Separable Equations

A **separable equation** is a first-order differential equation that can be written in the form

\[ \frac{dy}{dx} = g(x)f(y) \]

The name *separable* comes from the fact that the expression on the right side can be “separated” into a function of \( x \) and a function of \( y \). Equivalently, if \( f(y) \neq 0 \), we could write

\[ \frac{dy}{dx} = \frac{g(x)}{h(y)} \]  \hspace{1cm} (1)

where \( h(y) = 1/f(y) \). To solve this equation we rewrite it in the differential form

\[ h(y)dy = g(x)dx \]

so that all \( y \)'s are on one side of the equation and all \( x \)'s are on the other side. Then we integrate both sides of the equation:

\[ \int h(y)dy = \int g(x)dx \]  \hspace{1cm} (2)

Equation 2 defines \( y \) implicitly as a function of \( x \). In some cases we may be able to solve for \( y \) in terms of \( x \).

We use the Chain Rule to justify this procedure: If \( h \) and \( g \) satisfy (2), then

\[ \frac{d}{dx} \left( \int h(y)dy \right) = \frac{d}{dx} \left( \int g(x)dx \right) \]

so

\[ \frac{d}{dy} \left( \int h(y)dy \right) \frac{dy}{dx} = g(x) \]

and

\[ h(y)\frac{dy}{dx} = g(x) \]

**EXAMPLE:**

(a) Solve the differential equation \( \frac{dy}{dx} = \frac{x^2}{y^2} \).

Solution: We write the equation in terms of differentials and integrate both sides:

\[ y^2dy = x^2dx \]

\[ \int y^2dy = \int x^2dx \]

\[ \frac{1}{3}y^3 = \frac{1}{3}x^3 + C \]

hence

\[ y = \sqrt[3]{x^3 + 3C} \]

which can be rewritten as

\[ y = \sqrt[3]{x^3 + K} \]

where \( K = 3C \).
(b) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Solution: If we put $x = 0$ in the general solution in part (a), we get $y(0) = \sqrt[3]{K}$. To satisfy the initial condition $y(0) = 2$, we must have $\sqrt[3]{K} = 2$ and so $K = 8$. Thus the solution of the initial-value problem is

$$y = \sqrt[3]{x^3 + 8}$$

EXAMPLE: Solve the differential equation $\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$.

Solution: Writing the equation in differential form and integrating both sides, we have

$$(2y + \cos y)dy = 6x^2dx$$

$$\int(2y + \cos y)dy = \int 6x^2dx$$

$$y^2 + \sin y = 2x^3 + C$$

where $C$ is a constant. The last equation gives the general solution implicitly. In this case it’s impossible to solve the equation to express $y$ explicitly as a function of $x$.

EXAMPLE: Solve the equation $y' = x^2y$.

Solution 1: We have

$$y' = x^2y \implies y' - x^2y = 0 \implies y' + (-x^2)y = 0$$

This is the homogeneous first-order linear differential equation with $a(x) = -x^2$, therefore

$$y(x) = c \exp \left(- \int a(x)dx \right) = c \exp \left(- \int (-x^2)dx \right) = c \exp \left(\int x^2dx \right) = ce^{x^3/3}$$

where $c$ is any real number.

Solution 2: First we rewrite the equation using Leibniz notation $\frac{dy}{dx} = x^2y$. If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = x^2dx \quad y \neq 0$$

$$\int \frac{dy}{y} = \int x^2dx$$

$$\ln |y| = \frac{x^3}{3} + C$$

This equation defines $y$ implicitly as a function of $x$. But in this case we can solve explicitly for $y$ as follows:

$$|y| = e^{|\ln y|} = e^{(x^3/3)+C} = e^C e^{x^3/3} \implies y = \pm e^C e^{x^3/3}$$

We can easily verify that the function $y = 0$ is also a solution of the given differential equation. So we can write the general solution in the form

$$y = Ae^{x^3/3}$$

where $A$ is an arbitrary constant ($A = e^C$, or $A = -e^C$, or $A = 0$).