In this chapter we will consider simultaneous first-order differential equations in several variables, that is, equations of the form

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(t, x_1, \ldots, x_n) \\
\frac{dx_2}{dt} &= f_2(t, x_1, \ldots, x_n) \\
& \vdots \\
\frac{dx_n}{dt} &= f_n(t, x_1, \ldots, x_n)
\end{align*}
\]

(1)

In addition to equation (1), we will often impose initial conditions on the functions \(x_1(t), \ldots, x_n(t)\). These will be of the form

\[
x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0
\]

(1’)

Equation (1), together with the initial conditions (1’), is referred to as an initial-value problem.

First-order systems of differential equations also arise from higher-order equations for a single variable \(y(t)\). Every \(n\)th-order differential equation for the single variable \(y\) can be converted into a system of \(n\) first-order equations for the variables

\[
x_1(t) = y, \quad x_2(t) = \frac{dy}{dt}, \quad x_3(t) = \frac{d^2y}{dt^2}, \quad \ldots, \quad x_n(t) = \frac{d^{n-1}y}{dt^{n-1}}
\]

EXAMPLE: Convert the differential equation

\[
4 \frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = 0
\]

into a system of 2 first-order equations.

Solution: Let

\[
x_1(t) = y \quad \text{and} \quad x_2(t) = \frac{dy}{dt}
\]

From this it immediately follows that

\[
\frac{dx_1}{dt} = \frac{dy}{dt} = x_2(t) \quad \text{and} \quad \frac{dx_2}{dt} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{d^2y}{dt^2}
\]

But the original equation implies

\[
4 \frac{d^2y}{dt^2} = -\frac{dy}{dt} - 3y
\]

\[
\frac{d^2y}{dt^2} = -\frac{\frac{dy}{dt} + 3y}{4}
\]

therefore

\[
\frac{dx_2}{dt} = -\frac{x_2 + 3x_1}{4}
\]

So

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= -\frac{x_2 + 3x_1}{4}
\end{align*}
\]
EXAMPLE: Convert the differential equation

\[ a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0(t)y = 0 \]  

(2)

into a system of \( n \) first-order equations.

Solution: Let

\[ x_1(t) = y \]
\[ x_2(t) = \frac{dy}{dt} = \frac{dx_1}{dt} \]
\[ x_3(t) = \frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{dx_2}{dt} \]
\[ x_4(t) = \frac{d^3 y}{dt^3} = \frac{d}{dt} \left( \frac{d^2 y}{dt^2} \right) = \frac{dx_3}{dt} \]
\[ \vdots \]
\[ x_n(t) = \frac{d^{n-1} y}{dt^{n-1}} = \frac{d}{dt} \left( \frac{d^{n-2} y}{dt^{n-2}} \right) = \frac{dx_{n-1}}{dt} \]
\[ \frac{d^n y}{dt^n} = \frac{d}{dt} \left( \frac{d^{n-1} y}{dt^{n-1}} \right) = \frac{dx_n}{dt} \]

From this and (2) it follows that

\[ a_n(t) \frac{d^n y}{dt^n} + a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0(t)y = 0 \]

\[ a_n(t) \frac{d^n y}{dt^n} = -a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} - a_{n-2}(t) \frac{d^{n-2} y}{dt^{n-2}} - \ldots - a_0(t)y \]

\[ a_n(t) \frac{d^n y}{dt^n} = -\frac{a_{n-1}(t) \frac{d^{n-1} y}{dt^{n-1}} + a_{n-2}(t) \frac{d^{n-2} y}{dt^{n-2}} + \ldots + a_0(t)y}{a_n(t)} \]

\[ \frac{dx_n}{dt} = -\frac{a_{n-1}(t)x_n + a_{n-2}(t)x_{n-1} + \ldots + a_0(t)x_1}{a_n(t)} \]

So

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\vdots & \\
\frac{dx_{n-1}}{dt} &= x_n \\
\frac{dx_n}{dt} &= -\frac{a_{n-1}(t)x_n + a_{n-2}(t)x_{n-1} + \ldots + a_0(t)x_1}{a_n(t)} 
\end{align*}
\]
Example: Convert the initial-value problem
\[
\frac{d^3 y}{dt^3} + \left(\frac{dy}{dt}\right)^2 + 3y = e^t; \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0 \quad (3)
\]
into an initial-value problem for the variables \(y, \frac{dy}{dt},\) and \(\frac{d^2 y}{dt^2}\).

