The Method of Variation of Parameters

Consider the second-order linear nonhomogeneous differential equation

\[ y'' + a_1 y' + a_2 y = F \]  \hspace{1cm} (1)

where we assume that \( a_1, a_2, \) and \( F \) are continuous on an interval \( I. \) Suppose that \( y = y_1(x) \) and \( y = y_2(x) \) are two linearly independent solutions to the associated homogeneous equation

\[ y'' + a_1 y' + a_2 y = 0 \]  \hspace{1cm} (2)

on \( I, \) so that the general solution to equation (2) on \( I \) is

\[ y_c = c_1 y_1(x) + c_2 y_2(x) \]

The variation-of-parameters method consists of replacing the constants \( c_1 \) and \( c_2 \) by functions \( u_1(x) \) and \( u_2(x) \) (that is, we allow the parameters \( c_1 \) and \( c_2 \) to vary), determined in such a way that the resulting function

\[ y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \]  \hspace{1cm} (3)

is a particular solution to equation (1).

Differentiating equation (3) with respect to \( x \) yields

\[ y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \]

It is tempting to differentiate this expression once more and then substitute into equation (1) to determine \( u_1 \) and \( u_2. \) However, if we did this, the resulting expression for \( y_p'' \) would involve second derivatives of \( u_1 \) and \( u_2, \) hence we would have complicated our problem. Since \( y_p \) contains two unknown functions, whereas equation (1) gives only one condition for determining them, we have the freedom to impose a further constraint on \( u_1 \) and \( u_2. \) In order to eliminate second derivatives of \( u_1 \) and \( u_2 \) arising in \( y_p \) we try for solutions of the form (3) satisfying the constraint

\[ u_1'y_1 + u_2'y_2 = 0 \]  \hspace{1cm} (4)

The expression for \( y_p' \) then reduces to

\[ y_p' = u_1'y_1 + u_2'y_2 \]

so that

\[ y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \]

Substituting into equation (1) and collecting terms yields (see Appendix I)

\[ u_1(y_1'' + a_1 y_1' + a_2 y_1) + u_2(y_2'' + a_1 y_2' + a_2 y_2) + (u_1'y_1' + u_2'y_2') = F \]

The terms multiplying \( u_1 \) and \( u_2 \) vanish, since \( y_1 \) and \( y_2 \) each solve \( y'' + a_1 y' + a_2 y = 0. \) We therefore require that

\[ u_1'y_1' + u_2'y_2' = F \]  \hspace{1cm} (5)

Consequently, \( y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \) is a solution to equation (1), provided that \( u_1 \) and \( u_2 \) satisfy equations (4) and (5). That is,

\[
\begin{align*}
    y_1u_1' + y_2u_2' &= 0 \\
    y_1'u_1 + y_2'u_2 &= F
\end{align*}
\]  \hspace{1cm} (6)
This is a linear system of equations for the unknowns \( u_1' \) and \( u_2' \). The matrix of coefficients of this system has determinant

\[
\begin{vmatrix}
y_1 & y_2 \\
y_1' & y_2'
\end{vmatrix} = y_1 y_2' - y_2 y_1'
\]

which is the Wronskian, \( W[y_1, y_2](x) \), of \( y_1 \) and \( y_2 \). Since \( y_1 \) and \( y_2 \) are linearly independent on \( I \), \( W[y_1, y_2](x) \) is nonzero on \( I \) and hence the system (6) has a unique solution for \( u_1' \) and \( u_2' \). Indeed, applying Cramer’s rule to (6) (see Appendix II) yields

\[
\begin{align*}
\frac{u_1'(x)}{W[y_1, y_2](x)} &= -\frac{y_2(x)F(x)}{W[y_1, y_2](x)}, \\
\frac{u_2'(x)}{W[y_1, y_2](x)} &= \frac{y_1(x)F(x)}{W[y_1, y_2](x)}
\end{align*}
\]

(7)

Finally, we obtain \( u_1(t) \) and \( u_2(t) \) by integrating the right-hand sides of (7).

