z]_t - [Z,Z]_s)\lambda^2 \\ &= [Y+\lambda Z, Y+\lambda Z]_t - [Y+\lambda Z, Y+\lambda Z]_s . \end{aligned} \tag{$*$}$$

There is a set $\Omega_0 \in \mathcal{F}_\infty$ of full measure $\mathbb{P}[\Omega_0] = 1$ on which for all rational λ and all rational pairs $s \le t$ the equality $(*)$ obtains and thus $p(\lambda)$ is positive. The description shows that its complement N belongs to $\mathcal{A}_{\infty\sigma}$. By right-continuity, etc., $(*)$ is true and $p(\lambda) \ge 0$ for all real λ, all pairs $s \le t$, and all $\omega \in \Omega_0$.

Henceforth an $\omega \in \Omega_0$ is fixed. The positivity of $p(\lambda)$ gives $B^2 \le AC$, which implies that

$$|[Y,Z]_{t_i} - [Y,Z]_{t_{i-1}}| \le ([Y,Y]_{t_i} - [Y,Y]_{t_{i-1}})^{1/2} \cdot ([Z,Z]_{t_i} - [Z,Z]_{t_{i-1}})^{1/2}$$

for any partition $\{\ldots t_{i-1} \le t_i \ldots\}$. For any $r_i, s_i \in \mathbb{R}$ we get therefore

$$\sum_i r_i s_i \cdot ([Y,Z]_{t_i} - [Y,Z]_{t_{i-1}}) \le \sum_i |r_i s_i| |([Y,Z]_{t_i} - [Y,Z]_{t_{i-1}})|$$

$$\le \sum_i |r_i| ([Y,Y]_{t_i} - [Y,Y]_{t_{i-1}})^{1/2} \cdot |s_i| ([Z,Z]_{t_i} - [Z,Z]_{t_{i-1}})^{1/2} .$$

Schwarz's inequality applied to the right-hand side yields

$$\int_0^\infty uv \; d[Y,Z] \le \Big(\int_0^\infty u^2 \; d[Y,Y] \Big)^{1/2} \cdot \Big(\int_0^\infty v^2 \; d[Z,Z] \Big)^{1/2} \qquad (**)$$

at ω for the special functions $u = \sum r_i \cdot (t_{i-1}, t_i]$ and $v = \sum s_i \cdot (t_{i-1}, t_i]$ on the half-line. The collection of processes for which the inequality $(**)$ holds is evidently closed under taking limits of convergent sequences, so it holds for all Borel functions (theorem A.3.4). Replacing u by $|u|$ and v by $|v|h$, where h is a Borel version of the Radon–Nikodym derivative $d[Y,Z]/d\text{¦}[Y,Z]\text{¦}$, produces

$$\int_0^\infty |uv| \; d\text{¦}[Y,Z]\text{¦} \le \Big(\int_0^\infty u^2 \; d[Y,Y] \Big)^{1/2} \cdot \Big(\int_0^\infty v^2 \; d[Z,Z] \Big)^{1/2} .$$

The proof of the other two inequalities is identical to this one. ∎

Exercise 3.8.10 (Kunita–Watanabe) (i) Except possibly on an evanescent set

$$S[Y+Z] \le S[Y] + S[Z] \quad \text{and} \quad |S[Y] - S[Z]| \le S[Y-Z] ;$$

$$\sigma[Y+Z] \le \sigma[Y] + \sigma[Z] \quad \text{and} \quad |\sigma[Y] - \sigma[Z]| \le \sigma[Y-Z] .$$

Consequently, $\| |\sigma[Y] - \sigma[Z]|_T^\star \|_{L^p} \le \| S[Y-Z]_T^\star \|_{L^p} \le K_p^{(3.8.6)} \text{¦}(Y-Z)^T\text{¦}_{\mathcal{I}^p}$ for all $p > 0$ and all stopping times T.

(ii) For any stopping time T and $1/r = 1/p + 1/q > 0$,

$$\Big\| \text{¦}[Y,Z]\text{¦}_T \Big\|_{L^r} \le \|S_T[Y]\|_{L^p} \cdot \|S_T[Z]\|_{L^q} ,$$

$$\Big\| \text{¦}^c[Y,Z]\text{¦}_T \Big\|_{L^r} \le \|\sigma_T[Y]\|_{L^p} \cdot \|\sigma_T[Z]\|_{L^q} ,$$

$$\Big\| \text{¦}^j[Y,Z]\text{¦}_T \Big\|_{L^r} \le \Big\| \sqrt{{}^j[Y,Y]_T} \Big\|_{L^p} \cdot \Big\| \sqrt{{}^j[Z,Z]_T} \Big\|_{L^q} .$$

Exercise 3.8.11 Let M, N be càdlàg locally square-integrable martingales. There are arbitrarily large stopping times T such that

$$\mathbb{E}[M_T \cdot N_T] = \mathbb{E}[[M,N]_T] \quad \text{and} \quad \mathbb{E}[M_T^{\star 2}] \le 4 \cdot \mathbb{E}[[M,M]_T] \, .$$

Exercise 3.8.12 Let V be an L^0-integrator whose paths have finite variation, and let $V = {}^c V + {}^j V$ be its decomposition into a continuous and a pure jump process (theorem 2.4.4). Then $\sigma[V] = [{}^c V, {}^c V] = 0$ and, since $\Delta V_0 = V_0$,

$$[V,V]_t = [{}^j V, {}^j V]_t = \sum_{0 \le s \le t} (\Delta V_s)^2 \, .$$

Also, ${}^c[Z,V] = 0, \quad [Z,V]_t = {}^j[Z,V]_t = \sum_{0 \le s < t} \Delta Z_s \Delta V_s$

and

$$Z_T \cdot V_T - Z_S \cdot V_S = \int_{S+}^{T} Z \, dV + \int_{S+}^{T} V_{\text{.-}} \, dZ$$

for any other L^0-integrator Z and any two stopping times $S \le T$.

Exercise 3.8.13 A continuous local martingale of finite variation is nearly constant.

Exercise 3.8.14 Let Y, Z be L^p-integrators, $p > 0$, and $\mathcal{S} = \{0 = S_0 \le S_1 \le \cdots\}$ a stochastic partition[8] with $S_\infty = \infty$. Then for any stopping time T

$$\left\| \sup_{s \le T} \left| [Y,Z]_s - \Big(Y_0 Z_0 + \sum_{0 \le k < \infty} (Y_s^{S_{k+1}} - Y_s^{S_k})(Z_s^{S_{k+1}} - Z_s^{S_k}) \Big) \right| \right\|_{L^p}$$

$$\le C_p^{\star(2.3.5)} \Big(\left\| (Y - Y^{\mathcal{S}})_{\text{.-}} \cdot ((0,T]] \right\|_{Z-p}^{*} + \left\| (Z - Z^{\mathcal{S}})_{\text{.-}} \cdot ((0,T]] \right\|_{Y-p}^{*} \Big) \, .$$

In particular, if $\mathcal{S}$ is chosen so that on its intervals both Y and Z vary by less than δ, then

$$\left\| \sup_{s \le T} \left| [Y,Z]_s - \Big(Y_0 Z_0 + \sum_{0 \le k < \infty} (Y_s^{S_{k+1}} - Y_s^{S_k})(Z_s^{S_{k+1}} - Z_s^{S_k}) \Big) \right| \right\|_{L^p}$$

$$\le C_p^{\star} \Big(\lceil \delta Z^T \rceil_{\mathcal{I}^p} + \lceil \delta Y^T \rceil_{\mathcal{I}^p} \Big) \, .$$

If δ runs through a sequence (δ_n) with $\sum_n \lceil \delta_n Z^n \rceil_{\mathcal{I}^p} + \lceil \delta_n Y^n \rceil_{\mathcal{I}^p} < \infty$, then the sums of equation (3.8.5) nearly converge to the square bracket uniformly on bounded time-intervals.

Exercise 3.8.15 A complex-valued process $Z = X + iY$ is an L^p-integrator if its real and imaginary parts are, and the size of Z shall be the size of the vector (X,Y). We define the square function $S[Z]$ as $[Z,\overline{Z}]^{1/2} = \big([X,X] + [Y,Y]\big)^{1/2}$. Using exercise 3.8.8 reprove the square function estimates of theorem 3.8.4:

$$\|S_T[Z]\|_p \le K_p \cdot \lceil Z^T \rceil_{\mathcal{I}^p} \tag{3.8.9}$$

and

$$\|S_T[Z]\|_{[\alpha]} \le K_0 \cdot \lceil Z^T \rceil_{[\alpha\kappa_0]}$$

and show that for two complex integrators Z, Z'

$$Z_t \cdot Z'_t = \int_{0+}^{t} Z_{\text{.-}} \, dZ' + \int_{0+}^{t} Z'_{\text{.-}} \, dZ + [Z,Z']_t \, , \tag{3.8.10}$$

with the stochastic integral and bracket being defined so as to be complex-linear in either of their two arguments.

Proposition 3.8.16 *(i) A standard Wiener process* W *has bracket* $[W,W]_t = t$. *(ii) A standard* d*-dimensional Wiener process* $\boldsymbol{W} = (W^1, \ldots, W^d)$ *has bracket*

$$[W^\eta, W^\theta]_t = \delta^{\eta\theta} \cdot t \,.$$

Proof. (i) By exercise 2.5.4, $W_t^2 - t$ is a continuous martingale on the natural filtration of W. So is $W_t^2 - [W,W]_t = 2\int_0^t W\,dW$, because for $T_n = n \wedge \inf\{t : |W_t| \ge n\} \xrightarrow[n\to\infty]{} \infty$ the stopped process $(W{*}W)^{T_n}$ is the indefinite integral in the sense L^2 of the bounded integrand[6] $W \cdot [\![0, T_n]\!]$ against the L^2-integrator W. Then their difference $[W,W]_t - t$ is a local martingale as well. Since this continuous process has finite variation, it equals the value 0 that it has at time 0 (exercise 3.8.13).

(ii) If $\eta \ne \theta$, then W^η and W^θ are independent martingales on the natural filtrations $\mathcal{F}_.[W^\eta]$ and $\mathcal{F}_.[W^\theta]$, respectively. It is trivial to check that then $W^\eta \cdot W^\theta$ is a martingale on the filtration $t \mapsto \mathcal{F}_t[W^\eta] \vee \mathcal{F}_t[W^\theta]$. Since $[W^\eta, W^\theta] = W^\eta \cdot W^\theta - W^\eta{*}W^\theta - W^\theta{*}W^\eta$ is a continuous local martingale of finite variation, it must vanish. ∎

Definition 3.8.17 *We shall say that two paths* $X_.(\omega)$ *and* $X_.(\omega')$ *of the continuous integrator* X *describe the same arc in* $\mathbb{R}^n$ *if there is an increasing invertible continuous function* $t \mapsto t'$ *from* $[0,\infty)$ *onto itself so that* $X_t(\omega) = X_{t'}(\omega') \quad \forall\, t$. *We shall also say that* $X_.(\omega)$ *and* $X_.(\omega')$ ***describe the same arc via*** $t \mapsto t'$.

Exercise 3.8.18 (i) Suppose F is a $d \times n$-matrix of uniformly continuous functions on $\mathbb{R}^n$. There exist a version of $F(X){*}X$ and a nearly empty set after the removal of which the following holds: whenever $X_.(\omega)$ and $X_.(\omega')$ describe the same arc in $\mathbb{R}^n$ via $t \mapsto t'$ then $\big(F(X){*}X\big)_.(\omega)$ and $\big(F(X){*}X\big)_.(\omega')$ describe the same arc in $\mathbb{R}^d$, also via $t \mapsto t'$.

(ii) Let $\boldsymbol{W}$ be a standard d-dimensional Wiener process. There is a nearly empty set after removal of which any two paths $\boldsymbol{W}_.(\omega) = \boldsymbol{W}_.(\omega')$ describing the same arc actually agree: $\boldsymbol{W}_t(\omega) = \boldsymbol{W}_t(\omega')$ for all t.

The Square Bracket of an Indefinite Integral

Proposition 3.8.19 *Let* Y, Z *be* L^0*-integrators,* T *a stopping time, and* X *a locally* Z*–0-integrable process. Then*

(i) $$[Y,Z]^T = [Y^T, Z^T] = [Y, Z^T] \qquad \textit{almost surely, and}$$

(ii) $$[Y, X{*}Z] = X{*}[Y,Z] \qquad \textit{up to indistinguishability.}$$

Here $X{*}[Y,Z]$ *is understood as an indefinite Lebesgue–Stieltjes integral.*

Proof. (i) For the first equality write

$$[Y,Z]^T = (YZ)^T - (Y_{.-}{*}Z)^T - (Z_{.-}{*}Y)^T$$

by exercise 3.7.15: $$= Y^T Z^T - Y_{.-}{*}Z^T - Z_{.-}{*}Y^T$$

by exercise 3.7.16: $$= Y^T Z^T - Y^T_{.-}{*}Z^T - Z^T_{.-}{*}Y^T = [Y^T, Z^T]\,.$$

The second equality of (i) follows from the computation

$$[Y - Y^T, Z^T] = (Y - Y^T) \cdot Z^T - (Y_{.-} - Y^T_{.-}){*}Z^T - Z^T_{.-}{*}(Y - Y^T) = 0 \ .$$

(ii) Equality (i) applied to the stopping times $T \wedge t$ yields $\big[Y, [\![0, T]\!]{*}Z\big] = [\![0, T]\!]{*}[Y, Z]$. Taking differences gives $\big[Y, (\!(S, T]\!]{*}Z\big] = (\!(S, T]\!]{*}[Y, Z]$ for stopping times $S \le T$. Taking linear combinations shows that (ii) holds for elementary integrands X. Let $\mathfrak{L}^Z$ denote the class of locally Z–0-integrable predictable processes X that meet the following description: for any L^0-integrator Y there is a nearly empty set outside which the indefinite Lebesgue–Stieltjes integral $X{*}[Y, Z]$ exists and agrees with $[Y, X{*}Z]$. This is a vector space containing $\mathcal{E}$. Let $X^{(n)}$ be an increasing sequence in $\mathfrak{L}^Z_+$, and assume that its pointwise limit is locally Z–0-integrable. From the inequality of Kunita–Watanabe

$$\int_0^t X^{(n)} \, d\,\vert[Y, Z]\vert \le S_t[Y] \cdot \left(\int_0^t X^{(n)2} \, d[Z, Z] \right)^{1/2} \ .$$

Since $X^{(n)} \in \mathfrak{L}^Z$, the random variable on the far right equals $S_t\big[X^{(n)}{*}Z\big]$. Exercise 3.7.14 allows this estimate of its size: for all $\alpha > 0$

$$\Big\| S_t\big[X^{(n)}{*}Z\big] \Big\|_{[\alpha]} \le K_0 \cdot \Big\| X^{(n)} \Big\|^*_{Z^t\text{–}[\alpha\kappa_0]} \le K_0 \cdot \|X\|^*_{Z^t\text{–}[\alpha\kappa_0]} < \infty \ .$$

Thus
$$\int_0^t X^{(n)} \, d\vert[Y, Z]\vert \le K_0 \cdot S_t[Y] \cdot \|X\|^*_{Z^t\text{–}[\alpha\kappa_0]} < \infty \ . \qquad (*)$$

Hence $\sup_n \int_0^t X^{(n)} \, d\vert[Y, Z]\vert < \infty$ almost surely for all t, and the indefinite Lebesgue–Stieltjes integral $X{*}[Y, Z]$ exists except possibly on a nearly empty set. Moreover, $X{*}[Y, Z] = \lim X^{(n)}{*}[Y, Z]$ up to indistinguishability.

Exercise 3.7.14 can be put to further use:

$$\big\vert [Y, X{*}Z] - [Y, X^{(n)}{*}Z] \big\vert^\star_t = \big\vert [Y, \big(X - X^{(n)}\big){*}Z] \big\vert^\star_t$$

by 3.8.9:
$$\le S_t[Y] \cdot S_t\big[\big(X - X^{(n)}\big){*}Z\big]$$

and
$$\Big\| S_t\big[\big(X - X^{(n)}\big){*}Z\big] \Big\|_{[\alpha]} \le K_0 \cdot \big\vert \big(X - X^{(n)}\big){*}Z^t \big\vert_{[\alpha\kappa_0]} = K_0 \cdot \Big\| X - X^{(n)} \Big\|^*_{Z^t\text{–}[\alpha\kappa_0]} \ .$$

We replace $X^{(n)}$ by a subsequence such that $\| X - X^{(n)} \|^*_{Z^n\text{–}[2^{-n}]} < 2^{-n}$; the Borel–Cantelli lemma then allows us to conclude that $S_n\big[\big(X - X^{(n)}\big){*}Z\big] \to 0$ almost surely, so that

$$X{*}[Y, Z] = \lim X^{(n)}{*}[Y, Z] = \lim_{n \to \infty} [Y, X^{(n)}{*}Z] = [Y, X{*}Z]$$

uniformly on bounded time-intervals, except on a nearly empty set. Applying this to $X^{(n)} - X^{(1)}$ or $X^{(1)} - X^{(n)}$ shows that $\mathfrak{L}^Z$ is closed under pointwise convergent monotone sequences $X^{(n)}$ whose limits are locally Z–0-integrable. The Monotone Class Theorem A.3.5 then implies that $\mathfrak{L}^Z$ contains all bounded predictable processes, and the usual truncation argument shows that it contains in fact all predictable locally Z–0-integrable processes X.

If X is also Z–0-negligible, then thanks to (∗) $\int X\, d\,¦[Y,Z]¦$ is evanescent. If X is Z–0-negligible but not predictable, we apply this remark to a predictable envelope of $|X|$ and conclude again that $\int X\, d\,¦[Y,Z]¦$ is evanescent. The general case is done by sandwiching X between predictable lower and upper envelopes (page 125). ∎

Exercise 3.8.20 Let Y, Z be L^0-integrators. For any stopping time T and any locally Z–0-integrable process X

$$ {}^c[Y,Z]^T = {}^c[Y^T,Z^T] = {}^c[Y,Z^T] \quad\text{and}\quad {}^j[Y,Z]^T = {}^j[Y^T,Z^T] = {}^j[Y,Z^T]\ ; $$

also $\quad {}^c[Y, X{*}Z] = X{*}{}^c[Y,Z] \qquad\qquad$ and $\ {}^j[Y, X{*}Z] = X{*}{}^j[Y,Z]$.

Application: The Jump of an Indefinite Integral

Proposition 3.8.21 *Let Z be an L^0-integrator and X a Z–0-integrable process. There exists a nearly empty set $N \subset \Omega$ such that for all $\omega \notin N$*

$$ \Delta(X{*}Z)_t(\omega) = X_t(\omega)\cdot \Delta Z_t(\omega)\ , \qquad\qquad 0 \le t < \infty\ . $$

In other words, $\Delta(X{*}Z)$ is indistinguishable from $X\cdot\Delta Z$.