Solution: Let
\[
x_1(t) = y \\
x_2(t) = \frac{dy}{dt} = \frac{dx_1}{dt} \\
x_3(t) = \frac{d^2 y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{dx_2}{dt} \\
\frac{d^3 y}{dt^3} = \frac{d}{dt} \left( \frac{d^2 y}{dt^2} \right) = \frac{dx_3}{dt}
\]
From this and (3) it follows that
\[
\frac{d^3 y}{dt^3} + \left(\frac{dy}{dt}\right)^2 + 3y = e^t \\
\frac{d^3 y}{dt^3} = e^t - \left(\frac{dy}{dt}\right)^2 - 3y \\
\frac{dx_3}{dt} = e^t - x_2^2 - 3x_1
\]
So
\[
\begin{cases}
\frac{dx_1}{dt} = x_2 \\
\frac{dx_2}{dt} = x_3 \\
\frac{dx_3}{dt} = e^t - x_2^2 - 3x_1
\end{cases}
\]
Moreover, the functions \(x_1, x_2,\) and \(x_3\) satisfy the initial conditions
\[
x_1(0) = y(0) = 1, \quad x_2(0) = y'(0) = 0, \quad x_3(0) = y''(0) = 0
\]
The most general system of \(n\) first-order linear equations has the form
\[
\begin{cases}
\frac{dx_1}{dt} = a_{11}(t)x_1 + a_{12}(t)x_2 + \ldots + a_{1n}(t)x_n + g_1(t) \\
\frac{dx_2}{dt} = a_{21}(t)x_1 + a_{22}(t)x_2 + \ldots + a_{2n}(t)x_n + g_2(t) \\
\vdots \\
\frac{dx_n}{dt} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \ldots + a_{nn}(t)x_n + g_n(t)
\end{cases} \quad (4)
\]
If each of the functions \(g_1, g_2, \ldots g_n\) is identically zero, then the system (4) is said to be homogeneous; otherwise it is nonhomogeneous.
Now, even the homogeneous linear system with constant coefficients

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_1 + \ldots + a_{1n}x_n \\
\frac{dx_1}{dt} &= a_{21}x_1 + a_{22}x_1 + \ldots + a_{2n}x_n \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_1 + \ldots + a_{nn}x_n
\end{align*}
\]

is quite cumbersome to handle. This is especially true if \( n \) is large. Therefore, we seek to write these equations in as concise a manner as possible. To this end we introduce the concepts of \textit{vectors} and \textit{matrices}.

\textbf{DEFINITION: A vector}

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]

is a shorthand notation for the sequence of numbers \( x_1, \ldots, x_n \). The numbers \( x_1, \ldots, x_n \), are called the \textit{components} of \( x \). If \( x_1 = x_1(t), \ldots, x_n = x_n(t) \), then

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}
\]

is called a vector-valued function. Its derivative \( dx(t)/dt \) (often denoted by \( \dot{x}(t) \)) is the vector-valued function

\[
\begin{bmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt} \\
\vdots \\
\frac{dx_n(t)}{dt}
\end{bmatrix}
\]

\textbf{DEFINITION: A matrix}

\[
A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}
\]

is a shorthand notation for the array of numbers \( a_{ij} \) arranged in \( m \) rows and \( n \) columns.
We define the product of \( A \) with \( x \), denoted by \( Ax \), as the vector whose \( i \)th component is

\[
a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n, \quad i = 1, 2, \ldots, n
\]