EXAMPLE: Solve \( y'' + y = \sec x, -\pi/2 < x < \pi/2 \).

Solution: The characteristic polynomial is

\[ P(r) = r^2 + 1 \]

We have

\[ r^2 + 1 = 0 \implies r^2 = -1 \implies r = \pm \sqrt{-1} = 0 \pm 1 \cdot i \]

So, \( \alpha = 0 \) and \( \beta = 1 \). Consequently, two linearly independent solutions to the associated homogeneous equation are

\[ y_1(x) = e^{\alpha x} \cos \beta x = e^0 \cos(1 \cdot x) = \cos x \quad \text{and} \quad y_2(x) = e^{\alpha x} \sin \beta x = e^0 \sin(1 \cdot x) = \sin x \]

Thus, a particular solution to the given differential equation is

\[ y_p(x) = u_1 y_1 + u_2 y_2 = u_1 \cos x + u_2 \sin x \]

(8)

where \( u_1 \) and \( u_2 \) satisfy

\[
\begin{align*}
\begin{cases}
y_1 u_1' + y_2 u_2' = 0 \\
y_1' u_1' + y_2' u_2' = F
\end{cases} &\implies \begin{cases}
\cos x u_1' + \sin x u_2' = 0 \\
-\sin x u_1' + \cos x u_2' = \sec x
\end{cases}
\end{align*}
\]

Applying Cramer’s rule (or formulas (7)), the solution to this system is

\[
\begin{align*}
u_1' &= \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \\
\cos x & \sin x \\
-\sin x & \cos x \end{vmatrix} = \frac{0 - \sin x \sec x}{\cos^2 x + \sin^2 x} = -\sin x \sec x = -\frac{\sin x}{\cos x}, \\
u_2' &= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \\
\cos x & \sin x \\
-\sin x & \cos x \end{vmatrix} = \frac{\cos x \sec x - 0}{\cos^2 x + \sin^2 x} = \cos x \sec x = \cos x \frac{1}{\cos x} = 1
\end{align*}
\]
So, 
\[ u'_1 = -\frac{\sin x}{\cos x}, \quad u'_2 = 1 \]

Consequently,
\[ u_1(x) = \int \left( -\frac{\sin x}{\cos x} \right) dx = \int \frac{1}{\cos x} \cdot (-\sin x) dx = \int \frac{1}{u} du = \ln |u| = \ln |\cos x| \]

and
\[ u_2(x) = \int 1 dx = x \]

where we have set the integration constants to zero, since we require only one particular solution. Substitution into equation (8) yields
\[ y_p(x) = u_1 \cos x + u_2 \sin x \]
\[ = \ln |\cos x| \cdot \cos x + x \sin x \]
\[ = \ln(\cos x) \cdot \cos x + x \sin x \quad (-\pi/2 < x < \pi/2) \]

so that the general solution to the given differential equation is
\[ y(x) = c_1 \cos x + c_2 \sin x + \ln(\cos x) \cdot \cos x + x \sin x \]

EXAMPLE: Solve \( y'' + 4y' + 4y = e^{-2x} \ln x, \ x > 0. \)

Solution: The characteristic polynomial is
\[ P(r) = r^2 + 4r + 4 = r^2 + 2r \cdot 2 + 2^2 = (r + 2)^2 \]