Proof. If X is elementary, then the claim is obvious by inspection. In the general case we find a sequence $\big(X^{(n)}\big)$ of elementary integrands converging in Z–0-mean to X and so that their indefinite integrals nearly converge uniformly on bounded intervals to $X{*}Z$ (corollary 3.7.11). The path of $\Delta(X{*}Z)$ is thus nearly given by

$$ \Delta(X{*}Z) = \lim_{n\to\infty} X^{(n)}\cdot\Delta Z\ . $$

It is left to be shown that the path on the right is nearly equal to the path of $X\cdot\Delta Z$:

$$ \lim_{n\to\infty} X^{(n)}\cdot\Delta Z = X\cdot\Delta Z\ . \tag{?} $$

Since
$$ \sup_{0\le t<u}\Big|X_t\cdot\Delta Z_t - X^{(n)}_t\cdot\Delta Z_t\Big| \le \Big(\sum_{t<u}|X - X^{(n)}|^2_t\cdot(\Delta Z_t)^2\Big)^{1/2} $$
$$ = \Big(\int_0^{u-}|X - X^{(n)}|^2\, d{}^j[Z,Z]\Big)^{1/2} $$
$$ \le \Big(\int_0^{u-}|X - X^{(n)}|^2\, d[Z,Z]\Big)^{1/2} $$

by proposition 3.8.19: $= S_{u-}[(X - X^{(n)}) * Z]$,

$$\Big\| \sup_{0\le t<u} \Big| X_t \cdot \Delta Z_t - X_t^{(n)} \cdot \Delta Z_t \Big| \Big\|_{[\alpha]} \le \Big\| S_{u-}[(X - X^{(n)}) * Z] \Big\|_{[\alpha]}$$

by theorem 3.8.4: $\le K_0 \cdot \lceil (X - X^{(n)}) * Z \rceil_{[\alpha\kappa_0]}$

by lemma 3.7.5: $\le K_0 \cdot \Big\| X - X^{(n)} \Big\|^*_{Z-[\alpha\kappa_0]}$.

If we insist that $\| X - X^{(n)} \|^*_{Z-[2^{-n}\kappa_0]} < 2^{-n}/K_0$, as we may by taking a subsequence, then the previous inequality turns into

$$\mathbb{P}\Big[\sup_{0\le t<u} \Big| X_t \cdot \Delta Z_t - X_t^{(n)} \cdot \Delta Z_t \Big| > 2^{-n} \Big] < 2^{-n} ,$$

and the Borel–Cantelli lemma implies that

$$\Big[\limsup_{n\to\infty} \sup_{0\le t<u} \Big| X_t \cdot \Delta Z_t - X_t^{(n)} \cdot \Delta Z_t \Big| > 0 \Big] \in \mathcal{F}_u$$

is negligible. Equation (?) thus holds nearly. ∎

Proposition 3.8.22 *Let Y, Z be L^0-integrators, with Y previsible. At any finite stopping time T,*

$$\int_0^T \Delta Y \, dZ = \sum_{0\le s\le T} \Delta Y_s \cdot \Delta Z_s \ , \qquad (3.8.11)$$

the sum nearly converging absolutely. Consequently (recall that $Z_{0-} \overset{\text{def}}{=} 0$)

$$Y_T \cdot Z_T = Y_0 \cdot Z_0 + \int_{0+}^T Y \, dZ + \int_{0+}^T Z_{.-} \, dY + {}^c[Y, Z]_T$$
$$= \int_0^T Y \, dZ + \int_0^T Z_{.-} \, dY + {}^c[Y, Z]_T \ .$$

Proof. The integral on the left exists since ΔY is previsible and has finite maximal function $\Delta Y_T^\star \le 2Y_T^\star$ (lemma 2.3.2 and theorem 3.7.17). Let $\epsilon > 0$ and define $S'_0 = 0$, $S'_{n+1} = \inf\{t > S'_n : |\Delta Y_t| \ge \epsilon\}$. Next fix an instant t and let S_n be the reduction of S'_n to $[S'_n \le t] \in \mathcal{F}_{S'_n-}$. Since the graph of S'_{n+1} is the intersection of the previsible sets $(\!(S'_n, S'_{n+1}]\!]$ and $[|\Delta Y| \ge \epsilon]$, the S'_n are predictable stopping times (theorem 3.5.13). Thanks to lemma 3.5.15 (iv), so are the S_n. Also, the S_n have disjoint graphs. Thanks to theorem 3.5.14,[6]

$$\int_0^t \bigcup_n [\![S_n]\!] \cdot \Delta Y \, dZ = \sum_n \int_0^t [\![S_n]\!] \cdot \Delta Y \, dZ = \sum_n \Delta Y_{S_n} \cdot \Delta Z_{S_n} \ .$$

We take the limit as $\epsilon \to 0$ and arrive at the claim, at least at the instant t. We apply this to the stopped process Z^T and let $t \to \infty$: equation (3.8.11) holds in general. The absolute convergence of the sum follows from the inequality of Kunita–Watanabe.

The second claim follows from the definition (3.8.8) of the continuous and pure jump square brackets by

$$Y_T \cdot Z_T = \int_0^T Y_{.-}\, dZ + \int_0^T Z_{.-}\, dY + {}^c[Y,Z]_T + \sum_{0 \le s \le T} \Delta Y_s \cdot \Delta Z_s \ . \qquad \blacksquare$$

Corollary 3.8.23 *Let V be a right-continuous previsible process of integrable total variation. For any bounded martingale M and stopping times $S \le T$*

$$\mathbb{E}[M_T V_T - M_S V_S] = \mathbb{E}\Big[\int_{S+}^T M_{.-}\, dV\Big] \ . \tag{3.8.12}$$

Proof. Let $T' \le T$ be a bounded stopping time such that V is bounded on $[\![0, T']\!]$ (corollary 3.5.16). Since by exercise 3.8.12 ${}^c[M,V] = 0$, proposition 3.8.22 gives

$$M_{T' \vee S} V_{T' \vee S} - M_S V_S = \int_{S+}^{T' \vee S} M_{.-}\, dV + \int_{S+}^{T' \vee S} V\, dM \ .$$

The term on the far right has expectation 0. Now take $T' \uparrow T$. $\blacksquare$

It is shown in proposition 4.3.2 on page 222 that equation (3.8.12) actually characterizes the predictable processes among the right-continuous processes of finite variation.

Exercise 3.8.24 (i) A previsible local martingale of finite variation is constant.

(ii) The bracket $[V, M]$ of a previsible process V of finite variation and a local martingale M is a local martingale of finite variation.

Exercise 3.8.25 (i) Let Z be an L^0-integrator, T a random time, and f a random variable. If $f \cdot [\![T]\!]$ is Z–0-integrable, then $\int f \cdot [\![T]\!]\, dZ = f \cdot \Delta Z_T$.

(ii) Let Y, Z be L^0-integrators and assume that Y is Z-measurable. Then for any almost surely finite stopping time T

$$Y_T \cdot Z_T = Y_0 \cdot Z_0 + \int_{0+}^T Y\, dZ + \int_{0+}^T Z_{.-}\, dY + {}^c[Y,Z]_T \ .$$

3.9 Itô's Formula

Itô's formula is the stochastic analog of the Fundamental Theorem of Calculus. It identifies the increasing process $A[\Phi; \boldsymbol{Z}]$ of theorem 3.8.1 in terms of the second derivatives of Φ and the square brackets of $\boldsymbol{Z}$. Namely,

Theorem 3.9.1 (Itô) *Let $D \subset \mathbb{R}^d$ be open and let $\Phi : D \to \mathbb{R}$ be a twice continuously differentiable function. Let $\boldsymbol{Z} = (Z^\eta)^{\eta=1\ldots d}$ be a d-vector of L^0-integrators such that the paths of both $\boldsymbol{Z}$ and its left-continuous ver-*

sion $\boldsymbol{Z}_{\bullet-}$ stay in D at all times. Then $\Phi(\boldsymbol{Z})$ is an L^0-integrator, and for any nearly finite stopping time T

$$\Phi(\boldsymbol{Z}_T) = \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z}_{\bullet-})\, dZ^\eta + \int_0^1 (1-\lambda) \int_{0+}^{T} \Phi_{;\eta\theta}\big(\boldsymbol{Z}_{\bullet-} + \lambda\Delta\boldsymbol{Z}\big)\, d[Z^\eta, Z^\theta]\, d\lambda \tag{3.9.1}$$

$$= \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z}_{\bullet-})\, dZ^\eta + \frac{1}{2}\int_{0+}^{T} \Phi_{;\eta\theta}(\boldsymbol{Z}_{\bullet-})\, d^c[Z^\eta, Z^\theta] + \sum_{0<s\le T} \Big(\Phi(\boldsymbol{Z}_s) - \Phi(\boldsymbol{Z}_{s-}) - \Phi_{;\eta}(\boldsymbol{Z}_{s-})\cdot \Delta Z_s^\eta\Big)\,, \tag{3.9.2}$$

the last sum nearly converging absolutely.[14] It is often convenient to write equation (3.9.2) in its ***differential form***: in obvious notation

$$d\Phi(Z) = \Phi_{;\eta}(\boldsymbol{Z}_{\bullet-})dZ^\eta + \frac{1}{2}\Phi_{;\eta\theta}(\boldsymbol{Z}_{\bullet-})d^c[Z^\eta, Z^\theta] + \big(\Delta\Phi(\boldsymbol{Z}) - \Phi_{;\eta}(\boldsymbol{Z}_{\bullet-})\Delta Z^\eta\big)\,.$$

Proof. That $\Phi \circ \boldsymbol{Z}$ is an L^0-integrator is evident from (3.9.2) in conjunction with lemma 3.7.5. The $d[Z^\eta, Z^\theta]$-integral in (3.9.1) has to be read as a pathwise Lebesgue–Stieltjes integral, of course, since its integrand is not in general previsible. Note also that the two expressions (3.9.1) and (3.9.2) for $\Phi(\boldsymbol{Z}_T)$ agree. Namely, since the continuous part $d^c[Z^\eta, Z^\theta]$ of the square bracket does not charge the instants t – at most countable in number – where $\Delta\boldsymbol{Z}_t \ne 0$,

$$\int_0^1 (1-\lambda) \int_{0+}^{T} \Phi_{;\eta\theta}\big(\boldsymbol{Z}_{\bullet-} + \lambda\Delta\boldsymbol{Z}\big)\, d^c[Z^\eta, Z^\theta]\, d\lambda$$

$$= \int_0^1 (1-\lambda) \int_{0+}^{T} \Phi_{;\eta\theta}\big(\boldsymbol{Z}_{\bullet-}\big)\, d^c[Z^\eta, Z^\theta]\, d\lambda = \frac{1}{2}\int_{0+}^{T} \Phi_{;\eta\theta}\big(\boldsymbol{Z}_{\bullet-}\big) d^c[Z^\eta, Z^\theta]\,.$$

This leaves

$$\int_0^1 (1-\lambda) \int_{0+}^{T} \Phi_{;\eta\theta}\big(\boldsymbol{Z}_{\bullet-} + \lambda\Delta\boldsymbol{Z}\big)\, d^j[Z^\eta, Z^\theta]\, d\lambda = \sum_{0<s\le T} \int_0^1 (1-\lambda)\Phi_{;\eta\theta}\big(\boldsymbol{Z}_{s-} + \lambda\Delta\boldsymbol{Z}_s\big)\Delta Z_s^\eta \Delta Z_s^\theta\, d\lambda\,,$$

the jump part, which by Taylor's formula of order two (A.2.42) equals the sum in (3.9.2).

To start on the proof proper observe that any linear combination of two functions in $C^2(D)$ (see proposition A.2.11 on page 372) that satisfy equation (3.9.2) again satisfies this equation: such functions form a vector space $\mathcal{I}$. We leave it as an exercise in bookkeeping to show that $\mathcal{I}$ is also closed

[14] Subscripts after semicolons denote partial derivatives, e.g., $\Phi_{;\eta} \stackrel{\text{def}}{=} \frac{\partial\Phi}{\partial x^\eta}$, $\Phi_{;\eta\theta} \stackrel{\text{def}}{=} \frac{\partial^2\Phi}{\partial x^\eta \partial x^\theta}$.

under multiplication, so that it is actually an algebra. Since every coordinate function $\boldsymbol{z} \mapsto z^\eta$ is evidently a member of $\mathcal{I}$, every polynomial belongs to $\mathcal{I}$.

By proposition A.2.11 on page 372 there exists a sequence of polynomials P_k that converges to Φ uniformly on compact subsets of D, and such that every first and second partial of P_k also converges to the corresponding first or second partial of Φ, uniformly on every compact subset of D. Now observe this: the image of the path $\boldsymbol{Z}_{\bullet}(\omega)$ on $[0,t]$ has compact closure $\Gamma(\omega)$ in D, for every $\omega \in \Omega$ and $t > 0$ – this is immediate from the fact that the path is càdlàg and the assumption that it and its left-continuous version stay in D at all times t. Since $(P_k)_{;\eta} \to \Phi_{;\eta}$ uniformly on $\Gamma(\omega)$, the maximal process G_η of the process $\sup_k |(P_k)_{;\eta}(\boldsymbol{Z})|$ is finite at (t,ω), and this holds for all $t < \infty$, all $\omega \in \Omega$, and $\eta = 1,\ldots,d$ (see convention 2.3.5). By theorem 3.7.17 on page 137, the previsible processes $G_{\eta .-}$ are therefore Z^η-integrable in the sense L^0 and can serve as "the dominators" in the DCT: according to the latter $\Phi_{;\eta}(\boldsymbol{Z}_{.-}) = \lim (P_k)_{;\eta}(\boldsymbol{Z}_{.-})$ in $\lceil\!\lceil\ \rceil\!\rceil^*_{Z^\eta\text{–}0}$-mean

and
$$\int_0^t (P_k)_{;\eta}(\boldsymbol{Z}_{.-})\, dZ^\eta \xrightarrow[k\to\infty]{} \int_0^t \Phi_{;\eta}(\boldsymbol{Z}_{.-})\, dZ^\eta \text{ in measure.} \tag{$*$}$$

Clearly
$$(P_k)_{;\eta\theta}\big(\boldsymbol{Z}_{.-} + \lambda\Delta\boldsymbol{Z}\big) \xrightarrow[k\to\infty]{} \Phi_{;\eta\theta}\big(\boldsymbol{Z}_{.-} + \lambda\Delta\boldsymbol{Z}\big) \tag{$**$}$$
uniformly up to time t. Now equation (3.9.1) is known to hold with P_k replacing Φ. The facts $(*)$ and $(**)$ allow us to take the limit as $k \to \infty$ and to conclude that (3.9.1) persists on Φ. Finally, càdlàg processes that nearly agree at any instant t nearly agree at any nearly finite stopping time T: (3.9.1) is established. ▮

The Doléans–Dade Exponential

Here is a computation showing how Itô's theorem can be used to solve a stochastic differential equation.

Proposition 3.9.2 *Let Z be an L^0-integrator. There exists a unique right-continuous process $\mathscr{E} = \mathscr{E}[Z]$ with $\mathscr{E}_0 = 1$ satisfying $d\mathscr{E} = \mathscr{E}_{.-}\, dZ$ on $[\![0,\infty)\!)$, that is to say*
$$\mathscr{E}_t = 1 + \int_0^t \mathscr{E}_{.-}\, dZ \quad \textit{for } t \ge 0$$
or, equivalently,
$$\mathscr{E} = 1 + \mathscr{E}_{.-} * Z\ . \tag{3.9.3}$$
It is given by
$$\mathscr{E}_t[Z] = e^{Z_t - Z_0 - {}^c[Z,Z]_t/2} \cdot \prod_{0<s\le t} \big(1 + \Delta Z_s\big) e^{-\Delta Z_s} \tag{3.9.4}$$
*and is called the **Doléans–Dade**, or **stochastic**, **exponential** of Z.*

Proof. There is no loss of generality in assuming that $Z_0 = 0$; neither $\mathscr{E}$ nor the right-hand side $'\mathscr{E}$ of equation (3.9.4) change if Z is replaced by $Z - Z_0$.

Set
$$T_1 \stackrel{\text{def}}{=} \inf\{t : {}'\mathscr{E}_t \le 0\} = \inf\{t : \Delta Z_t \le -1\}\ ,$$
$${}^1Z \stackrel{\text{def}}{=} Z^{T_1-} = (Z - \Delta Z_{T_1})^{T_1}\ ,$$ "the process Z ***stopped just before*** T_1,"

and
$${}^1L_t \stackrel{\text{def}}{=} {}^1Z_t - {}^c[{}^1Z, {}^1Z]_t/2 + \textstyle\sum_{0<s\le t}\big(\ln(1+\Delta {}^1Z_s) - \Delta {}^1Z_s\big)\ .$$

Since $\big|\ln(1{+}u){-}u\big| \le 2u^2$ for $|u| < 1/2$ and $\sum_{0<s\le t}\big|\Delta Z_s\big|^2 < \infty$, the sum on the right converges absolutely, exhibiting ${}^1L$ as an L^0-integrator. Therefore so is ${}^1\mathscr{E} \stackrel{\text{def}}{=} e^{{}^1L}$. A straightforward application of Itô's formula shows that ${}^1\mathscr{E} = {}'\mathscr{E}^{T_1-}$ satisfies ${}^1\mathscr{E} = 1 + {}^1\mathscr{E}_{.-}{*}{}^1Z$. A simple calculation of jumps invoking proposition 3.8.21 then reveals that ${}'\mathscr{E}$ satisfies ${}'\mathscr{E}^{T_1} = 1 + {}'\mathscr{E}^{T_1}_{.-}{*}Z^{T_1}$.

Next set $\quad T_2 \stackrel{\text{def}}{=} \inf\{t > T_1 : \Delta Z_t \le -1\}$,

$$ {}^2Z \stackrel{\text{def}}{=} Z^{T_2-} - Z^{T_1} $$

and $$ {}^2L_t \stackrel{\text{def}}{=} {}^2Z_t - {}^c[{}^2Z, {}^2Z]_t/2 + \sum_{0<s\le t}\big(\ln(1{+}\Delta{}^2Z_s) - \Delta{}^2Z_s\big) . $$

The same argument as above shows that ${}^2\mathscr{E} \stackrel{\text{def}}{=} e^{{}^2L} = 1{+}{}^2\mathscr{E}_{.-}{*}{}^2Z$. Now clearly ${}'\mathscr{E} = {}^1\mathscr{E}_{T_1} \cdot {}^2\mathscr{E}$ on $[\![T_1, T_2)\!)$, from which we conclude that ${}'\mathscr{E}$ satisfies (3.9.3) on $[\![0, T_2)\!)$, and by proposition 3.8.21 even on $[\![0, T_2]\!]$. We continue by induction and see that the right-hand side ${}'\mathscr{E}$ of (3.9.4) solves (3.9.3).

Exercise 3.9.3 Finish the proof by establishing the uniqueness and use the latter to show that $\mathscr{E}[Z]\cdot\mathscr{E}[Z'] = \mathscr{E}[Z + Z' + [Z, Z']]$ for any two L^0-integrators Z, Z'.