In other words, the \( i \)th component of \( Ax \) is the sum of the product of corresponding terms of the \( i \)th row of \( A \) with the vector \( x \). Thus,

\[
Ax = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{bmatrix}
\]

For example,

\[
\begin{bmatrix} 1 & -2 & 3 & 17 \\ 2 & -4 & 9 & 46 \\ 3 & -6 & 4 & 31 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 17x_4 \\ 2x_1 - 4x_2 + 9x_3 + 46x_4 \\ 3x_1 - 6x_2 + 4x_3 + 31x_4 \end{bmatrix}
\]

Finally, we observe that the left-hand sides of (5)

\[
\begin{align*}
\frac{dx_1}{dt} &= a_{11}x_1 + \ldots + a_{1n}x_n \\
\vdots \\
\frac{dx_n}{dt} &= a_{n1}x_1 + \ldots + a_{nn}x_n
\end{align*}
\]

are the components of the vector \( dx/dt \), while the right-hand sides of (5) are the components of the vector \( Ax \). Hence, we can write (5) in the concise form

\[
\dot{x} = \frac{dx}{dt} = Ax
\]

where

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}
\]

Moreover, if \( x_1(t), \ldots, x_n(t) \) satisfy the initial conditions

\[
x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0, \quad \ldots, \quad x_n(t_0) = x_n^0
\]

then \( x(t) \) satisfies the initial-value problem

\[
\dot{x} = Ax, \quad x(t_0) = x^0, \quad \text{where} \quad x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}
\]
For example, the system of equations

\[
\begin{align*}
\frac{dx_1}{dt} &= 3x_1 - 7x_2 + 9x_3 \\
\frac{dx_2}{dt} &= 15x_1 + x_2 - x_3 \\
\frac{dx_3}{dt} &= 7x_1 + 6x_3
\end{align*}
\]

can be written in the concise form

\[
\dot{x} = Ax
\]

where

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{bmatrix}
\]

and

\[
A = \begin{bmatrix} 3 & -7 & 9 \\ 15 & 1 & -1 \\ 7 & 0 & 6 \end{bmatrix}
\]

DEFINITION: Let \( c \) be a number and \( x \) a vector with \( n \) components \( x_1, \ldots, x_n \). We define \( cx \) to be the vector whose components are \( cx_1, \ldots, cx_n \), that is

\[
cx = c \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}
\]

DEFINITION: Let \( x \) and \( y \) be vectors with components \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_n \) respectively. We define \( x + y \) to be the vector whose components are \( x_1 + y_1, \ldots, x_n + y_n \), that is

\[
x + y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}
\]

THEOREM 1: Let \( x(t) \) and \( y(t) \) be two solutions of (6). Then

(a) \( cx(t) \) is a solution for any constant \( c \)

(b) \( x(t) + y(t) \) is again a solution.

An immediate corollary of Theorem 1 is that any linear combination of solutions of (6) is again a solution of (6).
EXAMPLE: Solve the system \( \dot{x} = Ax \), where
\[
x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}
\]

Solution 1: We first note that the system can be rewritten as
\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= -4x_1
\end{align*}
\]
(8)

Plugging in the first equation into the second one, we get
\[
\begin{align*}
x_2' &= -4x_1 \\
(x_1')' &= -4x_1 \\
x_1'' &= -4x_1
\end{align*}
\]
(9)

To find two linearly independent solutions of (9) we note that the characteristic equation is
\[
r^2 + 4 = 0
\]
therefore
\[
r^2 = -4 \quad \implies \quad r = \pm \sqrt{-4} = \pm \sqrt{4(-1)} = \pm 2i = 0 \pm 2i
\]