Thus, \( r = -2 \) is a repeated root of the characteristic equation, and therefore two linearly independent solutions to the associated homogeneous equation are
\[ y_1(x) = e^{-2x} \quad \text{and} \quad y_2(x) = xe^{-2x} \]

hence we seek a particular solution to the given differential equation of the form
\[ y_p(x) = u_1y_1 + u_2y_2 = u_1e^{-2x} + u_2xe^{-2x} \]

where \( u_1 \) and \( u_2 \) satisfy
\[ \begin{cases} y_1u'_1 + y_2u'_2 = 0 \\ y'_1u_1 + y'_2u_2 = F \end{cases} \implies \begin{cases} e^{-2x}u'_1 + xe^{-2x}u'_2 = 0 \\ -2e^{-2x}u'_1 + e^{-2x}(1 - 2x)u'_2 = e^{-2x} \ln x \end{cases} \]

WORK: \( (xe^{-2x})' = x'e^{-2x} + x(e^{-2x})' = 1 \cdot e^{-2x} + x(-2)e^{-2x} = e^{-2x}(1 - 2x) \)

Canceling out \( e^{-2x} \), we get
\[ \begin{cases} u'_1 + xu'_2 = 0 \\ -2u'_1 + (1 - 2x)u'_2 = \ln x \end{cases} \]
Applying Cramer’s rule (or formulas (7)), the solution to this system is

\[
\begin{align*}
      u'_1 &= \begin{vmatrix} 0 & x \\ \ln x & 1 - 2x \\ 1 & x \\ -2 & 1 - 2x \end{vmatrix} = \frac{0 - x \ln x}{1 \cdot (1 - 2x) - x \cdot (-2)} = \frac{-x \ln x}{1 - 2x + 2x} = -x \ln x \\
      u'_2 &= \begin{vmatrix} 1 & 0 \\ -2 \ln x & 1 \\ 1 & x \\ -2 & 1 - 2x \end{vmatrix} = \frac{1 \cdot \ln x - 0}{1 \cdot (1 - 2x) - x \cdot (-2)} = \frac{\ln x}{1 - 2x + 2x} = \ln x
\end{align*}
\]

So,

\[
u'_1 = -x \ln x, \quad u'_2 = \ln x
\]

Integrating both of these expressions by parts

\[
\int u dv = uv - \int v du
\]

we obtain

\[
u_1(x) = -\int x \ln x \, dx = \left[ \begin{array}{c}
\ln x = u \\
\frac{d}{dx}(\ln x) = du \\
\frac{1}{x} \, dx = dv
\end{array} \right] x \frac{d}{dx} = dv = -\ln x \cdot \frac{x^2}{2} + \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx
\]

\[
= -\frac{1}{2} x^2 \ln x + \frac{1}{2} \int x \, dx
\]

\[
= -\frac{1}{2} x^2 \ln x + \frac{1}{2} \cdot \frac{x^2}{2}
\]

\[
= -\frac{1}{2} x^2 \ln x + \frac{1}{4} x^2
\]

\[
= \frac{1}{4} x^2 \cdot 1 - \frac{1}{4} x^2 \cdot 2 \ln x
\]

\[
= \frac{1}{4} x^2 (1 - 2 \ln x)
\]

and

\[
u_2(x) = \int \ln x \, dx = \left[ \begin{array}{c}
\ln x = u \\
\frac{d}{dx}(\ln x) = du \\
\frac{1}{x} \, dx = dv
\end{array} \right] x = v
\]

\[
= \ln x \cdot x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx
\]

\[
= x \ln x - x
\]

\[
= x(\ln x - 1)
\]
So,

\[ u_1(x) = \frac{1}{4}x^2(1 - 2 \ln x), \quad u_2(x) = x(\ln x - 1) \]

Thus,

\[ y_p(x) = u_1e^{-2x} + u_2xe^{-2x} \]

\[ = \frac{1}{4}x^2(1 - 2 \ln x) \cdot e^{-2x} + x(\ln x - 1) \cdot xe^{-2x} \]

\[ = \frac{1}{4}x^2e^{-2x}(1 - 2 \ln x) + x^2e^{-2x}(\ln x - 1) \]

\[ = \frac{1}{4}x^2e^{-2x}(1 - 2 \ln x + \frac{1}{4}x^2(4 \ln x - 4)) \]

\[ = \frac{1}{4}x^2e^{-2x}(2 \ln x - 3) \]

