3.8
V = \{ p(t) = a_0 + a_1 t + a_2 t^2 : p(0) + 2p'(0) + 3p''(0) = 0 \}

1. First we show that V is a vector space by showing it's a subspace of the space of all real-valued functions:

   1. if \( p(t) = 0 \) then \( p(0) + 2p'(0) + 3p''(0) = 0 \) \( \Rightarrow 0 \in V \).

   2. if \( p(t) \) and \( q(t) \in V \), then \( p(0) + 2p'(0) + 3p''(0) = 0 \) and \( q(0) + 2q'(0) + 3q''(0) = 0 \).

   For \( p + q \), we have

   \[
   (p + q)(0) + 2(p + q)'(0) + 3(p + q)''(0) = p(0) + q(0) + 2p'(0) + 2q'(0) + 3(p''(0) + q''(0)) = 0.
   \]

   Thus \( V \) is closed under addition.

3. if \( p(t) \in V \), then \( p(0) + 2p'(0) + 3p''(0) = 0 \).

   For any constant \( c \in \mathbb{R} \),

   \[
   (cp)(0) + 2(cp)'(0) + 3(cp)''(0) = c p(0) + 2 c p'(0) + 3 c p''(0) = c (p(0) + 2p'(0) + 3p''(0)) = 0.
   \]

   Thus \( V \) is closed under scalar multiplication.

   Therefore \( V \) is a subspace of the space of all functions. \( V \) is a vector space.

   (You can also 'translate' the condition \( p(0) + 2p'(0) + 3p''(0) = 0 \) into \( a_0 + 2a_1 + 6a_2 = 0 \), as we did in the next part, and show that all polynomials \( a_0 + a_1 t + a_2 t^2 \) with \( a_0 + 2a_1 + 6a_2 = 0 \) form a vector space. This is equivalent.)

Next we find a basis for \( V \).

First, by plugging in \( p(t) = a_0 + a_1 t + a_2 t^2 \) to \( p(0) + 2p'(0) + 3p''(0) = 0 \),

we get:

\[
\begin{align*}
p(0) &= a_0, \\
p'(t) &= a_1 + 2a_2 t, & p'(0) &= a_1, \\
p''(t) &= 2a_2, & p''(0) &= 2a_2
\end{align*}
\]

\( \Rightarrow \) \( p(0) + 2p'(0) + 3p''(0) = 0 \) is equivalent to \( a_0 + 2a_1 + 6a_2 = 0 \).
Thus the vector space

\[ V = \{ p(t) = a_0 + a_1 t + a_2 t^2 : \quad a_0 + 2a_1 + 6a_2 = 0 \} \]

\[ = \{ a_0 + a_1 t + a_2 t^2 : \quad a_0 = -2a_1 - 6a_2 \} \]

Since we notice that \( a_0 + 2a_1 + 6a_2 = 0 \) restricts the coefficients such that one of them can be completely determined by the other.

Thus

\[ V = \{ a_0 + a_1 t + a_2 t^2 : \quad a_0 = -2a_1 - 6a_2 \} \]

\[ = \{ (-2a_1 - 6a_2) + a_1 t + a_2 t^2 \} \]

\[ = \{ a_1 (-2 + t) + a_2 (t^2 - 6) \} \]

Thus any \( p(t) \in V \) can be represented as

\[ p(t) = a_1 (-2 + t) + a_2 (t^2 - 6) \]

or to say, \( p(t) \) is a linear combination of \( (-2 + t) \) and \( (t^2 - 6) \).

Since \( (-2 + t) \) and \( (t^2 - 6) \) are linearly independent, (you can verify this by \( c_1 (-2 + t) + c_2 (t^2 - 6) = 0 \) or by observing that they are not scalar multiples of each other), they form a basis for \( V \).

Thus \( V \) is a vector space with dimension 2, and a set of basis for \( V \) is \( \{ t^2 - 6, \ t - 2 \} \).

We can also write \( a_1 = -\frac{1}{2} a_0 - 3a_2 \) or \( a_2 = -\frac{1}{6} a_0 - \frac{1}{3} a_1 \), and

\[ p(t) = a_0 + a_1 t + a_2 t^2 = a_0 + a_1 t + \left(-\frac{1}{6} a_0 - \frac{1}{3} a_1\right) t^2 \]

\[ = a_0 \left(1 - \frac{1}{6} t^2\right) + a_1 \left(t - \frac{1}{3} t^2\right) \]

then \( p(t) \) is a linear combination of \( \left(1 - \frac{1}{6} t^2\right) \) and \( t - \frac{1}{3} t^2 \).

Thus \( \{ 1 - \frac{t^2}{6}, \ t - \frac{t^2}{3} \} \) is also a set of basis for \( V \).
You can also observe that $V$ has dimension 2 (since one of the coefficients is determined by the other 1), then find two arbitrary polynomials in $V$ such that they are linearly independent. For example, since $a_0 + 2a_1 + 6a_2 = 0$ try $a_0 = 1$, $a_1 = -\frac{1}{2}$, $a_2 = \frac{3}{8}$, $\beta(t) = 1 - \frac{1}{2}t$.

$a_0 = 0$, $a_1 = 0$, $a_2 = -\frac{1}{6}$ \Rightarrow $\beta(t) = 1 - \frac{1}{6}t^2$.

Since $\beta_1, \beta_2$ are linearly independent and $\dim V = 2$, they form a basis of $V$.

You must notice that $P(0) + 2P'(0) + 3P''(0) = 0$. It is not a differential equation. It is not the same as $y + 2y' + 3y'' = 0$, since that means $y(t) + 2y'(t) + 3y''(t) = 0$ for all $t$, not only for $t = 0$. The vector space $V$ is not the space of solutions to the differential equation $y + 2y' + 3y'' = 0$.