Consequently, two linearly independent solutions to differential equation (9) are
\[
\begin{align*}
x_1^{(1)}(t) &= e^{0t} \cos 2t = \cos 2t \\
x_1^{(2)}(t) &= e^{0t} \sin 2t = \sin 2t
\end{align*}
\]

hence
\[
x_1(t) = c_1 x_1^{(1)}(t) + c_2 x_1^{(2)}(t) = c_1 \cos 2t + c_2 \sin 2t
\]
(10)

is the general solution of (9). But \( x_2(t) = x_1'(t) \) by the first equation of (8), therefore
\[
x_2(t) = (c_1 \cos 2t + c_2 \sin 2t)' = c_1(\cos 2t)' + c_2(\sin 2t)' = -2c_1 \sin 2t + 2c_2 \cos 2t
\]

It follows that
\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{bmatrix} = \begin{bmatrix} c_1 \cos 2t \\ -2c_1 \sin 2t \end{bmatrix} + \begin{bmatrix} c_2 \sin 2t \\ 2c_2 \cos 2t \end{bmatrix} = c_1 \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix}
\]
(11)

is a solution of (8). Moreover, this is the general solution, since (10) is the general solution of (9).

REMARK 1: In general, in order to solve the \( 2 \times 2 \) system
\[
\begin{align*}
x_1' &= ax_1 + bx_2 \\
x_2' &= cx_1 + dx_2
\end{align*}
\]
where \( b, c \) are not zero, we either solve the first equation for \( x_2 \) and plug it into the second equation or solve the second equation for \( x_1 \) and plug it into the first equation.

REMARK 2: Another way to solve (8) will be discussed in Section 3.9.
Solution 2: We solve the second equation of (8) for $x_1$

\[ x'_2 = -4x_1 \implies x_1 = -\frac{1}{4}x'_2 \] (12)

and plug it into the first equation of (8):

\[ x'_1 = x_2 \]
\[ \left( -\frac{1}{4}x'_2 \right)' = x_2 \]
\[ -\frac{1}{4}(x'_2)' = x_2 \]
\[ x''_2 = -4x_2 \]
\[ x''_2 + 4x_2 = 0 \] (13)

therefore (see Solution 1)

\[ x_2(t) = c_3 \cos 2t + c_4 \sin 2t \] (14)

is the general solution of (13). Plugging in this into (12), we get

\[
\begin{align*}
x_1(t) &= -\frac{1}{4}x'_2 = -\frac{1}{4}(c_3 \cos 2t + c_4 \sin 2t)' = -\frac{1}{4}(c_3(\cos 2t)' + c_4(\sin 2t)') \\
&= -\frac{1}{4}(-2c_3 \sin 2t + 2c_4 \cos 2t) \\
&= \frac{1}{2}c_3 \sin 2t - \frac{1}{2}c_4 \cos 2t
\end{align*}
\]

It follows that

\[
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c_3 \sin 2t - \frac{1}{2}c_4 \cos 2t \\ c_3 \cos 2t + c_4 \sin 2t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c_3 \sin 2t \\ c_3 \cos 2t \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}c_4 \cos 2t \\ c_4 \sin 2t \end{bmatrix} = c_3 \begin{bmatrix} \frac{1}{2} \sin 2t \\ \cos 2t \end{bmatrix} + c_4 \begin{bmatrix} -\frac{1}{2} \cos 2t \\ \sin 2t \end{bmatrix} = \frac{1}{2}c_3 \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix} - \frac{1}{2}c_4 \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix}
\]

(15)

is a solution of (8). Moreover, this is the general solution, since (14) is the general solution of (13).