Consequently, the general solution to the given differential equation is

\[ y(x) = c_1e^{-2x} + c_2xe^{-2x} + \frac{1}{4}x^2e^{-2x}(2 \ln x - 3) \]

\[ = e^{-2x} \left[ c_1 + c_2x + \frac{1}{4}x^2(2 \ln x - 3) \right] \]
Appendix I

We have

\[ y_p = u_1 y_1 + u_2 y_2 \]
\[ y'_p = u_1 y'_1 + u_2 y'_2 \]
\[ y''_p = u'_1 y'_1 + u_1 y''_1 + u'_2 y'_2 + u_2 y''_2 \]

therefore

\[ y''_p + a_1 y'_p + a_2 y_p = (u'_1 y'_1 + u_1 y''_1 + u'_2 y'_2 + u_2 y''_2) + a_1(u_1 y'_1 + u_2 y'_2) + a_2(u_1 y_1 + u_2 y_2) \]
\[ = u'_1 y'_1 + u_1 y''_1 + u'_2 y'_2 + u_2 y''_2 + a_1 u_1 y'_1 + a_1 u_2 y'_2 + a_2 u_1 y_1 + a_2 u_2 y_2 \]
\[ = u_1 y'_1 + a_1 u_1 y'_1 + a_2 u_1 y_1 + u_2 y'_2 + a_1 u_2 y'_2 + a_2 u_2 y_2 + u'_1 y'_1 + u'_2 y'_2 \]
\[ = u_1(y''_1 + a_1 y'_1 + a_2 y_1) + u_2 (y''_2 + a_1 y'_2 + a_2 y_2) + (u'_1 y'_1 + u'_2 y'_2) \]

So, since \( y''_p + a_1 y'_p + a_2 y_p = F \), it follows that

\[ u_1(y''_1 + a_1 y'_1 + a_2 y_1) + u_2 (y''_2 + a_1 y'_2 + a_2 y_2) + (u'_1 y'_1 + u'_2 y'_2) = F \]
Appendix II

THEOREM (Cramer’s Rule): Let $A$ be an invertible $n \times n$ matrix. For any $b$ in $\mathbb{R}^n$, the unique solution $x$ of $Ax = b$ has entries given by

$$x_i = \frac{\det A_i(b)}{\det A}, \quad i = 1, 2, \ldots, n$$

EXAMPLE: Solve using Cramer’s rule

$$\begin{cases} x_1 - 2x_2 = 1 \\ 3x_1 + 4x_2 = -7 \end{cases}$$

Solution: We have

$$x_1 = \frac{\begin{vmatrix} 1 & -2 \\ -7 & 4 \\ 1 & -2 \\ 3 & 4 \end{vmatrix}}{4 - (-6)} = \frac{4 - 14}{10} = -1$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -7 \\ 1 & -2 \\ 3 & 4 \end{vmatrix}}{10} = -1$$

EXAMPLE: Solve using Cramer’s rule

$$\begin{cases} y_1u'_1 + y_2u'_2 = 0 \\ y'_1u_1 + y'_2u_2 = F \end{cases}$$

Solution: We have

$$u'_1 = \frac{\begin{vmatrix} 0 & y_2 \\ F & y'_2 \\ y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}{W[y_1, y_2](x)} = \frac{0 \cdot y'_2 - y_2 \cdot F}{W[y_1, y_2](x)} = -\frac{y_2 \cdot F}{W[y_1, y_2](x)}$$

$$u'_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & F \\ y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}}{W[y_1, y_2](x)} = \frac{y_1 \cdot F - 0 \cdot y'_1}{W[y_1, y_2](x)} = \frac{y_1 \cdot F}{W[y_1, y_2](x)}$$

where

$$W[y_1, y_2](x) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y_2y'_1$$