Exercise 3.9.4 (i) The solution of $dX = X_{.-}\,dZ$, $X_0 = x$, is $X = x\cdot\mathscr{E}[Z]$.
(ii) If Z is bounded below and $S \stackrel{\text{def}}{=} \inf\{s : \Delta Z_s = -1\}$ then, for any $u < \infty$ and $\omega \in \Omega$, $t \mapsto \mathscr{E}_t[Z](\omega)$ is bounded away from zero for $0 \le t < S \wedge u$. ∎

Corollary 3.9.5 (Lévy's Characterization of a Standard Wiener Process) *Assume that* $\boldsymbol{M} = (M^1, \ldots, M^d)$ *is a d-vector of local martingales on some measured filtration* $(\mathcal{F}_., \mathbb{P})$ *and that* $\boldsymbol{M}$ *has the same bracket as a standard d-dimensional Wiener process:* $M^\eta_0 = 0$ *and* $[M^\eta, M^\theta]_t = \delta^{\eta\theta}\cdot t$ *for* $\eta, \theta = 1 \ldots d$. *Then* $\boldsymbol{M}$ *is a standard d-dimensional Wiener process.*

In fact, for any finite stopping time T, $\boldsymbol{N}_. \stackrel{\text{def}}{=} \boldsymbol{M}_{T+.} - \boldsymbol{M}_T$ *is a standard Wiener process on* $\mathcal{G}_. \stackrel{\text{def}}{=} \mathcal{F}_{T+.}$ *and is independent of* $\mathcal{F}_T$.

Proof. We do the case $d = 1$. The generalization to $d > 1$ is trivial (see example A.3.31 and proposition 3.8.16). Note first that $(N_0)^2 = (\Delta N_0)^2 = [N, N]_0 = 0$, so that $N_0 = 0$. Since $[N, N]_t = [M, M]_{T+t} - [M, M]_T = t$ and therefore ${}^j[N, N] = 0$, N is continuous. Thanks to Doob's optional stopping theorem 2.5.22, it is locally a bounded martingale on $\mathcal{G}_.$. Now let Γ denote the vector space of all functions $\gamma : [0, \infty) \to \mathbb{R}$ of compact support that have a continuous derivative $\dot\gamma$. We view $\gamma \in \Gamma$ as the cumulative distribution function of the measure $d\gamma_t = \dot\gamma_t dt$. Since γ has finite variation, $[N, \gamma] = 0$; since $N_0\gamma_0 = N_u\gamma_u = 0$ when u lies to the right of the support of $\dot\gamma$, $\int_0^\infty N\,d\gamma = -\int_0^\infty \gamma\,dN$. Proposition 3.9.2 exhibits

$$ e^{i\int_0^\infty N\,d\gamma + \int_0^\infty \gamma_s^2\,ds/2} = e^{-(i\gamma*N)_\infty - [i\gamma*N, i\gamma*N]_\infty/2} $$

as the value at ∞ of a bounded $\mathcal{G}$-martingale with expectation 1.

Thus $$ \mathbb{E}\Big[e^{i\int_0^\infty N\,d\gamma}\cdot A\Big] = e^{-\int_0^\infty \gamma_s^2\,ds/2}\cdot\mathbb{E}[A] \qquad (*) $$

for any $A \in \mathcal{G}_0 = \mathcal{F}_T$. Now the measures $d\gamma$, $\gamma \in \Gamma$, form the dual of the space $\mathscr{C}$: equation $(*)$ simply says that that $N_.$ is independent of $\mathcal{F}_T$ and that the characteristic function of the law of N is

$$\gamma \mapsto e^{-\int_0^t \gamma_s^2 \, ds/2} .$$

The same calculation shows that this function is also the characteristic function of Wiener measure $\mathbb{W}$ (proposition 3.8.16). The law of N is thus $\mathbb{W}$ (exercise A.3.35). ∎

Additional Exercises

Exercise 3.9.6 (Wiener Processes with Covariance) (i) Let $\boldsymbol{W}$ be a standard n-dimensional Wiener process as in 3.9.5 and U a constant $d{\times}n$-matrix. Then[15] $\boldsymbol{W}' \stackrel{\text{def}}{=} U\boldsymbol{W} = (U^\eta_\nu W^\nu)$ is a d-vector of Wiener processes with ***covariance matrix*** $B = (B^{\eta\theta})$ defined by

$$\mathbb{E}[W'^\eta_t \cdot W'^\theta_t] = \mathbb{E}[W'^\eta, W'^\theta]_t = tB^{\eta\theta} \stackrel{\text{def}}{=} t(UU^T)^{\eta\theta} = t\sum_\nu U^\eta_\nu U^\theta_\nu .$$

(ii) Conversely, suppose that $\boldsymbol{W}'$ is a d-vector of continuous local martingales that vanishes at 0 and has square function $[\boldsymbol{W}', \boldsymbol{W}']_t = tB$, B constant and (necessarily) symmetric and positive semidefinite. There exist an $n \in \mathbb{N}$ and a $d{\times}n$-matrix U such that $B^{\eta\theta} = \sum_{\nu=1}^n U^\eta_\nu U^\theta_\nu$. Then there exists a standard n-dimensional Wiener process $\boldsymbol{W}$ so that $\boldsymbol{W}' = U\boldsymbol{W}$.

(iii) The integrator size of $\boldsymbol{W}'$ can be estimated by $\lceil \boldsymbol{W}'^t \rceil_{\mathcal{I}^p} \approx \sqrt{t}\,\|B\|$ for $p > 0$, where $\|B\|$ is the operator size of $B : \ell^\infty \to \ell^1$, $\|B\| \stackrel{\text{def}}{=} \sup\{\zeta_\eta \zeta_\theta B^{\eta\theta} : |\boldsymbol{\zeta}|_{\ell^\infty} \le 1\}$.

Exercise 3.9.7 A standard Wiener process $\boldsymbol{W}$ starts over at any finite stopping time T; in fact, the process $t \mapsto \boldsymbol{W}_{T+t} - \boldsymbol{W}_T$ is again a standard Wiener process and is independent of $\mathcal{F}_T[\boldsymbol{W}]$.

Exercise 3.9.8 Let M be a continuous martingale on the right-continuous filtration $\mathcal{F}_.$ and assume that $M_0 = 0$ and $[M, M]_t \xrightarrow[t\to\infty]{} \infty$ almost surely. Set

$$T^\lambda = \inf\{t : [M, M]_t \ge \lambda\} \qquad \text{and} \qquad T^{\lambda+} = \inf\{t : [M, M]_t > \lambda\} .$$

Then $W_\lambda \stackrel{\text{def}}{=} M_{T^{\lambda+}}$ is a continuous martingale on the filtration $\mathcal{G}_\lambda \stackrel{\text{def}}{=} \mathcal{F}_{T^{\lambda+}}$ with $W_0 = 0$ and $[W, W]_\lambda = \lambda$ and consequently is a standard Wiener process on $\mathcal{G}_.$. Furthermore, if X is $\mathcal{G}$-predictable, then $X_{[M,M]}$ is $\mathcal{F}$-predictable; if X is also W–p-integrable, $0 \le p < \infty$, then $X_{[M,M]}$ is M–p-integrable and

$$\int X_\lambda \, dW_\lambda = \int X_{[M,M]_t} \, dM_t .$$

Conversely, if X is predictable on $\mathcal{F}_.$, then $X_{T^\cdot}$ is predictable on $\mathcal{G}_.$; and if X is M–p-integrable, then $X_{T^\cdot}$ is W–p-integrable and

$$\int X_t \, dM_t = \int X_{T^\lambda} \, dW_\lambda .$$

Definition 3.9.9 *Let X be an n-dimensional continuous integrator on the measured filtration $(\Omega, \mathcal{F}_., \mathbb{P})$, T a stopping time, and $H = \{x : \langle \xi | x \rangle = a\}$ a hyperplane*

[15] Einstein's convention, adopted, implies summation over the same indices in opposite positions.

*in $\mathbb{R}^n$ with equation $\langle\xi|x\rangle = a$. We call H **transparent** for the path $X_{.}(\omega)$ at the time T if the following holds: if $X_T(\omega) \in H$, then $T^{\pm} \stackrel{\text{def}}{=} \inf\{t > T : \langle\xi|X\rangle \gtrless a\} = T$ at ω. This expresses the idea that immediately after T the path $X_{.}(\omega)$ can be found strictly on both sides of H, "or oscillates between the two strict sides of H." In the opposite case, when the path $X_{.}(\omega)$ stays on one side of H for a strictly positive length of time, we call H **opaque** for the path.*

Exercise 3.9.10 Suppose that X is a continuous martingale under $\mathbb{P} \in \mathfrak{P}$ such that $\xi_\mu\xi_\nu[X^\mu, X^\nu]_t$ increases strictly to ∞ as $t \to \infty$, for every non-zero $\xi \in \mathbb{R}^n$.

(i) For any hyperplane H and finite stopping time T with $X_T \in H$, the paths for which H is opaque form a $\mathbb{P}$-nearly empty set.

(ii) Let S be a finite stopping time such that X_S depends continuously on the path (in the topology of uniform convergence on compacta), and H a hyperplane with equation $\langle\xi|x\rangle = a$ and not containing X_S. Then the stopping times

$$T \stackrel{\text{def}}{=} \inf\{t > S : X_t \in H\} \quad \text{and} \quad T^{\pm} \stackrel{\text{def}}{=} \inf\{t > T : \langle\xi|X_t\rangle \gtrless a\}$$

$\mathbb{P}$-nearly are finite, agree, and are continuous.

Girsanov Theorems

Girsanov theorems are results to the effect that the sum of a standard Wiener process and a suitably smooth and small process of finite variation, a "slightly shifted Wiener process," is again a standard Wiener process, provided the original probability $\mathbb{P}$ is replaced with a properly chosen locally equivalent probability $\mathbb{P}'$.

We approach this subject by investigating how much a martingale under $\mathbb{P}' \approx_{.} \mathbb{P}$ deviates from being a $\mathbb{P}$-martingale. We assume that the filtration satisfies the natural conditions under either of $\mathbb{P}, \mathbb{P}'$ and then under both (exercise 1.3.42). The restrictions $\mathbb{P}_t, \mathbb{P}'_t$ of $\mathbb{P}, \mathbb{P}'$ to $\mathcal{F}_t$ being by definition mutually absolutely continuous at finite times t, there are Radon–Nikodym derivatives (theorem A.3.22): $\mathbb{P}'_t = G'_t\mathbb{P}_t$ and $\mathbb{P}_t = G_t\mathbb{P}'_t$. Then G' is a $\mathbb{P}$-martingale, and G is a $\mathbb{P}'$-martingale. G, G' can be chosen right-continuous (proposition 2.5.13), strictly positive, and so that $G \cdot G' \equiv 1$. They have expectations $\mathbb{E}[G'_t] = \mathbb{E}'[G_t] = 1$, $0 \le t < \infty$. Here $\mathbb{E}'$ denotes the expectation with respect to $\mathbb{P}'$, of course. $\mathbb{P}'$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_\infty$ if and only if G' is uniformly $\mathbb{P}$-integrable (see exercises 2.5.2 and 2.5.14).

Lemma 3.9.11 (Girsanov–Meyer) *Suppose M' is a local $\mathbb{P}'$-martingale. Then $M'G'$ is a local $\mathbb{P}$-martingale, and*

$$M' = \Big(M'_0 - G_{.-}{*}[M', G']\Big) + \Big(G_{.-}{*}(M'G') - (M'G)_{.-}{*}G'\Big) . \tag{3.9.5}$$

Reversing the roles of $\mathbb{P}, \mathbb{P}'$ gives this information: if M is a local $\mathbb{P}$-martingale, then

$$\begin{aligned} M - G_{.-}{*}[M, G'] &= M + G'_{.-}{*}[M, G] \\ &= M_0 + G'_{.-}{*}(MG) - (MG')_{.-}{*}G , \end{aligned} \tag{3.9.6}$$

every one of the processes in (3.9.6) being a local $\mathbb{P}'$-martingale.

The point, which will be used below and again in the proof of proposition 4.4.1, is that the first summand in (3.9.5) is a process of finite variation and the second a local $\mathbb{P}$-martingale, being as it is the difference of indefinite integrals against two local $\mathbb{P}$-martingales.

Proof. Two easy manipulations show that G, G' are martingales with respect to $\mathbb{P}', \mathbb{P}$, respectively, and that a process N' is a $\mathbb{P}'$-martingale if and only if the product $N'G'$ is a $\mathbb{P}$-martingale. Localization exhibits $M'G'$ as a local $\mathbb{P}$-martingale.

Now $$M'G' = G'_{\bullet-}{*}M' + M'_{\bullet-}{*}G' + [G', M']$$

gives $$G_{\bullet-}{*}(M'G') = (\!(0,\infty)\!){*}M' + (GM')_{\bullet-}{*}G' + G_{\bullet-}{*}[G', M'] \;,$$

and exercise 3.7.9 produces the claim after sorting terms. The second equality in (3.9.6) is the same as equation (3.9.5) with the roles of $\mathbb{P}$, $\mathbb{P}'$ reversed and the finite variation process shifted to the other side. Inasmuch as $GG' = 1$, we have $0 = G_{\bullet-}{*}G' + G'_{\bullet-}{*}G + [G, G']$, whence $0 = G_{\bullet-}{*}[G', M] + G'_{\bullet-}{*}[G, M]$ for continuous M, which gives the first equality. ∎

Now to approach the classical Girsanov results concerning Wiener process, consider a standard d-dimensional Wiener process $\boldsymbol{W} = (W^1, \ldots, W^d)$ on the measured filtration $(\mathcal{F}_{\bullet}, \mathbb{P})$ and let $\boldsymbol{h} = (h^1, \ldots, h^d)$ be a locally bounded $\mathcal{F}_{\bullet}$-previsible process. Then clearly the indefinite integral

$$M \stackrel{\text{def}}{=} \boldsymbol{h}{*}\boldsymbol{W} \stackrel{\text{def}}{=} \textstyle\sum_{\eta=1}^{d} h^\eta {*} W^\eta$$

is a continuous locally bounded local martingale and so is its Doléans–Dade exponential (see proposition 3.9.2)

$$G'_t \stackrel{\text{def}}{=} \exp\Big(M_t - 1/2\int_0^t |\boldsymbol{h}|_s^2\, ds\Big) = 1 + \int_0^t G'_s\, dM_s \;.$$

G' is a strictly positive supermartingale and is a martingale if and only if $\mathbb{E}[G'_t] = 1$ for all $t > 0$ (exercise 2.5.23 (iv)). Its reciprocal $G \stackrel{\text{def}}{=} 1/G'$ is an L^0-integrator (exercise 2.5.32 and theorem 3.9.1).

Exercise 3.9.12 (i) If there is a locally Lebesgue square integrable function $\eta : [0,\infty) \to \mathbb{R}$ so that $|\boldsymbol{h}|_t \le \eta_t \quad \forall\, t$, then G' is a square integrable martingale; in fact, then clearly $\mathbb{E}[G'^2_t] \le \exp(\int_0^t \eta_s^2\, ds)$. (ii) If it can merely be ascertained that the quantity

$$\mathbb{E}\Big[\exp\Big(\frac{1}{2}\int_0^t |\boldsymbol{h}|_s^2\, ds\Big)\Big] = \mathbb{E}\big[\exp\big([M, M]_t/2\big)\big] \tag{3.9.7}$$

is finite at all instants t, then G' is still a martingale. Equation (3.9.7) is known as ***Novikov's condition***. The condition $\mathbb{E}[\exp([M, M]_t/b)] < \infty$ for some $b > 2$ and all $t \ge 0$ will not do in general.

After these preliminaries consider the "shifted Wiener process"

$$\boldsymbol{W}' \stackrel{\text{def}}{=} \boldsymbol{W} + \boldsymbol{H} \quad \text{, where} \quad \boldsymbol{H}_{\boldsymbol{.}} \stackrel{\text{def}}{=} \int_0^{\boldsymbol{.}} \boldsymbol{h}_s \, ds = [M, \boldsymbol{W}]_{\boldsymbol{.}} \; .$$

Assume for the moment that G' is a uniformly integrable martingale, so that there is a limit G'_∞ in mean and almost surely (2.5.14). Then $\mathbb{P}' \stackrel{\text{def}}{=} G'_\infty \mathbb{P}$ defines a probability absolutely continuous with respect to $\mathbb{P}$ and locally equivalent to $\mathbb{P}$. Now $\boldsymbol{H}$ equals $G{*}[G', \boldsymbol{W}]$ and thus $\boldsymbol{W}'$ is a vector of local $\mathbb{P}'$-martingales – see equation (3.9.6) in the Girsanov–Meyer lemma 3.9.11. Clearly $\boldsymbol{W}'$ vanishes at time 0 and has the same bracket as a standard Wiener process. Due to Lévy's characterization 3.9.5, $\boldsymbol{W}'$ is itself a standard Wiener process under $\mathbb{P}'$. The requirement of uniform integrability will be satisfied for instance when G' is $L^2(\mathbb{P})$-bounded, which in turn is guaranteed by part (i) of exercise 3.9.12 when the function η is Lebesgue square integrable. To summarize:

Proposition 3.9.13 (Girsanov — the Basic Result) *Assume that G' is uniformly integrable. Then $\mathbb{P}' \stackrel{\text{def}}{=} G'_\infty \mathbb{P}$ is absolutely continuous with respect to $\mathbb{P}$ on $\mathcal{F}_\infty$ and $\boldsymbol{W}'$ is a standard Wiener process under $\mathbb{P}'$.*

In particular, if there is a Lebesgue square integrable function η on $[0, \infty)$ such that $|\boldsymbol{h}_t(\omega)| \le \eta_t$ for all t and all $\omega \in \Omega$, then G' is uniformly integrable and moreover $\mathbb{P}$ and $\mathbb{P}'$ are mutually absolutely continuous on $\mathcal{F}_\infty$.

Example 3.9.14 The assumption of uniform integrability in proposition 3.9.13 is rather restrictive. The simple shift $W'_t = W_t + t$ is not covered. Let us work out this simple one-dimensional example in order to see what might and might not be expected under less severe restrictions. Since here $h \equiv 1$, we have $G'_t = \exp(W_t - t/2)$, which is a square integrable – but not square bounded, not even uniformly integrable – martingale. Nevertheless there is, for every instant t, a probability $\mathbb{P}'_t$ on $\mathcal{F}_t$ equivalent with the restriction $\mathbb{P}_t$ of $\mathbb{P}$ to $\mathcal{F}_t$, to wit, $\mathbb{P}'_t \stackrel{\text{def}}{=} G'_t \mathbb{P}_t$. The pairs $(\mathcal{F}_t, \mathbb{P}'_t)$ form a ***consistent family of probabilities*** in the sense that for $s < t$ the restriction of $\mathbb{P}'_t$ to $\mathcal{F}_s$ equals $\mathbb{P}'_s$. There is therefore a unique measure $\mathbb{P}'$ on the algebra $\mathcal{A}_\infty \stackrel{\text{def}}{=} \bigcup_t \mathcal{F}_t$ of sets, the ***projective limit***, defined unequivocally by

$$\begin{aligned} \mathbb{P}'[A] &\stackrel{\text{def}}{=} \mathbb{P}'_s[A] \quad \text{if} \quad A \in \mathcal{A}_\infty \quad \text{belongs to} \quad \mathcal{F}_s \; . \\ &= \mathbb{P}'_t[A] \quad \text{if} \quad A \quad\;\; \text{also belongs to} \quad \mathcal{F}_t \; . \end{aligned}$$

Things are looking up. Here is a damper,[16] though: $\mathbb{P}'$ cannot be absolutely continuous with respect to $\mathbb{P}$. Namely, since $\lim_{t\to\infty} W_t/t = 0$ $\mathbb{P}$-almost surely, the set $[\lim_{t\to\infty} W_t/t = -1]$ is $\mathbb{P}$-negligible; yet this set has $\mathbb{P}'$-measure 1, since it coincides with the set $[\lim_t W'_t/t = 0]$. *Mutatis*

[16] This point is occasionally overlooked in the literature.

mutandis we see that $\mathbb{P}$ is not absolutely continuous with respect to $\mathbb{P}'$ either. In fact, these two measures are disjoint.