REMARK: Put

\[ \tilde{c}_1 = -\frac{1}{2}c_4 \quad \text{and} \quad \tilde{c}_2 = \frac{1}{2}c_3 \]

then (15) becomes

\[
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \tilde{c}_1 \begin{bmatrix} \cos 2t \\ -2 \sin 2t \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} \sin 2t \\ 2 \cos 2t \end{bmatrix}
\]

so (15) can be rewritten as (11).
EXAMPLE: Solve the system

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 \\
\frac{dx_2}{dt} &= 2x_1 + x_2
\end{align*}
\]  

(16)

Solution 1: We solve the second equation of (16) for \( x_1 \)

\[
x_2' = 2x_1 + x_2 \implies x_2' - x_2 = 2x_1 \implies x_1 = \frac{1}{2}(x_2' - x_2)
\]  

(17)

and plug it into the first equation of (16):

\[
x_1' = x_1
\]

\[
\left(\frac{1}{2}(x_2' - x_2)\right)' = \frac{1}{2}(x_2' - x_2)
\]

\[
\frac{1}{2}(x_2' - x_2)' = \frac{1}{2}(x_2' - x_2)
\]

\[
\frac{1}{2}(x_2'' - x_2') = \frac{1}{2}(x_2' - x_2)
\]

\[
x_2'' - x_2' = x_2' - x_2
\]

\[
x_2'' - 2x_2' + x_2 = 0
\]  

(18)

To find two linearly independent solutions of (18) we note that the characteristic equation is

\[r^2 - 2r + 1 = 0 \implies (r - 1)^2 = 0\]

Thus, \( r = 1 \) is a repeated root. Consequently, two linearly independent solutions to differential equation (18) are

\[x_2^{(1)}(t) = e^t \quad \text{and} \quad x_2^{(2)}(t) = te^t\]

hence

\[x_2(t) = c_1x_2^{(1)}(t) + c_2x_2^{(2)}(t) = c_1e^t + c_2te^t\]  

(19)

is the general solution of (18). Plugging in this into (17), we get

\[x_1(t) = \frac{1}{2}(x_2' - x_2) = \frac{1}{2}\left((c_1e^t + c_2te^t)' - (c_1e^t + c_2te^t)\right) = \frac{1}{2}\left(c_1(e^t)' + c_2(te^t)' - c_1e^t - c_2te^t\right)\]

\[= \frac{1}{2}\left(c_1e^t + c_2(te^t + t(e^t)') - c_1e^t - c_2te^t\right)\]

\[= \frac{1}{2}\left(c_1e^t + c_2(e^t + te^t) - c_1e^t - c_2te^t\right)\]

\[= \frac{1}{2}\left(c_1e^t + c_2e^t + c_2te^t - c_1e^t - c_2te^t\right)\]

\[= \frac{1}{2}c_2e^t\]

It follows that

\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_2e^t/2 \\ c_1e^t + c_2te^t \end{bmatrix} = \begin{bmatrix} c_10 + c_2e^t/2 \\ c_1e^t + c_2te^t \end{bmatrix} = \begin{bmatrix} c_10 \\ c_1e^t \end{bmatrix} + \begin{bmatrix} c_2e^t/2 \\ c_2te^t \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} e^t/2 \\ te^t \end{bmatrix}
\]

