Section 2.1

6. \( y'' + ty' + y = 0 \) (*)

(a) \( y_1(t) = e^{-\frac{t^2}{2}} \).

We show that \( y_1 \) satisfies equation (*)

\[
\begin{align*}
    y_1' &= (-t)e^{-\frac{t^2}{2}} \\
    y_1'' &= (-1) e^{-\frac{t^2}{2}} + (-t)(-te^{-\frac{t^2}{2}}) = (t^2 - 1)e^{-\frac{t^2}{2}}.
\end{align*}
\]

Then \( y'' + ty' + y_1 = (t^2 - 1)e^{-\frac{t^2}{2}} + t(-te^{-\frac{t^2}{2}}) + e^{-\frac{t^2}{2}} \)
\[= (t^2 - 1 - t^3 + 1)e^{-\frac{t^2}{2}} \]
\[= 0. \quad \text{(1)} \]

Next we show that \( y_2(t) = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} \, ds \) satisfies equation (1).

Observe that \( y_2(t) = y_1(t) \cdot F(t) \) \text{(2)} where \( F(t) = \int_0^t e^{\frac{s^2}{2}} \, ds \).

From Fundamental Theorem of Calculus,
\[ F'(t) = e^{\frac{t^2}{2}}. \]

Therefore, \( y_2' = y_1 \cdot F + y_1F' = y_1F + e^{-\frac{t^2}{2}} \cdot \frac{t}{2} = y_1F + \frac{t}{2} \)
\[= y_1F + t. \quad \text{(3)} \]

\[ y_2'' = (y_1F + t)' \\
= y_1''F + y_1F' + 0 \\
= y_1''F + (-te^{-\frac{t^2}{2}}) \cdot e^{\frac{t^2}{2}} \\
= y_1''F - t. \quad \text{(4)} \]

Thus \( y_2'' + ty_2' + y_2 \) \text{ from } \text{(2)-(4)}
\[= F(y_1'' + ty_1' + y_1) - t + t = 0. \quad \text{(Since } y_1'' + ty_1' + y_1 = 0) \]
(b) \[ W[y_1, y_2](t) = y_1 y_2' - y_2 y_1' = y_1 (y_2' F + 1) - (y_1 F) y_2' \]
\[ = y_1 y_2' F + y_1 - y_1 y_2' F \]
\[ = y_1(t) \]
\[ = e^{-\frac{t^2}{2}} \]

(c) Since \( W[y_1, y_2](t) = e^{-\frac{t^2}{2}} \neq 0 \) for all \( t \), \( y_1 \) and \( y_2 \) form a fundamental set of solutions of (**) on \( (-\infty, \infty) \).

(d) \[ y'' + ty' + y = 0, \quad y(0) = 0, \quad y'(0) = 1. \quad (***) \]
\[ y(t) = C_1 y_1(t) + C_2 y_2(t). \]
Then \[ y'(t) = C_1 y_1'(t) + C_2 y_2'(t). \]
Plug in \( t = 0 \), \[ y_1(0) = 1, \quad y_1'(0) = 0, \]
\[ y_2(0) = y_2(0) f(0) = 0, \quad y_2'(0) = y_2(0) f(0) + 1 = 1 \]
\[ \Rightarrow \begin{cases} \quad 0 = C_1 \cdot 1 + C_2 \cdot 0 \\
1 = C_1 \cdot 0 + C_2 \cdot 1 \end{cases} \]
\[ \Rightarrow C_1 = 0, \quad C_2 = 1. \]
\[ \Rightarrow y(t) = y_2(t) = e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} ds \quad \text{is the solution} \]
\[ \text{to the initial value problem (***).} \]