The situation is actually even worse. Namely, in the previous argument the σ-additivity of $\mathbb{P}'$ was used, but this is by no means assured.[16] Roughly, Σ-additivity requires that the ambient space be "not too sparse," a feature Ω may miss. Assume for example that the underlying set Ω is the path space $\mathscr{C}$ with the $\mathbb{P}$-negligible Borel set $\{\omega : \limsup |\omega_t/t| > 0\}$ removed, W_t is of course evaluation: $W_t(\omega_.) = \omega_t$, and $\mathbb{P}$ is Wiener measure $\mathbb{W}$ restricted to Ω. The set function $\mathbb{P}'$ is additive on $\mathcal{A}_\infty$ but cannot be σ-additive. If it were, it would have a unique extension to the σ-algebra generated by $\mathcal{A}_\infty$, which is the Borel σ-algebra on Ω; $t \mapsto \omega_t + t$ would be a standard Wiener process under $\mathbb{P}'$ with $\mathbb{P}'\big[\{\omega : \lim(\omega_t + t)/t = 0\}\big] = 1$, yet Ω does not contain a single path ω with $\lim(\omega_t + t)/t = 0$!

On the positive side, the discussion suggests that if Ω is the full path space $\mathscr{C}$, then $\mathbb{P}'$ might in fact be σ-additive as the projective limit of tight probabilities (see theorem A.7.1 (v)). As long as we are content with having $\mathbb{P}'$ absolutely continuous with respect to $\mathbb{P}$ merely locally, there ought to be some "non-sparseness" or "fullness" condition on Ω that permits a satisfactory conclusion even for somewhat large $\boldsymbol{h}$. ∎

Let us approach the Girsanov problem again, with example 3.9.14 in mind. Now the collection $\mathfrak{T}'$ of stopping times T with $\mathbb{E}[G'_T] = 1$ is increasingly directed (exercise 2.5.23 (iv)), and therefore $\mathcal{A} \stackrel{\text{def}}{=} \bigcup_{T \in \mathfrak{T}'} \mathcal{F}_T$ is an algebra of sets. On it we define unequivocally the additive measure $\mathbb{P}'$ by $\mathbb{P}'[A] \stackrel{\text{def}}{=} \mathbb{E}[G'_S A]$ if $A \in \mathcal{A}$ belongs to $\mathcal{F}_S$, $S \in \mathfrak{T}$. Due to the optional stopping theorem 2.5.22, this definition is consistent. It looks more general than it is, however:

Exercise 3.9.15 In the presence of the natural conditions $\mathcal{A}$ generates $\mathcal{F}_\infty$, and for $\mathbb{P}'$ to be σ-additive G' must be a martingale.

Now one might be willing to forgo the σ-additivity of $\mathbb{P}'$ on $\mathcal{F}_\infty$ given as it is that it holds on "arbitrarily large" σ-subalgebras $\mathcal{F}_T$, $T \in \mathfrak{T}$. But probabilists like to think in terms of σ-additive measures, and without the σ-additivity some of the cherished facts about a Wiener process W', such as $\lim_{t\to\infty} W'_t/t = 0$ a.s., for example, are lost. We shall therefore have to assume that G' is a martingale, for instance by requiring the Novikov condition (3.9.7) on $\boldsymbol{h}$.

Let us now go after the "non-sparseness" or "fullness" of $(\Omega, \mathcal{F}_.)$ mentioned above. One can formulate a technical condition essentially to the effect that each of the $\mathcal{F}_t$ contain lots of compact sets; we will go a different route and give a definition[17] that merely spells out the properties we need, and then provide a plethora of permanence properties ensuring that this definition is usually met.

[17] As far as I know first used in Ikeda–Watanabe [40, page 176].

Definition 3.9.16 *(i) The filtration $(\Omega, \mathcal{F}_.)$ is **full** if whenever $(\mathcal{F}_t, \mathbb{P}_t)$ is a consistent family of probabilities (see page 164) on $\mathcal{F}_.$, then there exists a σ-additive probability $\mathbb{P}$ on $\mathcal{F}_\infty$ whose restriction to $\mathcal{F}_t$ is $\mathbb{P}_t$, $t \ge 0$.*

*(ii) The measured filtration $(\Omega, \mathcal{F}_., \mathbb{P})$ is **full** if whenever $(\mathcal{F}_t, \overline{\mathbb{P}}_t)$ is a consistent family of probabilities with $\overline{\mathbb{P}}_t \ll \mathbb{P}$ on $\mathcal{F}_t$,[18] $t < \infty$, then there exists a σ-additive probability $\overline{\mathbb{P}}$ on $\mathcal{F}_\infty$ whose restriction to $\mathcal{F}_t$ is $\overline{\mathbb{P}}_t$, $t \ge 0$. The measured filtration $(\Omega, \mathcal{F}_., \mathfrak{P})$ is full if every one of the measured filtrations $(\Omega, \mathcal{F}_., \mathbb{P})$, $\mathbb{P} \in \mathfrak{P}$, is full.*

Proposition 3.9.17 (The Prime Examples) *Fix a polish space (P, ρ). The cartesian product $P^{[0,\infty)}$ equipped with its basic filtration is full. The path spaces $\mathscr{D}_P$ and $\mathscr{C}_P$ equipped with their basic filtrations are full.* ▮

When making a stochastic model for some physical phenomenon, financial phenomenon, etc., one usually has to begin by producing a filtered measured space that carries a model for the drivers of the stochastic behavior – in this book this happens for instance when Wiener process is constructed to drive Brownian motion (page 11), or when Lévy processes are constructed (page 267), or when a Markov process is associated with a semigroup (page 351). In these instances the naturally appearing ambient space Ω is a path space $\mathscr{D}_P$ or $\mathscr{C}^d$ equipped with its basic full filtration. Thereafter though, in order to facilitate the stochastic analysis, one wishes to discard inconsequential sets from Ω and to go to the natural enlargement. At this point one hopes that fullness has permanence properties good enough to survive these operations. Indeed it has:

Proposition 3.9.18 *(i) Suppose that $(\Omega, \mathcal{F}_.)$ is full, and let $N \in \mathcal{A}_{\infty\sigma}$. Set $\Omega' \stackrel{\text{def}}{=} \Omega \backslash N$, and let $\mathcal{F}'_.$ denote the filtration induced on Ω', that is to say, $\mathcal{F}'_t \stackrel{\text{def}}{=} \{A \cap \Omega' : A \in \mathcal{F}_t\}$. Then $(\Omega', \mathcal{F}'_.)$ is full. Similarly, if the measured filtration $(\Omega, \mathcal{F}_., \mathfrak{P})$ is full and a $\mathfrak{P}$-nearly empty set N is removed from Ω, then the measured filtration induced on $\Omega' \stackrel{\text{def}}{=} \Omega \backslash N$ is full. (ii) If the measured filtration $(\Omega, \mathcal{F}_., \mathfrak{P})$ is full, then so is its natural enlargement. In particular, the natural filtration on canonical path space is full.*

Proof. (i) Let $(\mathcal{F}'_t, \mathbb{P}'_t)$ be a consistent family of σ-additive probabilities, with additive projective limit $\mathbb{P}'$ on the algebra $\mathcal{A}'_\infty \stackrel{\text{def}}{=} \bigcup_t \mathcal{F}'_t$. For $t \ge 0$ and $A \in \mathcal{F}_t$ set $\mathbb{P}_t[A] \stackrel{\text{def}}{=} \mathbb{P}'_t[A \cap \Omega']$. Then $(\mathcal{F}_t, \mathbb{P}_t)$ is easily seen to be a consistent family of σ-additive probabilities. Since $\mathcal{F}_.$ is full there is a σ-additive probability $\mathbb{P}$ that coincides with $\mathbb{P}_t$ on $\mathcal{F}_t$, $t \ge 0$. Now let $\mathcal{A}'_\infty \ni A'_n \downarrow \emptyset$. It is to be shown that $\mathbb{P}'[A'_n] \to 0$; any of the usual extension procedures will then provide the required σ-additive $\mathbb{P}'$ on $\mathcal{F}'_.$ that agrees with $\mathbb{P}'_t$ on $\mathcal{F}'_t$, $t \ge 0$. Now there are $A_n \in \mathcal{A}_\infty$ such that $A'_n = A_n \cap \Omega'$; they can be chosen to decrease as n increases, by replacing A_n with $\bigcap_{\nu \le n} A_\nu$ if necessary. Then there are $N_n \in \mathcal{A}_\infty$ with union N; they can be chosen to increase with n.

[18] I.e., a $\mathbb{P}$-negligible set belonging to $\mathcal{F}_t$ (!) is $\overline{\mathbb{P}}_t$-negligible.

There is an increasing sequence of instants t^n so that both $N_n \in \mathcal{F}_{t^n}$ and $A_n \in \mathcal{F}_{t^n}$, $n \in \mathbb{N}$. Now, since $\bigcap_n A_n \subseteq N$,

$$\begin{aligned}\lim \mathbb{P}'[A'_n] &= \lim \mathbb{P}'_{t^n}[A'_n] = \lim \mathbb{P}'_{t^n}[A_n \cap \Omega'] = \lim \mathbb{P}_{t^n}[A_n] \\ &= \lim \mathbb{P}[A_n] = \mathbb{P}\Big[\bigcap A_n\Big] \le \mathbb{P}[N] = \lim \mathbb{P}[N_n] \qquad (3.9.8) \\ &= \lim \mathbb{P}_{t^n}[N_n] = \lim \mathbb{P}'_{t^n}[N_n \cap \Omega'] = \lim \mathbb{P}'_{t^n}[\emptyset] = 0 .\end{aligned}$$

The proof of the second statement of (i) is left as an exercise.

(ii) It is easy to see that $(\Omega, \mathcal{F}_{.+})$ is full when $(\Omega, \mathcal{F}_.)$ is, so we may assume that $\mathcal{F}_.$ is right-continuous and only need to worry about the regularization. Let then $(\Omega, \mathcal{F}_., \mathfrak{P})$ be a full measured filtration and $(\mathcal{F}^{\mathfrak{P}}_t, \overline{\mathbb{P}}_t)$ a consistent family of σ-additive probabilities on $\mathcal{F}^{\mathfrak{P}}_.$, with additive projective limit $\overline{\mathbb{P}}$ on $\mathcal{A}^{\mathfrak{P}}_\infty \overset{\text{def}}{=} \bigcup_t \mathcal{F}^{\mathfrak{P}}_t$ and $\overline{\mathbb{P}}_t \ll \mathbb{P}$ on $\mathcal{F}^{\mathfrak{P}}_t$, $\mathbb{P} \in \mathfrak{P}, t \ge 0$. The restrictions $\overline{\mathbb{P}}^0_t$ of $\overline{\mathbb{P}}_t$ to $\mathcal{F}_t$ have a σ-additive extension $\overline{\mathbb{P}}^0$ to $\mathcal{F}_\infty$ that vanishes on $\mathbb{P}$-nearly empty sets, $\mathbb{P} \in \mathfrak{P}$, and thus is defined and σ-additive on $\mathcal{F}^{\mathbb{P}}_\infty$. On $\mathcal{A}^{\mathfrak{P}}_\infty$, $\overline{\mathbb{P}}^0$ coincides with $\overline{\mathbb{P}}$, which is therefore σ-additive. ∎

Imagine, for example, that we started off by representing a number of processes, among them perhaps a standard Wiener process $\boldsymbol{W}$ and a few Poisson point processes, canonically on the Skorohod path space: $\Omega = \mathscr{D}^n$. Having proved that the $\omega \in \Omega$ where the path $\boldsymbol{W}_.(\omega)$ is anywhere differentiable form a nearly empty set, we may simply throw them away; the remainder is still full. Similarly we may then toss out the ω where the Wiener paths violate the law of the iterated logarithm, the paths where the approximation scheme 3.7.26 for some stochastic integral fails to converge, etc. What we cannot throw away without risking complications are sets like $[\boldsymbol{W}_t(\cdot)/t \xrightarrow[t\to\infty]{} 0]$ that depend on the tail–σ-algebra of W; they may be negligible but may well not be nearly empty. With a modicum of precaution we have the Girsanov theorem in its most frequently stated form:

Theorem 3.9.19 (Girsanov's Theorem) *Assume that $\boldsymbol{W} = (W^1, \ldots, W^d)$ is a standard Wiener process on the full measured filtration $(\Omega, \mathcal{F}_., \mathbb{P})$, and let $\boldsymbol{h} = (h^1, \ldots, h^d)$ be a locally bounded previsible process. If the Doléans–Dade exponential G' of the local martingale $M \overset{\text{def}}{=} \boldsymbol{h}{*}\boldsymbol{W}$ is a martingale, then there is a unique σ-additive probability $\mathbb{P}'$ on $\mathcal{F}_\infty$ so that $\mathbb{P}' = G'_t\mathbb{P}$ on $\mathcal{F}_t$ at all finite instants t, and*

$$\boldsymbol{W}' \overset{\text{def}}{=} \boldsymbol{W} + [M, \boldsymbol{W}] = \boldsymbol{W} + \int_0^. \boldsymbol{h}_s \, ds$$

is a standard Wiener process under $\mathbb{P}'$.

Warning 3.9.20 In order to ensure a plentiful supply of stopping times (see exercise 1.3.30 and items A.5.10–A.5.21) and the existence of modifications with regular paths (section 2.3) and of cross sections (pages 436–440), most every author requires right off the bat that the underlying filtration $\mathcal{F}_{.}$ satisfy the so-called ***usual conditions***, which say that $\mathcal{F}_{.}$ is right-continuous and that every $\mathcal{F}_t$ contains every negligible set of $\mathcal{F}_\infty$ (!). This is achieved by making the basic filtration right-continuous and by throwing into $\mathcal{F}_0$ all subsets of negligible sets in $\mathcal{F}_\infty$. If the enlargement is effected this way, then theorem 3.9.19 fails, even when Ω is the full path space $\mathscr{C}$ and the shift is as simple as $h \equiv 1$, i.e., $W'_t = W_t + t$, as witness example 3.9.14. In other words, the usual enlargement of a full measured filtration may well not be full. If the enlargement is effected by adding into $\mathcal{F}_0$ only the nearly empty sets,[19] then all of the benefits mentioned persist and theorem 3.9.19 turns true.

We hope the reader will at this point forgive the painstaking (unusual but natural) way we chose to regularize a measured filtration.

The Stratonovich Integral

Let us revisit the algorithm (3.7.11) on page 140 for the pathwise approximation of the integral $\int_0^T X_{.-}\, dZ$. Given a threshold δ we would define stopping times S_k, $k = 0, 1, \ldots$, partitioning $(\!(0, T]\!]$ such that on $\mathcal{I}_k \stackrel{\text{def}}{=} (\!(S_k, S_{k+1}]\!]$ the integrand $X_{.-}$ did not change by more than δ. On each of the intervals $\mathcal{I}_k$ we would approximate the integral by the value of the right-continuous process X at the left endpoint S_k multiplied with the change $Z^T_{S_{k+1}} - Z^T_{S_k}$ of Z^T over $\mathcal{I}_k$. Then we would approximate the integral over $(\!(0, T]\!]$ by the sum over k of these local approximations. We said in remarks 3.7.27 (iii)–(iv) that the limit of these approximations as $\delta \to 0$ would serve as a perfectly intuitive *definition* of the integral, if integrands in $\mathfrak{L}$ were all we had to contend with – definition 2.1.7 identifies the condition under which the limit exists.

Now the practical reader who remembers the trapezoidal rule from calculus might at this point offer the following suggestion. Since from the definition (3.7.10) of S_{k+1} we know the value of X at that time already, a better local approximation to X than its value at the left endpoint might be the average

$$1/2\big(X_{S_k} + X_{S_{k+1}}\big) = X_{S_k} + 1/2\big(X_{S_{k+1}} - X_{S_k}\big)$$

of its values at the two endpoints. He would accordingly propose to define $\int_{0+}^T X\, dZ$ as

$$\lim_{\delta \to 0} \sum_{0 \le k \le \infty} \frac{X_{S_k} + X_{S_{k+1}}}{2} \cdot \big(Z^T_{S_{k+1}} - Z^T_{S_k}\big)$$

$$= \lim_{\delta \to 0} \sum_{0 \le k \le \infty} X_{S_k}\big(Z^T_{S_{k+1}} - Z^T_{S_k}\big) + \frac{1}{2} \lim_{\delta \to 0} \sum_{0 \le k \le \infty} \big(X_{S_{k+1}} - X_{S_k}\big)\big(Z^T_{S_{k+1}} - Z^T_{S_k}\big) .$$

The merit of writing it as in the second line above is that the two limits are actually known: the first one equals the Itô integral $\int_{0+}^T X\, dZ$, thanks to

[19] Think of them as the sets whose negligibility can be detected before the expiration of time.

theorem 3.7.26, and the second limit is $[X, Z^T]_T - [X, Z^T]_0$ – at least when X is an L^0-integrator (page 150). Our practical reader would be lead to the following notion:

Definition 3.9.21 *Let X, Z be two L^0-integrators and T a finite stopping time. The **Stratonovich integral** is defined by*

$$\int_0^T X\ \delta Z \stackrel{\mathrm{def}}{=} X_0 Z_0 + \frac{1}{2} \lim \sum_{0 \le k \le \infty} \big(X_{S_k} + X_{S_{k+1}}\big)\big(Z^T_{S_{k+1}} - Z^T_{S_k}\big) , \tag{3.9.9}$$

the limit being taken as the partition[8] $\mathcal{S} = \{0{=}S_0 \le S_1 \le S_2 \le \ldots \le S_\infty{=}\infty\}$ *runs through a sequence whose mesh goes to zero.* It can be computed in terms of the Itô integral as

$$\int_0^T X\ \delta Z = X_0 Z_0 + \int_0^T X_{\boldsymbol{.}-}\ dZ + \frac{1}{2}\big([X, Z]_T - [X, Z]_0\big) . \tag{3.9.10}$$

$X{\circ}Z$ denotes the corresponding indefinite integral $t \mapsto \int_0^t X\ \delta Z$:

$$X{\circ}Z = X_0 Z_0 + X_{\boldsymbol{.}-}{*}Z + 1/2\big([X, Z] - [X, Z]_0\big) .$$

Remarks 3.9.22 (i) The Itô and Stratonovich integrals not only apply to different classes of integrands, they also give different results when they happen to apply to the same integrand. For instance, when both X and Z are continuous L^0-integrators, then $X{\circ}Z = X{*}Z + 1/2[X, Z]$. In particular, $(W{\circ}W)_t = (W{*}W)_t + t/2$ (proposition 3.8.16).

