2.1. 3. \[ L[cy_1(t)] = \int^t_a \int^t_a S^2 y(s) ds \]

In order to show \( L \) is linear, I want to show

\[ L[c_1 y_1(t) + c_2 y_2(t)] = c_1 L[y_1(t)] + c_2 L[y_2(t)]. \]

\[ L[c_1 y_1(t) + c_2 y_2(t)] = \int^t_a S^2 (c_1 y_1(s) + c_2 y_2(s)) ds \]

\[ = \int^t_a S^2 c_1 y_1(s) + \int^t_a S^2 c_2 y_2(s) ds \]

\[ = c_1 \int^t_a S^2 y_1(s) ds + c_2 \int^t_a S^2 y_2(s) ds \]

\[ = c_1 L[y_1(t)] + c_2 L[y_2(t)]. \]

Note: The linearity of \( L \) is basically coming from the linearity of the integration.

4. \( L[y(t)] = y'' + p(t)y' + q(t)y(t). \)

We know \( L[t^2] = \int^t_a t + 1 \) & \( L[t] = -2t + 2. \)

If \( y(t) = t - 2t^2 \) solves \( y'' + p(t)y' + q(t)y = 0, \)

\[ L[t - 2t^2] = 0, \]

\[ L[t - 2t^2] = L[t] - 2L[t^2] = 2t + 2 - 2(t + 1) = 0. \]

Hence \( t - 2t^2 \) is a solution of \( y'' + p(t)y' + q(t)y = 0. \)

5a. Simply pluging \( y(t) = t - 2t^2 \) into \( 2t^2 y'' + 3ty' - y = 0 \)

to see if the equality holds.

b. \( W = y_1 y_2' - y_2 y_1' = \sqrt{E} \cdot \left( 1 - \frac{t}{2t^2} \right) - \frac{t}{2t^2} \left( \frac{1}{2t^2} \right) = \frac{1}{t^{3/2}} - \frac{1}{2t^{3/2}} = -\frac{3}{2t^{3/2}}. \)

So \( y_1, y_2 \) form a fundamental set of sol.

c. \( W[y_1, y_2] = \frac{3}{2t^{3/2}} \neq 0 \) for \( 0 < t < \infty. \)

d. \( y = c_1 y_1 + c_2 y_2 = c_1 \sqrt{E} + c_2 \frac{1}{t}. \) Since \( y(1) = 2, \ y'(1) = 1, \) I have \( y = 2 \sqrt{E} + \frac{1}{t}. \)