is a solution of (16). Moreover, this is the general solution, since (19) is the general solution of (18).
Solution 2: We solve the first equation of system (16). We have
\[
\frac{dx_1}{dt} = x_1 \\
\frac{dx_1}{x_1} = dt \\
\int \frac{dx_1}{x_1} dt = \int dt \\
\ln |x_1| = t + c \\
e^{\ln |x_1|} = e^{t+c} \\
|x_1| = e^{t+c} = e^c e^t
\]
so
\[
x_1 = \pm e^c e^t
\]
We can easily verify that the function \(x_1 = 0\) is also a solution of \(\frac{dx_1}{dt} = x_1\). So we can write the general solution in the form
\[
x_1 = c_3 e^t
\]
where \(c_3\) is an arbitrary constant (\(c_3 = e^c\), or \(c_3 = -e^c\), or \(c_3 = 0\)). We now substitute \(x_1 = c_3 e^t\) into the second equation of system (16):
\[
x'_2 = 2x_1 + x_2 \\
x'_2 = 2c_3 e^t + x_2 \\
x'_2 - x_2 = 2c_3 e^t
\]
This is a first-order linear differential equation (see Section 1.2). Here \(a(t) = -1\) so that
\[
\mu(t) = \exp \left( \int a(t) dt \right) = \exp \left( - \int dt \right) = e^{-t}
\]
Multiplying both sides of the equation \(x'_2 - x_2 = 2c_3 e^t\) by \(\mu(t)\) we obtain the equivalent equation
\[
e^{-t} (x'_2 - x_2) = 2c_3 \quad \text{or} \quad \frac{d}{dt} (e^{-t} x_2) = 2c_3
\]
Hence
\[
e^{-t} x_2 = \int 2c_3 dt = 2c_3 t + c_4
\]
so
\[
x_2 = (2c_3 t + c_4) e^t
\]
Therefore
\[
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_3 e^t \\ (2c_3 t + c_4) e^t \end{bmatrix} = \begin{bmatrix} c_3 e^t + c_4 0 \\ 2c_3 t e^t + c_4 e^t \end{bmatrix} = c_3 \begin{bmatrix} e^t \\ 2te^t \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ e^t \end{bmatrix} = 2c_3 \begin{bmatrix} e^t/2 \\ te^t \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ e^t \end{bmatrix}
\]
REMARK: Another way to solve (16) will be discussed in Section 3.10.
EXAMPLE: Solve the system
\[
\begin{align*}
\begin{cases} 
  x'_1 &= 7x_1 + 4x_2 \\
  x'_2 &= -3x_1 - x_2 
\end{cases}
\end{align*}
\tag{20}
\]

Solution 1: We solve the first equation of (20) for \( x \)

is a solution of (20). Moreover, this is the general solution, since (23) is the general solution of (22).

\[ x'_1 = 7x_1 + 4x_2 \implies x'_1 - 7x_1 = 4x_2 \implies x_2 = \frac{1}{4}(x'_1 - 7x_1) \tag{21} \]

and plug it into the second equation of (20):

\[
\begin{align*}
  x'_2 &= -3x_1 - x_2 \\
  \left( \frac{1}{4}(x'_1 - 7x_1) \right)' &= -3x_1 - \frac{1}{4}(x'_1 - 7x_1) \\
  \frac{1}{4}(x'_1 - 7x_1)' &= -3x_1 - \frac{1}{4}(x'_1 - 7x_1) \\
  (x'_1 - 7x_1)' &= -12x_1 - x'_1 + 7x_1 \\
  x''_1 - 7x'_1 &= -12x_1 - x'_1 + 7x_1 \\
  x''_1 - 6x'_1 + 5x_1 &= 0 \tag{22}
\end{align*}
\]

To find two linearly independent solutions of (22) we note that the characteristic equation is

\[ r^2 - 6r + 5 = 0 \implies (r - 1)(r - 5) = 0 \]

with the roots \( r_1 = 1 \) and \( r_2 = 5 \). Consequently, two linearly independent solutions to differential equation (22) are

\[ x^{(1)}_1(t) = e^t \quad \text{and} \quad x^{(2)}_1(t) = e^{5t} \]

hence

\[ x_1(t) = c_1 x^{(1)}_1(t) + c_2 x^{(2)}_1(t) = c_1 e^t + c_2 e^{5t} \tag{23} \]

is the general solution of (22). Plugging this into (21), we get

\[
\begin{align*}
x_2(t) &= \frac{1}{4}(x'_1 - 7x_1) = \frac{1}{4} \left( \left( c_1 e^t + c_2 e^{5t} \right)' - 7 \left( c_1 e^t + c_2 e^{5t} \right) \right) \\
&= \frac{1}{4} \left( c_1 (e^t)' + c_2 (e^{5t})' - 7c_1 e^t - 7c_2 e^{5t} \right) \\
&= \frac{1}{4} \left( c_1 e^t + c_2 e^{5t} 5t)' - 7c_1 e^t - 7c_2 e^{5t} \right) \\
&= \frac{1}{4} \left( c_1 e^t + 5c_2 e^{5t} - 7c_1 e^t - 7c_2 e^{5t} \right) \\
&= \frac{1}{4} \left( -6c_1 e^t - 2c_2 e^{5t} \right) \\
&= -\frac{3}{2}c_1 e^t - \frac{1}{2}c_2 e^{5t}
\end{align*}
\]