(ii) Which of the two integrals to use? The answer depends entirely on the purpose. Engineers and other applied scientists generally prefer the Stratonovich integral when the driver Z is continuous. This is partly due to the appeal of the "trapezoidal" definition (3.9.9) and partly to the simplicity of the formula governing coordinate transformations (theorem 3.9.24 below). The ultimate criterion is, of course, which integral better models the physical situation. It is claimed that the Stratonovich integral generally does. Even in pure mathematics – if there is such a thing – the Stratonovich integral is indispensable when it comes to coordinate-free constructions of Brownian motion on Riemannian manifolds, say.

(iii) So why not stick to Stratonovich's integral and forget Itô's? Well, the Dominated Convergence Theorem does not hold for Stratonovich's integral, so there are hardly any limit results that one can prove without resorting to equation (3.9.10), which connects it with Itô's. In fact, when it comes to a computation of a Stratonovich integral, it is generally turned into an Itô integral via (3.9.10), which is then evaluated.

(iv) An algorithm for the pathwise computation of the Stratonovich integral $X{\circ}Z$ is available just as for the Itô integral. We describe it in case both X and Z are L^p-integrators for some $p > 0$, leaving the case $p = 0$ to the reader. Fix a threshold $\delta > 0$. There is a partition $\mathcal{S} = \{0 = S_0 \le S_1 \le \cdots\}$ with $S_\infty \overset{\text{def}}{=} \sup_{k<\infty} S_k = \infty$ on whose intervals $[\![S_k, S_{k+1})\!)$ both X and Z vary by less than δ. For example, the recursive definition $S_{k+1} \overset{\text{def}}{=} \inf\{t > S_k : |X_t - X_{S_k}| \vee |Z_t - Z_{S_k}| > \delta\}$ produces one. The approximate

$$Y_t^{(\delta)} \overset{\text{def}}{=} \sum \frac{X_t^{S_k} + X_t^{S_{k+1}}}{2} \cdot \left(Z_t^{S_{k+1}} - Z_t^{S_k}\right)$$
$$= \sum X_{S_k}(Z_t^{S_{k+1}} - Z_t^{S_k}) + \frac{1}{2}\Big(X_0 Z_0 + \sum_{0<k<\infty} (X_t^{S_{k+1}} - X_t^{S_k})(Z_t^{S_{k+1}} - Z_t^{S_k})\Big)$$

has, by exercise 3.8.14,

$$\Big\|\Big| |(X \circ Z) - Y^{(\delta)}|_t^\star \Big|\Big\|_{L^p} \le 2C_p^{\star(2.3.5)}\left(\lceil \delta Z^t \rceil_{\mathcal{I}^p} + \lceil \delta X^t \rceil_{\mathcal{I}^p}\right) .$$

Thus, if δ runs through a sequence (δ_n) with

$$\textstyle\sum_n \lceil \delta_n Z^n \rceil_{\mathcal{I}^p} + \lceil \delta_n X^n \rceil_{\mathcal{I}^p} < \infty ,$$

then $Y^{(\delta)} \xrightarrow[n\to\infty]{} X \circ Z$ nearly, uniformly on bounded intervals. ∎

Practically, Stratonovich's integral is useful only when *the integrator is a continuous L^0-integrator*, so we shall assume this for the remainder of this subsection.

Given a partition $\mathcal{S}$ and a continuous L^0-integrator Z, let $\overline{Z}^{\mathcal{S}}$ denote the continuous process that agrees in the points S_k of $\mathcal{S}$ with Z and is linear in between. It is clearly not adapted in general: knowledge of $Z_{S_{k+1}}$ is contained in the definition of Z_t for $t \in (\!(S_k, S_{k+1}]\!]$. Nevertheless, the piecewise linear process $\overline{Z}^{\mathcal{S}}$ of finite variation is easy to visualize, and the approximate $Y_t^{(\delta)}$ above is nothing but the Lebesgue–Stieltjes integral $\int_0^t \overline{X}^{\mathcal{S}}\, d\overline{Z}^{\mathcal{S}}$, at least at the points $t = S_k$, $k = 1, 2, \ldots$. In other words, the approximation scheme above can be seen as an approximation of the Stratonovich integral by Lebesgue–Stieltjes integrals that are measurably parametrized by $\omega \in \Omega$.

Exercise 3.9.23 $X \cdot Z = X \circ Z + Z \circ X$, and so $\delta(XZ) = X\delta Z + Z\delta X$. Also, $X \circ (Y \circ Z) = (XY) \circ Z$.

Consider a differentiable curve $t \mapsto \boldsymbol{\zeta}_t = (\zeta_t^1, \ldots, \zeta_t^d)$ in $\mathbb{R}^d$ and a smooth function $\Phi : \mathbb{R}^d \to \mathbb{R}$. The Fundamental Theorem of Calculus says that

$$\Phi(\boldsymbol{\zeta}_t) = \Phi(\boldsymbol{\zeta}_0) + \int_0^t \Phi_{;\eta}(\boldsymbol{\zeta}_s)\, d\zeta_s^\eta .$$

It is perhaps the main virtue of the Stratonovich integral that a similarly simple formula holds for it:

Theorem 3.9.24 *Let $D \subset \mathbb{R}^d$ be open and convex, and let $\Phi : D \to \mathbb{R}$ be a twice continuously differentiable function. Let $\boldsymbol{Z} = (Z^\eta)_{\eta=1}^d$ be a d-vector* *of continuous L^0-integrators and assume that the path of $\boldsymbol{Z}$ stays in D at all times. Then $\Phi(\boldsymbol{Z})$ is an L^0-integrator, and for any almost surely finite stopping time T* [14]

$$\Phi(\boldsymbol{Z}_T) = \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z})\, \delta Z^\eta . \tag{3.9.11}$$

Proof. Recall our convention that $X_{0-} = 0$ for $X \in \mathfrak{D}$. Itô's formula gives

$$\Phi_{;\eta}(\boldsymbol{Z}) = \Phi_{;\eta}(\boldsymbol{Z}_0) + \Phi_{;\eta\theta}(\boldsymbol{Z})_{.-}{*}Z^\theta + 1/2\ \Phi_{;\eta\theta\iota}(\boldsymbol{Z}){*}{}^c[Z^\theta, Z^\iota] ,$$

so by exercise 3.8.12 and proposition 3.8.19

$${}^c\big[\Phi_{;\eta}(\boldsymbol{Z}), Z^\eta\big] = {}^c\big[\Phi_{;\eta\theta}(\boldsymbol{Z}){*}Z^\theta, Z^\eta\big] = \Phi_{;\eta\theta}(\boldsymbol{Z}){*}{}^c[Z^\eta, Z^\theta] .$$

Equation (3.9.10) produces

$$\int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z})\delta Z^\eta = \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z})\, dZ^\eta + 1/2 \int_{0+}^{T} \Phi_{;\eta\theta}(\boldsymbol{Z})\, d{}^c[Z^\eta, Z^\theta] ,$$

thus $$\Phi(\boldsymbol{Z}_T) = \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z})\, dZ^\eta + 1/2 \int_{0+}^{T} \Phi_{;\eta\theta}(\boldsymbol{Z})\, d{}^c[Z^\eta, Z^\theta] ,$$

i.e., $$\Phi(\boldsymbol{Z}_T) = \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z})\delta Z^\eta .$$

For this argument to work Φ must be thrice continuously differentiable. In the general case we find a sequence of smooth functions Φ^n that converge to Φ uniformly on compacta together with their first and second derivatives, use the penultimate equation above, and apply the Dominated Convergence Theorem. ∎

3.10 Random Measures

Very loosely speaking, a random measure is what one gets if the index η in a vector $\boldsymbol{Z} = (Z^\eta)$ of integrators is allowed to vary over a continuous set, the "auxiliary space," instead of the finite index set $\{1, \ldots, d\}$. Visualize for instance a drum pelted randomly by grains of sand (see [108]). At any surface element $d\eta$ and during any interval ds there is some random noise $\beta(d\eta, ds)$ acting on the surface; a suitable model for the noise β together with the appropriate differential equation should describe the effect of the action. We won't go into such a model here but only provide the mathematics to do so, since the integration theory of random measures is such a straightforward extension of the stochastic analysis above. The reader may look at this section as an overview or summary of the material offered so far, done through a slight generalization.

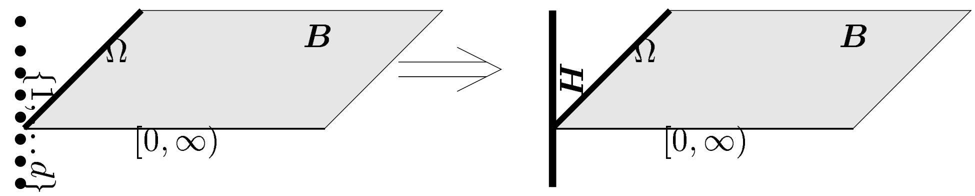

Figure 3.11 Going from a discrete to a continuous auxiliary space

Let us then fix an ***auxiliary space*** $\boldsymbol{H}$; this is to be a separable metrizable locally compact space[20] equipped with its natural class $\mathcal{E}[\boldsymbol{H}] \stackrel{\text{def}}{=} C_{00}(\boldsymbol{H})$ of elementary integrands and its Borel σ-algebra $\mathcal{B}^\bullet(\boldsymbol{H})$. This and the ambient measured filtration $(\Omega, \mathcal{F}_., \mathfrak{P})$ with accompanying base space $\boldsymbol{B}$ give rise to the ***auxiliary base space***

$$\check{\boldsymbol{B}} \stackrel{\text{def}}{=} \boldsymbol{H} \times \boldsymbol{B} \quad \text{with typical point} \quad \check{\eta} = (\eta, \varpi) = (\eta, s, \omega) \ ,$$

which is naturally equipped with the algebra of elementary functions

$$\check{\mathcal{E}} \stackrel{\text{def}}{=} \mathcal{E}[\boldsymbol{H}] \otimes \mathcal{E}[\mathcal{F}_.] = \Big\{ (\eta, \varpi) \mapsto \sum_i h^i(\eta) X^i(\varpi) : h^i \in \mathcal{E}[\boldsymbol{H}] \ , \ X^i \in \mathcal{E} \Big\} \ .$$

$\check{\mathcal{E}}$ is evidently self-confined, and its natural topology is the topology of confined uniform convergence (see item A.2.5 on page 370). Its confined uniform closure is a self-confined algebra and vector lattice closed under chopping (see exercise A.2.6). The sequential closure of $\check{\mathcal{E}}$ is denoted by $\check{\mathcal{P}}$ and consists of the ***predictable random functions***. One further bit of notation: for any stopping time T ,

$$[\![0, T]\!] \stackrel{\text{def}}{=} \boldsymbol{H} \times [\![0, T]\!] = \{(\eta, s, \omega) : 0 \le s \le T(\omega)\} \ .$$

The essence of a function F of finite variation is the measure dF that goes with it. The essence of an integrator $\boldsymbol{Z} = (Z^1, Z^2, \ldots, Z^d)$ is the vector measure $d\boldsymbol{Z}$, which maps an elementary integrand $\boldsymbol{X} = \check{X} = (X^\eta)^{\eta = 1 \ldots d}$ to the random variable $\int \boldsymbol{X}\, d\boldsymbol{Z}$; the vector $\boldsymbol{Z}$ of cumulative distribution functions is really but a tool in the investigation of $d\boldsymbol{Z}$. Viewing matters this way leads to a straightforward generalization of the notion of an integrator: a random measure with auxiliary space $\boldsymbol{H}$ should be a non-anticipating continuous linear and σ-additive map ζ from $\check{\mathcal{E}}$ to L^p . Such a map will have an ***elementary indefinite integral***[6]

$$(\check{X} * \zeta)_t \stackrel{\text{def}}{=} \zeta\big([\![0, t]\!] \cdot \check{X}\big) \ , \qquad \check{X} \in \check{\mathcal{E}} \ , \ t \ge 0 \ .$$

It is convenient to replace the requirement of σ-additivity by a weaker condition, which is easier to check yet implies it, just as was done for integrators in definition 2.1.7, (RC-0), and proposition 3.3.2, and to employ this definition:

[20] For most of the analysis below it would suffice to have $\boldsymbol{H}$ Suslin, and to take for $\mathcal{E}[\boldsymbol{H}]$ a self-confined algebra or vector lattice closed under chopping of bounded Borel functions that generates the Borels – see [94].

Definition 3.10.1 *Let $0 \le p < \infty$. An L^p-random measure with auxiliary space $\boldsymbol{H}$ is a linear map $\zeta : \check{\mathcal{E}} \to L^p$ having the following properties:*

(i) ζ maps order-bounded sets of $\check{\mathcal{E}}$ to topologically bounded sets[21] *in L^p.*

(ii) The indefinite integral $\check{X}{*}\zeta$ of any $\check{X} \in \check{\mathcal{E}}$ is right-continuous in probability and adapted and satisfies

$$\big([\![0,T]\!] \cdot \check{X}\big){*}\zeta = \big(\check{X}{*}\zeta\big)^T , \qquad T \in \mathfrak{T} . \tag{3.10.1}$$

A few comments and amplifications are in order. If the probability $\mathbb{P}$ must be specified, we talk about an $L^p(\mathbb{P})$-random measure. If $p = 0$, we also speak simply of a ***random measure*** instead of an L^0-random measure.

The continuity condition (i) means that

for every order interval[6] $\quad [-\check{Y},\check{Y}] \overset{\text{def}}{=} \big\{\check{X} \in \check{\mathcal{E}} : \big|\check{X}\big| \le \check{Y}\big\}$

its image $\quad \zeta\big([-\check{Y},\check{Y}]\big)$ is bounded[21] in L^p .

The integrator size of ζ is then naturally measured by the quantities[6]

$$\lceil\!\lceil \zeta^{h,t} \rceil\!\rceil_{\mathcal{I}^p} \overset{\text{def}}{=} \sup\left\{ \left\lceil\!\left\lceil \int \check{X}\, d\zeta \right\rceil\!\right\rceil_p : \check{X} \in \check{\mathcal{E}},\ |\check{X}(\eta,\varpi)| \le h(\eta) \cdot 1_{[\![0,t]\!]}(\varpi) \right\} ,$$

where $h \in \mathcal{E}_+[\boldsymbol{H}]$ and $t \ge 0$.

If $$\lceil\!\lceil \lambda\zeta^{h,\infty} \rceil\!\rceil_{\mathcal{I}^p} \xrightarrow[\lambda\to 0]{} 0 \qquad \text{for all } h \in \mathcal{E}_+[\boldsymbol{H}] ,$$

then ζ is reasonably called a ***global L^p-random measure***.

If $$\lceil\!\lceil \lambda\zeta^{1,t} \rceil\!\rceil_{\mathcal{I}^p} \xrightarrow[\lambda\to 0]{} 0 \qquad \text{at all } t ,$$

then ζ is ***spatially bounded***.

Equation (3.10.1) means that ζ is non-anticipating and generalizes with a little algebra to $(X{\cdot}\check{X}){*}\zeta = X{*}(\check{X}{*}\zeta)$ for $X \in \mathcal{E}$, $\check{X} \in \check{\mathcal{E}}$. This in conjunction with the continuity (i) shows that $\check{X}{*}\zeta$ is an L^p-integrator for every $\check{X} \in \check{\mathcal{E}}$ and is σ-additive in L^p-mean (proposition 3.3.2).

An L^0-random measure ζ is ***locally an L^p-random measure*** or is a ***local L^p-random measure*** if there are arbitrarily large stopping times T so that the ***stopped random measure***

$$\zeta^T \overset{\text{def}}{=} [\![0,T]\!]{\cdot}\zeta : \check{X} \mapsto \zeta\big([\![0,T]\!]\check{X}\big)$$

has $$\lceil\!\lceil ([\![0,T]\!]\lambda\zeta)^{h,\infty} \rceil\!\rceil_{\mathcal{I}^p} \xrightarrow[\lambda\to 0]{} 0$$

for all $h \in \mathcal{E}_+[\boldsymbol{H}]$. (This extends the notion for integrators – see exercise 3.3.4.) A random measure ζ ***vanishes at zero*** if $(\check{X}{*}\zeta)_0 = 0$ for every elementary integrand $\check{X} \in \check{\mathcal{E}}$.

[21] This amounts to saying that ζ is continuous from $\check{\mathcal{E}}$ to the target space, $\check{\mathcal{E}}$ being given the topology of confined uniform convergence (see item A.2.5 on page 370).

σ-Additivity

For the integration theory of a random measure its σ-additivity is indispensable, of course. It comes from the following result, which is the analog of proposition 3.3.2 and has a somewhat technical proof.

Lemma 3.10.2 *An L^p-random measure is σ-additive in p-mean, $0 \le p < \infty$.*

Proof. We shall use on two occasions the Gelfand transform $\widehat{\zeta}$ (see corollary A.2.7 on page 370). It is a linear map from a space $C_{00}(\widehat{\boldsymbol{B}})$, $\widehat{\boldsymbol{B}}$ locally compact, to L^p that maps order-bounded sets to bounded sets and therefore has the usual Daniell extension featuring the Dominated Convergence Theorem (pages 88–105).[22]

First the case $1 \le p < \infty$. Let g be any element of the dual $L^{p'}$ of L^p. Then $\theta(\check{X}) \stackrel{\text{def}}{=} \langle g|\zeta(\check{X})\rangle$ defines a scalar measure θ of finite variation on $\check{\mathcal{E}}$ that is marginally σ-additive on $\mathcal{E}$. Indeed, for every $H \in \mathcal{E}[\boldsymbol{H}]$ the functional $\mathcal{E} \ni X \mapsto \theta(H \otimes X) = \langle g| \int X\, d(H{*}\zeta)\rangle$ is σ-additive on the grounds that $H{*}\zeta$ is an L^p-integrator. By corollary A.2.8, $\theta = \langle g|\zeta\rangle$ is σ-additive on $\check{\mathcal{E}}$. As this holds for any g in the dual of L^p, ζ is σ-additive in the weak topology $\sigma(L^p, L^{p'})$. Due to corollary A.2.7, ζ is σ-additive in p-mean.

Now to the case $0 \le p < 1$. It is to be shown that $\zeta(\check{X}_n) \to 0$ in $L^p(\mathbb{P})$ whenever $\check{\mathcal{E}} \ni \check{X}_n \downarrow 0$ (exercise 3.1.5). There is a function $\check{H} \in \check{\mathcal{E}}$ that equals 1 on $[\check{X}_1 > 0]$. The random measure $\zeta' : \check{X} \to \zeta(\check{H} \cdot \check{X})$ has domain $\check{\mathcal{E}}' \stackrel{\text{def}}{=} \check{\mathcal{E}} + \mathbb{R}$, algebra of bounded functions containing the constants. On the $\check{X}_n$ both ζ and ζ' agree. According to exercise 4.1.8 on page 195 and proposition 4.1.12 on page 206, there is a probability $\mathbb{P}' \approx \mathbb{P}$ so that $\zeta' : \check{\mathcal{E}}' \to L^2(\mathbb{P}')$ is bounded. From proposition 3.3.2 and the first part of the proof we know now that ζ' and then ζ is σ-additive in the topology of $L^2(\mathbb{P}')$. Therefore $\zeta(\check{X}_n) \to 0$ in $L^0(\mathbb{P}) = L^0(\mathbb{P}')$: ζ is σ-additive in $\mathbb{P}$-probability. We invoke corollary A.2.7 again to produce the σ-additivity of ζ in $L^p(\mathbb{P})$. ∎

The extension theory of a random measure is entirely straightforward. We sketch here an overview, leaving most details to the reader – no novel argument is required, simply apply sections 3.1–3.6 *mutatis perpauculis mutandis*.

Suppose then that ζ is an L^p-random measure for some $p \in [0, \infty)$. In view of definition 3.10.1 and lemma 3.10.2, ζ is an L^p-valued linear map on a self-confined algebra of bounded functions, is continuous in the topology of confined uniform convergence, and is σ-additive. Thus there exists an extension of ζ that satisfies the Dominated Convergence Theorem. It is obtained by the utterly straightforward construction and application of THE Daniell mean

$$\check{F} \mapsto \left\lceil\!\left\lceil \check{F} \right\rceil\!\right\rceil^*_{\zeta-p} \stackrel{\text{def}}{=} \inf_{\substack{\check{H} \in \check{\mathcal{E}}^\uparrow, \\ \check{H} \ge |\check{F}|}} \ \sup_{\substack{\check{X} \in \check{\mathcal{E}}, \\ |\check{X}| \le \check{H}}} \left\lceil\!\left\lceil \int \check{X}\, d\zeta \right\rceil\!\right\rceil_{L^p}$$

[22] To be utterly precise, $\widehat{\zeta}$ is a linear map on $\widehat{\check{\mathcal{E}}} \subset C_{00}(\widehat{\boldsymbol{B}})$ that maps order-bounded sets to bounded sets, but it is easily seen to have an extension to C_{00} with the same property.

on $(\check{\boldsymbol{B}}, \check{\mathcal{E}})$ and is written in integral notation as

$$\check{F} \mapsto \int \check{F}\, d\zeta = \int_{\check{\boldsymbol{B}}} F(\eta, s)\, \zeta(d\eta, ds)\ , \qquad \check{F} \in \mathfrak{L}^1[\zeta\text{–}p]\ .$$

Here $\mathfrak{L}^1[\zeta\text{–}p] \stackrel{\text{def}}{=} \mathfrak{L}^1[\llbracket\ \rrbracket^*_{\zeta-p}]$, the closure of $\check{\mathcal{E}}$ under $\llbracket\ \rrbracket^*_{\zeta-p}$, is the collection of ***ζ–p-integrable random functions***. On it the DCT holds. For every $\check{F} \in \mathfrak{L}^1[\zeta\text{–}p]$ the process $\check{F}*\zeta$, whose value at t is variously written as

$$\check{F}*\zeta(t) = \int [\![0, t]\!] \check{F}\, d\zeta = \int_0^t \check{F}(\eta, s)\, \zeta(d\eta, ds)\ ,$$

is an L^p-integrator (of classes) and thus has a nearly unique càdlàg representative, which is chosen for $\check{F}*\zeta$; and for all bounded predictable X

$$\int X\, d\big(\check{F}*\zeta\big) = \int X \cdot \check{F}\, d\zeta$$

or
$$X*(\check{F}*\zeta) = (X \cdot \check{F})*\zeta\ . \tag{3.10.2}$$

Hence
$$\ulcorner \check{F}*\zeta \urcorner_{\mathcal{I}^p} = \llbracket \check{F} \rrbracket^*_{\zeta-p} = \llbracket \check{F} \rrbracket_{\mathfrak{L}^1[\zeta-p]}$$

and
$$\Big\| |\check{F}*\zeta|^\star_\infty \Big\|_{L^p} \le C_p^{\star(2.3.5)} \cdot \llbracket \check{F} \rrbracket^*_{\zeta-p}\ . \tag{3.10.3}$$

A random function $\check{F}$ is of course defined to be ζ–p-measurable[23] if it is "largely as smooth as an elementary integrand from $\check{\mathcal{E}}$" (see definition 3.4.2). Egoroff's Theorem holds. A function $\check{F}$ in the algebra and vector lattice of measurable functions, which is sequentially closed and generated by its idempotents, is ζ–p-integrable if and only if $\llbracket \lambda\check{F} \rrbracket^*_{\zeta-p} \xrightarrow[\lambda\to 0]{} 0$. The ***predictable random functions*** $\check{\mathcal{P}} \stackrel{\text{def}}{=} \check{\mathcal{E}}^\sigma$ provide upper and largest lower envelopes, and $\llbracket\ \rrbracket^*_{\zeta-p}$ is regular in the sense of corollary 3.6.10 on page 128. And so on.

If ζ is merely a local L^p-random measure, then $\check{X}*\zeta$ is a local L^p-integrator for every bounded predictable $\check{X} \in \check{\mathcal{P}} \stackrel{\text{def}}{=} \check{\mathcal{E}}^\sigma$.

Law and Canonical Representation

For motivation consider first an integrator $\boldsymbol{Z} = (Z^1, \ldots, Z^d)$. Its essence is the random measure $d\boldsymbol{Z}$; yet its law was defined as the image of the pertinent probability $\mathbb{P}$ under the "vector of cumulative distribution functions" considered as a map Φ from Ω to the path space $\mathscr{D}^d$. In the case of a general random measure ζ there is no such thing as the "collection of cumulative distribution functions." But in the former case there is another way of looking at Φ. Let h^η denote the indicator function of the singleton set $\{\eta\} \subset \boldsymbol{H} = \{1, \ldots, d\}$. The collection $\mathcal{H} \stackrel{\text{def}}{=} \{h^\eta : \{\eta\} \subset \boldsymbol{H}\}$ has the property that its linear span is dense in (in fact is all of) $\mathcal{E}[\boldsymbol{H}]$, and we might as well interpret Φ as the map that sends $\omega \in \Omega$ to the vector $\big((h*\boldsymbol{Z})_{\bullet}(\omega) : h \in \mathcal{H}\big)$ of indefinite integrals.

[23] This notion does not depend on p; see corollary 3.6.11 on page 128.

This can be emulated in the case of a random measure. Namely, let $\mathcal{H} \subset \mathcal{E}[\boldsymbol{H}]$ be a collection of functions whose linear span is dense in $\mathcal{E}[\boldsymbol{H}]$, in the topology of confined uniform convergence. Such $\mathcal{H}$ can be chosen countable, due to the σ-compactness of $\boldsymbol{H}$, and most often will be. For every $h \in \mathcal{H}$ pick a càdlàg version of the indefinite integral $h{*}\zeta$ (theorem 2.3.4 on page 62). The map

$$\zeta^{\mathcal{H}} : \omega \mapsto \big((h{*}\zeta)_{\bullet}(\omega) : h \in \mathcal{H}\big)$$

sends Ω into the set $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$ of càdlàg paths having values in $\mathbb{R}^{\mathcal{H}}$. In case $\mathcal{H}$ is countable $\mathbb{R}^{\mathcal{H}}$ equals ℓ^0, the Fréchet space of scalar sequences, and then $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}} = \mathscr{D}_{\ell^0}$ is polish under the Skorohod topology (theorem A.7.1 on page 445); the map $\zeta^{\mathcal{H}}$ is measurable if $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$ is equipped with the Borel σ-algebra for the Skorohod topology; and ***the law*** $\zeta^{\mathcal{H}}[\mathbb{P}]$ ***is a tight probability*** on $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$.

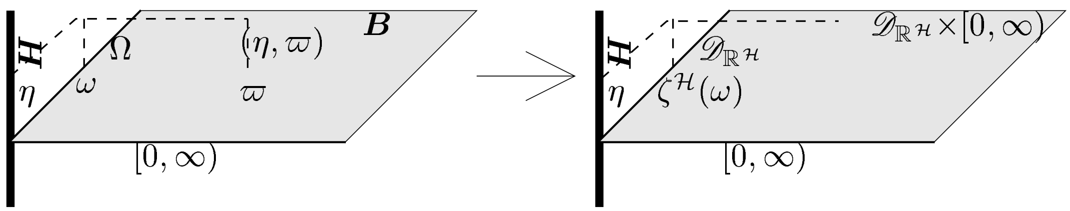

Figure 3.12 The canonical representation

Remark 3.10.3 Two different people will generally pick different càdlàg versions of the integrators $(h{*}\zeta)_{\bullet}$, $h \in \mathcal{H}$. This will affect the map $\zeta^{\mathcal{H}}$ only in a nearly empty set and thus will not change the law. More disturbing is this observation: two different people will generally pick different sets $\mathcal{H} \in \mathcal{E}[\boldsymbol{H}]$ with dense linear span, and that will affect both $\zeta^{\mathcal{H}}$ and the law. If $\boldsymbol{H}$ is discrete, there is a canonical choice of $\mathcal{H}$: take all singletons.[7] One might think of the choice $\mathcal{H} = \mathcal{E}[\boldsymbol{H}]$ in the general case, but that has the disadvantage that now path space $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$ is not polish in general and the law cannot be ascertained to be tight. To see why it is desirable to have the path space polish, consider a Wiener random measure β (see definition 3.10.5). Then if β is realized canonically on a polish path space, whose basic filtration is full (theorem A.7.1), we have a chance at Girsanov's theorem (theorem 3.10.8).[24] We shall therefore do as above and simply state which choice of $\mathcal{H}$ enters the definition of the law.

For fixed $(\Omega, \mathcal{F}_{\bullet}, \mathbb{P})$, $\boldsymbol{H}$, $\mathcal{H}$ countable, and ζ, we have now an ad hoc map $\zeta^{\mathcal{H}}$ from Ω to $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$ and have declared the law of ζ to be the image of $\mathbb{P}$ under this map. This is of course justified only if "the map $\zeta^{\mathcal{H}}$ of ζ is detailed enough" to capture the essence of ζ. Let us see that it does. To this end some notation. For $h \in \mathcal{H}$ let π^h denote the projection of $\ell^0 = \mathbb{R}^{\mathcal{H}}$ onto its h^{th} component and $\pi^h_{\bullet}$ the projection of a path in $\mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$ onto its h^{th} component. Then $\zeta^h_{\bullet}(\omega) \stackrel{\text{def}}{=} \pi^h_{\bullet} \circ \zeta^{\mathcal{H}}(\omega)$ is the h-component of $\zeta^{\mathcal{H}}(\omega)$. This

[24] The measured filtration appearing in the construction of a Wiener random measure on page 177, is it by any chance full? I do not know.

is a scalar càdlàg path. The basic filtration on ${}^0\Omega \overset{\text{def}}{=} \mathscr{D}_{\mathbb{R}^{\mathcal{H}}}$ is denoted by ${}^0\mathcal{F}^0_{\bullet}$, with elementary integrands ${}^0\mathcal{E} \overset{\text{def}}{=} \mathcal{E}[{}^0\mathcal{F}^0_{\bullet}]$. Its counterpart on the given probability space is the ***basic filtration*** $\mathcal{F}^0_{\bullet}[\zeta]$ of ζ, defined as the filtration generated by the càdlàg processes $h{*}\zeta$, $h \in \mathcal{H}$. A simple sequential closure argument shows that a function f on Ω is measurable on $\mathcal{F}^0_t[\zeta]$ if and only if it is of the form $f = \underline{f} \circ \zeta^{\mathcal{H}}$, $\underline{f} \in {}^0\mathcal{F}^0_t$. Indeed, the collection of functions f of this form is closed under pointwise limits of sequences and contains the functions $\omega \mapsto (h{*}\zeta)_s(\omega)$, $h \in \mathcal{H}$, $s \le t$, which generate $\mathcal{F}^0_t[\zeta]$. Also, if $f = \underline{f} \circ \zeta^{\mathcal{H}} = \underline{f}' \circ \zeta^{\mathcal{H}}$, then $[\underline{f} \ne \underline{f}']$ is negligible for the law $\zeta^{\mathcal{H}}[\mathbb{P}]$ of ζ. With $\zeta^{\mathcal{H}} : \Omega \to {}^0\Omega$ there go the maps

$$I{\times}\zeta^{\mathcal{H}} : \boldsymbol{B} \to {}^0\boldsymbol{B} \overset{\text{def}}{=} \mathbb{R}_+ \times {}^0\Omega ,$$

which does $$(s,\omega) \mapsto (s, \zeta^{\mathcal{H}}(\omega)) ,$$

and $$I{\times}I{\times}\zeta^{\mathcal{H}} : \check{\boldsymbol{B}} \to {}^0\check{\boldsymbol{B}} \overset{\text{def}}{=} \boldsymbol{H} \times [0,\infty) \times \Omega = \boldsymbol{H} \times {}^0\boldsymbol{B} ,$$

which does $$(\eta, s, \omega) \mapsto (\eta, s, \zeta^{\mathcal{H}}(\omega)) .$$

It is easily seen that a process X is $\mathcal{F}^0_{\bullet}[\zeta]$-predictable if and only if it is of the form $X = \underline{X} \circ I{\times}\zeta^{\mathcal{H}}$, where $\underline{X}$ is ${}^0\mathcal{F}^0_{\bullet}$-predictable, and that a random function $\check{F}$ is predictable if and only if it is of the form $\check{F} = \underline{\check{F}} \circ I{\times}I{\times}\zeta^{\mathcal{H}}$ with $\underline{\check{F}}$ ${}^0\mathcal{F}^0_{\bullet}$-predictable. With these notations in place consider an elementary function $\underline{\check{X}} = \sum_i h^i \underline{X}^i \in \check{\mathcal{E}}[{}^0\mathcal{F}^0_{\bullet}]$. Then $\check{X} \overset{\text{def}}{=} \underline{\check{X}} \circ I{\times}I{\times}\zeta^{\mathcal{H}} \in \check{\mathcal{E}}\big[\mathcal{F}^0_{\bullet}[\zeta]\big]$ and

$${}^0\zeta\big(\underline{\check{X}}\big)\big(\zeta^{\mathcal{H}}(\omega)\big) \overset{\text{def}}{=} \textstyle\sum_i \big(\underline{X}^i{*}\pi^{h_i}_{\bullet}\big)\big(\zeta^{\mathcal{H}}(\omega)\big)$$

with $X^i \overset{\text{def}}{=} \underline{X}^i \circ I{\times}\zeta^{\mathcal{H}}$: $$= \textstyle\sum_i X^i{*}(h^i{*}\zeta)(\omega) = \check{X}{*}\zeta(\omega)$$

for nearly every $\omega \in \Omega$. From this it is evident that

$${}^0\zeta\big(\underline{\check{X}}\big) \overset{\text{def}}{=} \textstyle\sum_i \underline{X}^i{*}\zeta^{h_i}$$

defines a random measure ${}^0\zeta$ on ${}^0\check{\mathcal{E}}$ that mirrors ζ in this sense:

$${}^0\zeta(\underline{\check{X}}) = \zeta\big(\underline{\check{X}} \circ I{\times}I{\times}\zeta^{\mathcal{H}}\big) , \qquad \underline{\check{X}} \in {}^0\check{\mathcal{E}} .$$

${}^0\zeta$ is defined on a *full filtration.* We call it the ***canonical representation of the random measure*** ζ – despite the arbitrariness in the choice of $\mathcal{H}$.

Exercise 3.10.4 The regularization of the right-continuous version of the basic filtration $\mathcal{F}^0_{\bullet}[\zeta]$ is called the ***natural filtration*** of ζ and is denoted by $\mathcal{F}_{\bullet}[\zeta]$. Every indefinite integral $\check{F}{*}\zeta$, $\check{F} \in \mathfrak{L}^1[\zeta{-}p]$, is adapted to it.

Example: Wiener Random Measure

Compare the following construction with exercise 1.2.16 on page 20. On the auxiliary space $\boldsymbol{H}$ let ν be a positive Radon measure, and on $\check{\boldsymbol{H}} \overset{\text{def}}{=} \boldsymbol{H} \times [0,\infty)$ let μ denote the product of ν and Lebesgue measure λ. Let $\{\phi_n\}$ be an orthonormal basis of $L^2(\mu)$ and ξ_n independent Gaussians distributed $N(0,1)$ and defined on some measured space $(\Omega, \mathcal{F}, \mathbb{P})$. The map $U : \phi_n \mapsto \xi_n$ extends to a linear isometry of $L^2(\mu)$ into $L^2(\mathbb{P})$ that takes orthogonal elements of $L^2(\mu)$ to independent random variables in $L^2(\mathbb{P})$; $U(f)$ has distribution $N(0, \|f\|^2_{L^2(\mu)})$ for $f \in L^2(\mu)$. The restriction of U to relatively compact Borel subsets[7] of $\check{\boldsymbol{H}}$ is an $L^2(\mathbb{P})$-valued set function, and aficionados of the Riesz representation theorem may wish to write the value of U at $f \in L^2(\mu)$ as

$$U(f) = \int f(\eta, s)\, U(d\eta, ds) \; .$$

We shall now equip Ω with a suitable filtration $\mathcal{F}^0_{\bullet}$. To this end fix a countable subset $\mathcal{H} \subset \mathcal{E}[\boldsymbol{H}]$ whose linear span is dense in $\mathcal{E}[\boldsymbol{H}]$ in the topology of confined uniform convergence (item A.2.5). For $h \in \mathcal{H}$ set

$$U^h_t \stackrel{\text{def}}{=} \int h(\eta) 1_{[0,t]}(s)\, U(d\eta, ds) \; , \qquad t \ge 0 \; .$$

This produces a countable number of Wiener processes. By theorem 1.2.2 (ii) we can arrange things so that after removal of a nearly empty set every one of the $U^h_{\bullet}$, $h \in \mathcal{H}$, has continuous paths. (If we want, we can employ Gram–Schmid and have the $U^h_{\bullet}$ standard and independent.) We now let $\mathcal{F}^0_t$ be the σ-algebra generated by the random variables U^h_s, $h \in \mathcal{H}, s \le t$, to obtain the sought-after filtration $\mathcal{F}^0_{\bullet}$. It is left to the reader to check that, for every relatively compact Borel set $B \subset \boldsymbol{H} \times [0,t]$, $U(B)$ differs negligibly from some set in $\mathcal{F}^0_t$ [Hint: use the Hardy mean (3.10.4) below].