It follows that

\[
\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{5t} \\ -3c_1 e^t/2 - c_2 e^{5t}/2 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ -3c_1 e^t/2 \end{bmatrix} + \begin{bmatrix} c_2 e^{5t} \\ -c_2 e^{5t}/2 \end{bmatrix} = c_1 \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix} + c_2 \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}
\]

is a solution of (20). Moreover, this is the general solution, since (23) is the general solution of (22).
Solution 2: We solve the second equation of (20) for $x_1$

$$x'_2 = -3x_1 - x_2 \implies x'_2 + x_2 = -3x_1 \implies x_1 = -\frac{1}{3} (x'_2 + x_2) \quad (24)$$

and plug it into the first equation of (20):

$$x'_1 = 7x_1 + 4x_2$$

$$\left(-\frac{1}{3} (x'_2 + x_2)\right)' = -\frac{7}{3} (x'_2 + x_2) + 4x_2$$

$$-\frac{1}{3} (x'_2 + x_2)' = -\frac{7}{3} (x'_2 + x_2) + 4x_2$$

$$-\frac{1}{3} x'_2 = 7 \left( x'_2 + x_2 \right) - 12x_2$$

$$x''_2 + x'_2 = 7x'_2 + 7x_2 - 12x_2$$

$$x'' - 6x'_2 + 5x_2 = 0 \quad (25)$$

therefore (see Solution 1)

$$x_2(t) = c_3 e^t + c_4 e^{5t} \quad (26)$$

is the general solution of (25). Plugging in this into (24), we get

$$x_1(t) = -\frac{1}{3} (x'_2 + x_2) = -\frac{1}{3} \left( \left( c_3 e^t + c_4 e^{5t} \right)' + (c_3 e^t + c_4 e^{5t}) \right)$$

$$= -\frac{1}{3} \left( c_3 (e^t)' + c_4 (e^{5t})' + c_3 e^t + c_4 e^{5t} \right)$$

$$= -\frac{1}{3} \left( c_3 e^t + c_4 e^{5t} (5t)' + c_3 e^t + c_4 e^{5t} \right)$$

$$= -\frac{1}{3} \left( c_3 e^t + 5c_4 e^{5t} + c_3 e^t + c_4 e^{5t} \right)$$

$$= -\frac{1}{3} \left( 2c_3 e^t + 6c_4 e^{5t} \right)$$

$$= -\frac{2}{3} c_3 e^t - 2c_4 e^{5t}$$

It follows that

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -2c_3 e^t/3 - 2c_4 e^{5t} \\ c_3 e^t + c_4 e^{5t} \end{bmatrix}$$

$$= \begin{bmatrix} -2c_3 e^t/3 \\ c_3 e^t \end{bmatrix} + \begin{bmatrix} -2c_4 e^{5t} \\ c_4 e^{5t} \end{bmatrix}$$

$$= c_3 \begin{bmatrix} -2e^t/3 \\ e^t \end{bmatrix} + c_4 \begin{bmatrix} -2e^{5t} \\ e^{5t} \end{bmatrix} = -\frac{2}{3} c_3 \begin{bmatrix} e^t \\ -3e^t/2 \end{bmatrix} - 2c_4 \begin{bmatrix} e^{5t} \\ -e^{5t}/2 \end{bmatrix}$$

is a solution of (20). Moreover, this is the general solution, since (26) is the general solution of (25).

REMARK: Another way to solve (20) will be discussed in Section 3.8.