To construct the Wiener random measure β we take for the elementary functions $\mathcal{E}[\boldsymbol{H}]$ the step functions over the relatively compact Borels of $\boldsymbol{H}$ instead of $C_{00}[\boldsymbol{H}]$; this eases visualization a little. With that, an elementary integrand $\check{X} \in \mathcal{E}[\boldsymbol{H}] \otimes \mathcal{E}[\mathcal{F}_{\bullet}]$ can be written as a finite sum

$$\check{X}(\eta, s; \omega) = \textstyle\sum_\imath h_\imath(\eta) \cdot X_\imath(s, \omega) \; , \qquad h_\imath \in \mathcal{E}[\boldsymbol{H}] \; , \; X_\imath \in \mathcal{E}[\mathcal{F}_{\bullet}] \; ,$$

or as

$$\check{X}(\eta, s, \omega) = \textstyle\sum_i f_i(\omega) \cdot R_i(\eta, s) \; ,$$

where the R_i are mutually disjoint rectangles of $\check{\boldsymbol{H}} = \boldsymbol{H} \times [0, \infty)$ and f_i is a simple function measurable on the σ-algebra that goes with the left edge[25] of R_i. In fact, things can clearly be arranged so that the R_i all are rectangles of the form $B_i \times (t_{i-1}, t_i]$, where the B_i are from a fixed partition of $\boldsymbol{H}$ into disjoint relatively compact Borels and the t_i are from a fixed partition $0 \le t_1 < \ldots < t_N < +\infty$ of $[0, \infty)$. For such $\check{X}$ define now

$$\beta(\check{X})(\omega) = \int \check{X}(\eta, s, \omega)\, U(d\eta, ds; \omega)$$
$$= \textstyle\sum_i f_i(\omega)\, U(R_i)(\omega) \; .$$

[25] Meaning the left endpoint of the projection of R_i on $[0, \infty)$.

The first line asks us to apply, for every fixed $\omega \in \Omega$, the $L^2(\mathbb{P})$-valued measure U to the integrand $(\eta, s) \mapsto \check{X}(\eta, s; \omega)$, showing that the definition does not depend on the particular representation of $\check{X}$, and implying the linearity[26] of β. The second line allows for an easy check that β is an L^2-random measure. Namely, consider

$$\mathbb{E}\big[\big(\beta(\check{X})\big)^2\big] = \sum_{i,j}\mathbb{E}\big[f_i f_j \cdot U(R_i) \cdot U(R_j)\big] . \tag{$*$}$$

Now if the left edge[25] of R_i is strictly less than the left edge of R_j, then even the right edge of R_i is less than the left edge of R_j, and the random variables $f_i f_j, U(R_i), U(R_j)$ are independent. Since the latter two have mean zero, the expectation $\mathbb{E}[f_i f_j \cdot U(R_i) \cdot U(R_j)]$ vanishes. It does so even if R_i and R_j have the same left edge[25] and $i \neq j$. Indeed, then R_i and R_j are disjoint, again $f_i f_j, U(R_i), U(R_j)$ are independent, and the previous argument applies. The cross-terms in $(*)$ thus vanish so that

$$\mathbb{E}\Big[\big(\beta(\check{X})\big)^2\Big] = \sum_i \mathbb{E}\Big[f_i^2\big(U(R_i)\big)^2\Big] = \sum_i \mathbb{E}\big[f_i^2\big] \cdot \mathbb{E}\Big[\big(U(R_i)\big)^2\Big]$$

$$= \sum_i \mathbb{E}\big[f_i^2\big] \cdot \mu(R_i) = \mathbb{E}\Big[\int \check{X}^2(\eta, s; \cdot)\mu(d\eta, ds)\Big] .$$

Therefore

$$F \mapsto \|F\|_{\beta\text{–}2}^{\mathcal{H}*} \stackrel{\text{def}}{=} \Big(\int^* F_t^2(\eta, s, \omega)\, \mu(d\eta, ds)\mathbb{P}(d\omega)\Big)^{1/2} \tag{3.10.4}$$

is a mean majorizing the linear map $\beta : \check{\mathcal{E}} \to L^2$, the ***Hardy mean***. From this it is obvious that β has an extension to all previsible $\check{X}$ with $\|\check{X}\|_{\beta\text{–}2}^{\mathcal{H}*} < \infty$, an extension reasonably denoted by $\check{X} \mapsto \int \check{X}\, d\beta$. β evidently meets the following description and shows that there are instances of it:

Definition 3.10.5 *A random measure β with auxiliary space $\boldsymbol{H}$ is a **Wiener random measure** if $h{*}\beta, h'{*}\beta$ are independent Wiener processes whenever the functions $h, h' \in \mathcal{E}[\boldsymbol{H}] \stackrel{\text{def}}{=} C_{00}[\boldsymbol{H}]$ have disjoint support.*

Here are a few features of Wiener random measure. Their proofs are left as exercises.

Theorem 3.10.6 (The Structure of Wiener Random Measures) *(i) The integral extension of β has the property that $h{*}\beta$, $h'{*}\beta$ are independent Wiener processes whenever h, h' are disjoint relatively compact Borel sets. Thus the set function $\nu : B \mapsto [B{*}\beta, B{*}\beta]_1$ is a positive σ-additive measure on $\mathcal{B}^\bullet[\boldsymbol{H}]$, called the **intensity rate of** β. For $h, h' \in L^2(\nu)$, $h{*}\beta$ and $h'{*}\beta$ are Wiener processes with $[h{*}\beta, h'{*}\beta]_t = t \cdot \int h(\eta)h'(\eta)\nu(d\eta)$; if h and h' are orthogonal in the Hilbert space $L^2(\nu)$, then $h{*}\beta, h'{*}\beta$ are independent.*

(ii) The Daniell mean $\|\ \|_{\beta\text{–}2}^*$ and the Hardy mean $\|\ \|_{\beta\text{–}2}^{\mathcal{H}*}$ agree on elementary integrands and therefore on $\check{\mathcal{P}}$. Consequently, $\mathfrak{L}^1[\beta\text{–}2]$ is the Hilbert space $L^2(\nu{\times}\lambda{\times}\mathbb{P})$, and the map $\check{X} \mapsto \int \check{X}\ d\beta$ is an isometry of $\mathfrak{L}^1[\beta\text{–}2]$ onto a

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[26] As a map into *classes* modulo negligible functions.

closed subspace of $L^2[\mathbb{P}]$ *(its range is the subspace of all functions with expectation zero; see theorem 3.10.9). For* $\check{X} \in \mathfrak{L}^1[\beta\text{–}2]$*,* $\int \check{X}\, d\beta$ *is normal with mean zero and standard deviation* $\|\check{X}\|^{\mathcal{H}*}_{\beta\text{–}2}$*. (That* β *is an* L^1*-random measure with* $\mathbb{E}[\int \check{X}\, d\beta] = 0 \quad \forall\, \check{X} \in \check{\mathcal{E}}$ *makes it a* ***martingale random measure****.)*

(iii) β *is an* L^p*-random measure for all* $p < \infty$*. It is continuous in the sense that* $\check{X}*\beta$ *has a nearly continuous version, for all* $\check{X} \in \check{\mathcal{P}}_b$*. It is spatially bounded if and only if* $\nu(\boldsymbol{H})$ *is finite. For* $\boldsymbol{H} = \{1, \ldots, d\}$ *and* ν *counting measure,* β *is a standard d-dimensional Wiener process.*

Theorem 3.10.7 (Lévy Characterization of a Wiener Random Measure) *Suppose* β *is a local martingale random measure such that, for every* $h \in \mathcal{E}[\boldsymbol{H}]$*,* $[h*\beta, h*\beta]_t/t$ *is a constant, and* $[h*\beta, h'*\beta]_t = 0$ *if* $h, h' \in \mathcal{E}[\boldsymbol{H}]$ *have disjoint support. Then* β *is a Wiener random measure whose intensity rate is given by* $\nu(h) = \mathbb{E}[(h*\beta)_1^2] = [h*\beta, h*\beta]_1$*.*

Theorem 3.10.8 (The Girsanov Theorem for Wiener Random Measure) *Assume the measured filtration* $(\Omega, \mathcal{F}_{\bullet}, \mathbb{P})$ *is full, and* β *is a Wiener random measure on* $(\Omega, \mathcal{F}_{\bullet}, \mathbb{P})$ *with intensity rate* ν*. Suppose* $\check{H}$ *is a predictable random function such that the Doléans–Dade exponential* G' *of* $\check{H}*\beta$ *is a martingale. Then there is a unique* σ*-additive probability* $\mathbb{P}'$ *on* $\mathcal{F}_\infty$ *so that* $\mathbb{P}' = G'_t\mathbb{P}$ *on* $\mathcal{F}_t$ *at all finite instants* t*, and*

$$\beta'(d\eta, ds) \stackrel{\text{def}}{=} \beta(d\eta, ds) + \check{H}_s(\eta)\nu(d\eta)ds$$

is under $\mathbb{P}'$ *a Wiener random measure with intensity rate* ν*.*

Theorem 3.10.9 (Martingale Representation for Wiener Random Measure) *Every function* $f \in L^2(\mathcal{F}_\infty[\beta], \mathbb{P})$ *is the sum of the constant* $\mathbb{E}f$ *and a random variable of the form* $\int \check{X}\, d\beta$*,* $\check{X} \in \mathfrak{L}^1[\beta\text{–}2]$*. Thus every square integrable* $(\mathcal{F}_{\bullet}[\beta], \mathbb{P})$*-martingale* $M_{\bullet}$ *has the form*

$$M_t = M_0 + \int_0^t \check{X}(\eta, s)\; \beta(d\eta, ds)\;, \qquad\qquad \check{X}\cdot[\![0, t]\!] \in \mathfrak{L}^1[\beta\text{–}2]\;,\; t \geq 0\;.$$

For more on this interesting random measure see corollary 4.2.16 on page 219.

Example: The Jump Measure of an Integrator

Given a vector $\boldsymbol{Z} = (Z^1, Z^2, \ldots, Z^d)$ of L^0-integrators let us define, for any function H that is measurable on $\mathcal{B}^\bullet(\mathbb{R}^d_*) \times \mathcal{P}$ [27] – a predictable random function – and any stopping time T, the number

$$\int_0^T H_s(\boldsymbol{y}; \omega)\; \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds; \omega) \stackrel{\text{def}}{=} \sum_{0 \leq s \leq T(\omega)} H_s\big(\Delta \boldsymbol{Z}_s(\omega); \omega\big)\;,$$

or, suppressing mention of ω as usual,

$$\int_0^T H_s(\boldsymbol{y})\; \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds) = \sum_{0 \leq s \leq T} H_s(\Delta \boldsymbol{Z}_s)\;. \tag{3.10.5}$$

The sum will in general diverge. However, if H is the sure function

$$h_0(\boldsymbol{y}) \stackrel{\text{def}}{=} |\boldsymbol{y}|^2 \wedge 1\;,$$

[27] It is customary to denote by $\mathbb{R}^d_*$ the ***punctured*** d***-space***: $\mathbb{R}^d_* \stackrel{\text{def}}{=} \mathbb{R}^d \setminus \{0\}$. We identify functions on $\mathbb{R}^d_*$ with functions on $\mathbb{R}^d$ that vanish at the origin. The generic point of the auxiliary space $\boldsymbol{H} = \mathbb{R}^d_*$ is denoted by $\boldsymbol{y} = (y^\eta)$ in this subsection.

then the sum will converge absolutely, since

$$|h_0(\Delta \boldsymbol{Z}_s)| \le \sum_{1\le\eta\le d} |\Delta Z_s^\eta|^2 = \sum_\eta \Delta[Z^\eta, Z^\eta]_s \ .$$

If H is a bounded predictable random function that is majorized in absolute value by a multiple of h_0, then the sum (3.10.5) will exist wherever T is finite. Let us call such a function a ***Hunt function***. Their collection clearly forms a vector lattice and algebra of predictable random functions. The integral notation in equation (3.10.5) is justified by the observation that the map

$$H \mapsto \sum_{0\le s\le t} H_s(\Delta \boldsymbol{Z}_s)$$

is, ω for ω, a positive σ-additive linear functional on the Hunt functions; in fact, it is a sum of point masses supported by the points $(\Delta \boldsymbol{Z}_s, s) \in \mathbb{R}_*^d \times [0,\infty)$ at the countable number of instants s at which the path $\boldsymbol{Z}_{\boldsymbol{.}}(\omega)$ jumps:

$$\jmath_{\boldsymbol{Z}} = \sum_{\substack{0\le s\le\infty \\ \Delta \boldsymbol{Z}_s \ne 0}} \delta_{(\Delta \boldsymbol{Z}_s, s)} \ .$$

This ω-dependent measure $\jmath_{\boldsymbol{Z}}$ is called the ***jump measure*** of $\boldsymbol{Z}$. With $\boldsymbol{H} \overset{\text{def}}{=} \mathbb{R}_*^d$, the map $\check{X} \mapsto \int \check{X}(\boldsymbol{y}, s)\ \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$ is clearly a random measure. We identify it with $\jmath_{\boldsymbol{Z}}$. It integrates a priori more than the elementary integrands $\check{\mathcal{E}} \overset{\text{def}}{=} C_{00}[\mathbb{R}_*^d] \otimes \mathcal{E}$, to wit, any Hunt function whose carrier is bounded in time. For a Hunt function H the ***indefinite integral*** $H{*}\jmath_{\boldsymbol{Z}}$ is defined by

$$\big(H{*}\jmath_{\boldsymbol{Z}}\big)_t = \int_{[\![0,t]\!]} H_s(\boldsymbol{y})\ \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\ , \quad \text{where } [\![0,t]\!] \overset{\text{def}}{=} \mathbb{R}_*^d \times [\![0,t]\!]$$

is the cartesian product of punctured d-space with the stochastic interval $[\![0,t]\!]$. Clearly $H{*}\jmath_{\boldsymbol{Z}}$ is an adapted[28] process of finite variation $|H|{*}\jmath_{\boldsymbol{Z}}$ and has bounded jumps. It is therefore a local L^p-integrator for all $p > 0$ (exercise 4.3.8). Indeed, let $S_t = \sum_\eta [Z^\eta, Z^\eta]_t$; at the stopping times $T^n = \inf\{t : S_t \ge n\}$, which tend to infinity, $\lceil H{*}\jmath_{\boldsymbol{Z}} \rceil_{T^n} = \big(|H|{*}\jmath_{\boldsymbol{Z}}\big)_{T^n}$ is bounded by $(n{+}1)\|H/h_0\|_\infty$. This fact explains the prominence of the Hunt functions. For any $q \ge 2$ and all $t < \infty$,

$$\int_{[\![0}^{t]\!]} |\boldsymbol{y}|^q\ \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds) \le \Big(\sum_{1\le\eta\le d} {}^j[Z^\eta, Z^\eta]_t\Big)^{q/2} \le \Big(\sum_{1\le\eta\le d} S_t[Z^\eta]\Big)^q$$

is nearly finite; if the components Z^η of $\boldsymbol{Z}$ are L^q-integrators, then this random variable is evidently integrable. The next result is left as an exercise.

[28] Extend exercise 1.3.21 (iii) on page 31 slightly so as to cover *random* Hunt functions H.

Proposition 3.10.10 *Formula (3.9.2) can be rewritten in terms of* $\jmath_{\boldsymbol{Z}}$ *as*[15]

$$\Phi(\boldsymbol{Z}_T) = \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z}_{.-})\, dZ^\eta + \frac{1}{2}\int_{0+}^{T} \Phi_{;\eta\theta}(\boldsymbol{Z}_{.-})\, d{}^c[Z^\eta, Z^\theta] \tag{3.10.6}$$

$$+ \int_{0}^{T} \Big(\Phi(\boldsymbol{Z}_{s-} + \boldsymbol{y}) - \Phi(\boldsymbol{Z}_{s-}) - \Phi_{;\eta}(\boldsymbol{Z}_{s-}) \cdot y^\eta\Big)\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$$

$$= \Phi(\boldsymbol{Z}_0) + \int_{0+}^{T} \Phi_{;\eta}(\boldsymbol{Z}_{.-})\, dZ^\eta + \frac{1}{2}\int_{0+}^{T} \Phi_{;\eta\theta}(\boldsymbol{Z}_{.-})\, d[Z^\eta, Z^\theta]$$

$$+ \int_{0+}^{T} R^3_\Phi(\boldsymbol{Z}_{s-}, \boldsymbol{y})\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\ , \tag{3.10.7}$$

if Φ *is thrice continuously differentiable, where*

$$R^3_\Phi(\boldsymbol{z}, \boldsymbol{y}) = \Phi(\boldsymbol{z} + \boldsymbol{y}) - \Phi(\boldsymbol{z}) - \Phi_{;\eta}(\boldsymbol{z})y^\eta - \frac{1}{2}\Phi_{;\eta\theta}(\boldsymbol{z})y^\eta y^\theta$$

$$= \int_0^1 \frac{(1-\lambda)^2}{2} \Phi'_{;\eta\theta\iota}(\boldsymbol{z} + \lambda\boldsymbol{y})y^\eta y^\theta y^\iota\, d\lambda\ ,$$

or, when Φ *is* n*-times continuously differentiable:*

$$R^3_\Phi(\boldsymbol{z}, \boldsymbol{y}) = \sum_{\nu=3}^{n-1} \frac{1}{\nu!} \Phi_{;\eta_1\ldots\eta_\nu}(\boldsymbol{z}) y^{\eta_1} \cdots y^{\eta_\nu}$$

$$+ \int_0^1 \frac{(1-\lambda)^{n-1}}{(n-1)!} \Phi_{;\eta_1\ldots\eta_n}(\boldsymbol{z}_1 + \lambda\boldsymbol{y}) y^{\eta_1} \cdots y^{\eta_n}\, d\lambda\ .$$

Exercise 3.10.11 $h'_0 : \boldsymbol{y} \mapsto \int_{[|\boldsymbol{\zeta}|\le 1]} |e^{i\langle\boldsymbol{\zeta}|\boldsymbol{y}\rangle} - 1|^2\, d\boldsymbol{\zeta}$ defines another prototypical sure Hunt function in the sense that h'_0/h_0 is both bounded and bounded away from zero.

Exercise 3.10.12 Let H, H' be previsible Hunt functions and T a stopping time. Then (i)

$$[H{*}\jmath_{\boldsymbol{Z}}, H'{*}\jmath_{\boldsymbol{Z}}] = HH'{*}\jmath_{\boldsymbol{Z}}\ .$$

(ii) For any bounded predictable process X the product XH is a Hunt function and

$$\int_{[\![0,T]\!]} X_s H_s(\boldsymbol{y})\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds) = \int_{[\![0,T]\!]} X_s\, d(H{*}\jmath_{\boldsymbol{Z}})_s\ .$$

In fact, this equality holds whenever either side exists.

(iii)
$$\Delta(H{*}\jmath_{\boldsymbol{Z}})_t = H_t(\Delta\boldsymbol{Z}_t)\ , \qquad t \ge 0\ ,$$

and
$$(H' * \jmath_{H*\jmath_{\boldsymbol{Z}}})_T = \int_{[\![0,T]\!]} H'_s(H_s(\boldsymbol{y}))\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)$$

as long as merely $|H'_s(\boldsymbol{y})| \le const \cdot |\boldsymbol{y}|$. For any bounded predictable process X

(iv)
$$\int_{[\![0,T]\!]} H_s(\boldsymbol{y})\, \jmath_{X*\boldsymbol{Z}}(d\boldsymbol{y}, ds) = \int_{[\![0,T]\!]} H_s(X_s{\cdot}\boldsymbol{y})\, \jmath_{\boldsymbol{Z}}(d\boldsymbol{y}, ds)\ ,$$

and if $X = X^2$ is a set, then[27] $\jmath_{X*\boldsymbol{Z}} = X \cdot \jmath_{\boldsymbol{Z}}$.

Strict Random Measures and Point Processes

The jump measure $\jmath_{\boldsymbol{Z}}$ of an integrator $\boldsymbol{Z}$ actually is a strict random measure in this sense:

Definition 3.10.13 *Let $\zeta : \Omega \to \mathfrak{M}^*[\check{\boldsymbol{H}}]$ be a family of σ-additive measures on $\check{\boldsymbol{H}} \stackrel{\text{def}}{=} \boldsymbol{H} \times [0, \infty)$, one for every $\omega \in \Omega$. If the ordinary integral*

$$\check{X} \mapsto \int_{\check{\boldsymbol{H}}} \check{X}(\eta, s; \omega)\, \zeta(d\eta, ds; \omega)\,, \qquad \check{X} \in \check{\mathcal{E}}\,,$$

*computed ω–by–ω, is a random measure, then the linear map of the previous line is identified with ζ and is called a **strict random measure**.*

These are the random measures treated in [50] and [53]. The Wiener random measure of page 179 is in some sense as far from being strict as one can get. The definitions presented here follow [8]. Kurtz and Protter [61] call our random measures "standard semimartingale random measures" and investigate even more general objects.

Exercise 3.10.14 If ζ is a strict random measure, then $\check{F}{*}\zeta$ can be computed ω–by–ω when the random function $\check{F} \in \mathfrak{L}^1[\zeta\text{–}p]$ is predictable (meaning that $\check{F}$ belongs to the sequential closure $\check{\mathcal{P}} \stackrel{\text{def}}{=} \check{\mathcal{E}}^\sigma$ of $\check{\mathcal{E}}$, the collection of functions measurable on $\mathcal{B}^*(\boldsymbol{H}) \otimes \mathcal{P}$). There is a nearly empty set outside which all the indefinite integrals (integrators) $\check{F}{*}\zeta$ can be chosen to be simultaneously càdlàg. Also the maps $\check{\mathcal{P}} \ni \check{X} \mapsto \check{X}{*}\zeta(\omega)$ are linear at every $\omega \in \Omega$ – not merely as maps from $\check{\mathcal{P}}$ to *classes* of measurable functions.

Exercise 3.10.15 An integrator is a random measure whose auxiliary space is a singleton, but it is a strict random measure only if it has finite variation.

Example 3.10.16 (Sure Random Measures) Let μ be a positive Radon measure on $\check{\boldsymbol{H}} \stackrel{\text{def}}{=} \boldsymbol{H} \times [0, \infty)$. The formula $\zeta(\check{X})(\omega) \stackrel{\text{def}}{=} \int_{\check{\boldsymbol{H}}} \check{X}(\eta, s, \omega)\mu(d\eta, ds)$ defines a simple strict random measure ζ. In particular, when μ is the product of a Radon measure ν on $\boldsymbol{H}$ with Lebesgue measure ds then this reads

$$\zeta(\check{X})(\omega) = \int_0^\infty \int_{\boldsymbol{H}} \check{X}(\eta, s, \omega)\, \nu(d\eta) ds\,. \tag{3.10.8}$$

Actually, the jump measure $\jmath_{\boldsymbol{Z}}$ of an integrator $\boldsymbol{Z}$ is even more special. Namely, its value on a set $\check{A} \subset \check{\boldsymbol{B}}$ is an integer, the number of jumps whose size lies in $\check{A}$. More specifically, $\jmath_{\boldsymbol{Z}}(\,\cdot\,;\omega)$ is the sum of point masses on $\check{\boldsymbol{H}}$:

Definition 3.10.17 *A positive strict random measure ζ is called a **point process** if $\zeta(d\check{\eta}\,;\omega)$ is, for every $\omega \in \Omega$, the sum of point masses $\delta_{\check{\eta}}$. We call the point process ζ **simple** if almost surely $\zeta\big(\boldsymbol{H} \times \{t\}\big) \le 1$ at all instants t – this means that* $\operatorname{supp}\zeta \cap \big(\boldsymbol{H} \times \{t\}\big)$ *contains at most one point.* A simple point process clearly is described entirely by the random point set $\operatorname{supp}\zeta$, whence the name.

Exercise 3.10.18 For a simple point process ζ and $\check{F} \in \check{\mathcal{P}} \cap \mathfrak{L}^1[\zeta\text{–}p]$

$$\Delta(\check{F}{*}\zeta)_t = \int_{\boldsymbol{H} \times \{t\}} \check{F}(\eta, s)\, \zeta(d\eta, ds) \quad \text{and} \quad [\check{F}{*}\zeta, \check{F}{*}\zeta] = \check{F}^2{*}\zeta\,.$$

Example: Poisson Point Processes

Suppose again that we are given on our separable metrizable locally compact auxiliary space $\boldsymbol{H}$ a positive Radon measure ν. Let $\check{B} \in \mathcal{B}^\bullet(\check{\boldsymbol{H}})$ with $\nu\times\lambda(\check{B}) < \infty$ and set $\mu \stackrel{\text{def}}{=} \check{B}\cdot(\nu\times\lambda)$. Next let N be a random variable distributed Poisson with mean $|\mu| \stackrel{\text{def}}{=} \mu(1) = \nu\times\lambda(\check{B})$ and let Y_i, $i = 0, 1, 2, \ldots$, be random variables with values in $\check{\boldsymbol{H}}$ that have distribution $\mu/|\mu|$. They and N are chosen to form an independent family and live on some probability space $(\Omega^\mu, \mathcal{F}^\mu, \mathbb{P}^\mu)$. We use these data to define a point process π^μ as follows: for $\check{F} : \check{\boldsymbol{H}} \to \mathbb{R}$ set

$$\pi^\mu(\check{F}) \stackrel{\text{def}}{=} \sum_{\nu=0}^{N} \check{F}(Y_\nu) = \sum_{\nu=0}^{N} \delta_{Y_\nu}(\check{F}) . \tag{3.10.9}$$

In other words, pick independently N points from $\check{\boldsymbol{H}}$ according to the distribution $\mu/|\mu|$, and let π^μ be the sum of the δ-masses at these points. To check the distribution of π^μ, let $\check{A}_k \subset \check{\boldsymbol{H}}$ be mutually disjoint and let us show that $\pi^\mu(\check{A}_1), \ldots, \pi^\mu(\check{A}_K)$ are independent and Poisson with means $\mu(\check{A}_1), \ldots, \mu(\check{A}_K)$, respectively. It is convenient to set $\check{A}_0 \stackrel{\text{def}}{=} \left(\bigcup_{k=1}^K \check{A}_k\right)^c$ and $p_k \stackrel{\text{def}}{=} \mathbb{P}^\mu[Y_0 \in \check{A}_k] = \mu(\check{A}_k)/|\mu|$. Fix natural numbers $n_0, \ldots, n_K$ and set $n = n_0 + \cdots + n_K$. The event

$$\left[\pi^\mu(\check{A}_0) = n_0, \ldots, \pi^\mu(\check{A}_K) = n_K\right]$$

occurs precisely when of the first n points Y_ν n_0 fall into $\check{A}_0$, n_1 fall into $\check{A}_1$, ..., and n_K fall into $\check{A}_K$, and has (multinomial) probability

$$\binom{n}{n_0 \cdots n_K} p_0^{n_0} \cdots p_K^{n_K}$$

of occurring given that $N = n$. Therefore

$$\begin{aligned}
&\mathbb{P}^\mu\left[\pi^\mu(\check{A}_0) = n_0, \ldots, \pi^\mu(\check{A}_K) = n_K\right] \\
&= \mathbb{P}^\mu\left[N = n, \pi^\mu(\check{A}_0) = n_0, \ldots, \pi^\mu(\check{A}_K) = n_K\right] \\
\text{by independence:}\quad &= e^{-|\mu|} \frac{|\mu|^n}{n!} \cdot \binom{n}{n_0 \cdots n_K} p_0^{n_0} \cdots p_K^{n_K} \\
&= e^{-\mu(\check{A}_0)} \frac{|\mu(\check{A}_0)|^{n_0}}{n_0!} \cdots e^{-\mu(\check{A}_K)} \frac{|\mu(\check{A}_K)|^{n_K}}{n_K!} \\
&= \prod_{k=0}^{K} e^{-\mu(\check{A}_k)} \frac{|\mu(\check{A}_k)|^{n_k}}{n_k!} .
\end{aligned}$$

Summing over n_0 produces

$$\mathbb{P}^\mu\left[\pi^\mu(\check{A}_1) = n_1, \ldots, \pi^\mu(\check{A}_K) = n_K\right] = \prod_{k=1}^{K} e^{-\mu(\check{A}_k)} \frac{|\mu(\check{A}_k)|^{n_k}}{n_k!} ,$$

showing that the random variables $\pi^\mu(\check{A}_1), \ldots, \pi^\mu(\check{A}_K)$ are independent Poisson random variables with means $\mu(\check{A}_1), \ldots, \mu(\check{A}_K)$, respectively.

To finish the construction we cover $\check{\boldsymbol{H}}$ by countably many mutually disjoint relatively compact Borel sets B^k, set $\mu^k \stackrel{\text{def}}{=} B^k(\nu \times \lambda)$, and denote by π^k the corresponding Poisson random measures just constructed, which live on probability spaces $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$. Then we equip the cartesian product $\Omega \stackrel{\text{def}}{=} \prod_k \Omega^k$ with the product σ-algebra $\mathcal{F} \stackrel{\text{def}}{=} \bigotimes_k \mathcal{F}^k$, on which the natural probability is of course the product $\mathbb{P} \stackrel{\text{def}}{=} \prod_k \mathbb{P}^k$. It is left as an exercise in bookkeeping to show that $\pi \stackrel{\text{def}}{=} \sum_k \pi^k$ meets the following description:

Definition 3.10.19 *A point process π with auxiliary space $\boldsymbol{H}$ is called a **Poisson point process** if, for any two disjoint relatively compact Borel sets $B, B' \subset \boldsymbol{H}$, the processes $B{*}\pi, B'{*}\pi$ are independent and Poisson.*

Theorem 3.10.20 (Structure of Poisson Point Processes) $\nu : B \mapsto \mathbb{E}[(B{*}\pi)_1]$ *is a positive σ-additive measure on $\mathcal{B}^\bullet[\boldsymbol{H}]$, called the **intensity rate**. Whenever $h \in L^1(\nu)$, the indefinite integral $h{*}\pi$ is a process with independent stationary increments, is an L^p-integrator for all $p > 0$, and has square bracket $[h{*}\pi, h{*}\pi] = h^2{*}\pi$. If $h, h' \in L^1(\nu)$ have disjoint carriers, then $h{*}\pi, h'{*}\pi$ are independent. Furthermore,*

$$\widehat{\pi}(\check{X}) \stackrel{\text{def}}{=} \int \check{X}_s(\eta)\, \nu(d\eta) ds$$

*defines a strict random measure $\widehat{\pi}$, called the **compensator** of π. Also, $\widetilde{\pi} \stackrel{\text{def}}{=} \pi - \widehat{\pi}$ is a strict martingale random measure, called **compensated Poisson point process**. The $\pi, \widehat{\pi}, \widetilde{\pi}$ are L^p-random measures for all $p > 0$.*

The Girsanov Theorem for Poisson Point Processes

Let π be a Poisson point process with intensity rate ν on $\boldsymbol{H}$ and intensity $\widehat{\pi} = \nu \times \lambda$ on $\check{\boldsymbol{H}} \stackrel{\text{def}}{=} \boldsymbol{H} \times [0, \infty)$. A ***predictable transformation*** of $\boldsymbol{H}$ is a map $\Gamma : \check{\boldsymbol{B}} \to \check{\boldsymbol{B}}$ of the form

$$\big(\eta, s, \omega\big) \mapsto \big(\gamma(\eta, s; \omega), s, \omega\big) ,$$

where $\gamma : \check{\boldsymbol{B}} \to \boldsymbol{H}$ is predictable, i.e., $\check{\mathcal{P}}/\mathcal{B}^\bullet(\boldsymbol{H})$-measurable. Then Γ is clearly $\check{\mathcal{P}}/\check{\mathcal{P}}$-measurable. Let us fix such Γ, and assume the following:

(i) The given measured filtration $(\mathcal{F}_\bullet, \mathbb{P})$ is full (see definition 3.9.16).

(ii) Γ is invertible and Γ^{-1} is $\check{\mathcal{P}}/\check{\mathcal{P}}$-measurable.

(iii) $\gamma[\nu] \ll \nu$, with bounded Radon–Nikodym derivative $\check{D} \stackrel{\text{def}}{=} \dfrac{d\gamma[\nu]}{d\nu} \in \check{\mathcal{P}}$.

(iv) $\check{Y} \stackrel{\text{def}}{=} \check{D} - 1$ is a "Hunt function:" $\sup_{s,\omega} \int \check{Y}^2(\eta, s, \omega)\, \nu(d\eta) < \infty$.

Then $M \stackrel{\text{def}}{=} \check{Y}{*}\widetilde{\pi}$ is a martingale, and is a local L^p-integrator for all $p > 0$ on the grounds that its jumps are bounded (corollary 4.4.3). Consider the stochastic exponential $G' \overset{\text{def}}{=} 1 + G'_{\bullet-}{*}M = 1 + (G'_{\bullet-}\check{Y}){*}\widetilde{\pi}$ of M. Since $\Delta M \geq -1$, we have $G' \geq 0$.

$$\text{Now} \quad \mathbb{E}\big[[G', G']_t\big] = \mathbb{E}\Big[1 + \big(G'^2_{\bullet-}{*}[M,M]\big)_t\Big]$$
$$= \mathbb{E}\Big[1 + \big((G'_{\bullet-}\check{Y})^2{*}\pi\big)_t\Big] = \mathbb{E}\Big[1 + \big((G'_{\bullet-}\check{Y})^2{*}\widehat{\pi}\big)_t\Big]$$
$$\leq \mathbb{E}\Big[1 + \int_0^t G'^2_{\bullet-}(s) \int_{\boldsymbol{H}} \check{Y}^2(\eta, s)\, \nu(d\eta)\, ds\Big] ,$$

$$\text{and so} \qquad \mathbb{E}\big[G'^{\star 2}_t\big] \leq \mathit{const}\Big(1 + \mathit{const}\int_0^t \mathbb{E}\big[G'^{\star 2}_s\big]\, ds\Big) .$$

By Gronwall's lemma A.2.35, G' is a square integrable martingale, and the fullness provides a probability $\mathbb{P}'$ on $\mathcal{F}_\infty$ whose restriction to the $\mathcal{F}_t$ is $G'_t\mathbb{P}$.

Let us now compute the compensator $\widehat{\pi}'$ of π with respect to $\mathbb{P}'$. For $\check{H} \in \check{\mathcal{P}}_b$ vanishing after t we have

$$\mathbb{E}'\big[(\check{H}{*}\widehat{\pi}')_t\big] = \mathbb{E}'\big[(\check{H}{*}\pi)_t\big] = \mathbb{E}\big[G'_t \cdot (\check{H}{*}\pi)_t\big]$$
$$= \mathbb{E}\Big[\big((G'_{\bullet-}\check{H}){*}\pi\big)_t\Big] + \mathbb{E}\Big[\big((\check{H}{*}\pi)_{\bullet-}{*}G'\big)_t\Big] + \mathbb{E}\Big[[G', \check{H}{*}\pi]_t\Big]$$
$$= \mathbb{E}\Big[\big((G'_{\bullet-}\check{H}){*}\widehat{\pi}\big)_t\Big] + 0 + \mathbb{E}\Big[G'_{\bullet-}{*}[\check{Y}{*}\widetilde{\pi}, \check{H}{*}\widetilde{\pi}]_t\Big]$$
$$\text{by 3.10.18:} \qquad = \mathbb{E}\Big[\big((G'_{\bullet-}\check{H}){*}\widehat{\pi}\big)_t\Big] + \mathbb{E}\Big[\big(G'_{\bullet-}\check{Y}\check{H}{*}\pi\big)_t\Big]$$
$$\text{as } 1 + \check{Y} = \check{D}: \qquad = \mathbb{E}\Big[\big(G'_{\bullet-}\check{D}\check{H}{*}\widehat{\pi}\big)_t\Big] = \mathbb{E}\Big[\big(G'_{\bullet-}{*}(\check{D}\check{H}{*}\widehat{\pi})\big)_t\Big]$$
$$= \mathbb{E}\Big[G'_t \cdot (\check{D}\check{H}{*}\widehat{\pi})_t\Big] = \mathbb{E}'\Big[(\check{D}\check{H}{*}\widehat{\pi})_t\Big] ;$$

$$\text{so} \quad \mathbb{E}'\big[(\check{H}{*}\widehat{\pi}')_t\big] = \mathbb{E}'\Big[\big(\check{H}{*}(\check{D}\widehat{\pi})\big)_t\Big] .$$

$$\text{Therefore} \qquad \widehat{\pi}' = \check{D}\widehat{\pi} = \Gamma[\widehat{\pi}] \ , \text{ i.e., } \ \widehat{\Gamma^{-1}[\pi]}' = \Gamma^{-1}[\widehat{\pi}'] = \widehat{\pi} .$$

In other words, replacing $\check{H}$ by $\check{H} \circ \Gamma^{-1} \in \check{\mathcal{P}}$ gives

$$\mathbb{E}'\big[(\check{H}{*}\Gamma^{-1}[\pi])_t\big] = \mathbb{E}'\Big[\big((\check{H} \circ \Gamma^{-1}){*}(\check{D}\widehat{\pi})\big)_t\Big] = \mathbb{E}'\Big[\big(\check{H}{*}\Gamma^{-1}[\check{D}\widehat{\pi}]\big)_t\Big]$$
$$\text{as } \Gamma[\widehat{\pi}] = \check{D}\widehat{\pi}: \qquad = \mathbb{E}'\Big[\big(\check{H}{*}\widehat{\pi}\big)_t\Big] \ , \text{ therefore}$$

Theorem 3.10.21 *Under the assumptions above the "shifted Poisson point process" $\Gamma^{-1}[\pi]$ is a Poisson point process with respect to $\mathbb{P}'$, of the same intensity rate ν that π had under $\mathbb{P}$. Consequently, the law of $\Gamma^{-1}[\pi]$ under $\mathbb{P}'$ agrees with the law of π under $\mathbb{P}$.*

Repeated Footnotes: 95[1] 110[4] 116[6] 125[7] 139[8] 158[14] 161[15] 164[16] 173[21] 178[25] 180[27]