Dimension of a Vector Space

DEFINITION: The vector \( y \) defined by
\[
y = c_1 v_1 + \ldots + c_p v_p
\]
where \( v_1, \ldots, v_p \) are vectors from a vector space \( V \) and \( c_1, \ldots, c_p \) are scalars, is called a linear combination of vectors \( v_1, \ldots, v_p \).

DEFINITION: The set of all combinations of vectors \( v_1, \ldots, v_p \) from a vector space \( V \) is denoted by \( \text{Span}\{v_1, \ldots, v_p\} \) and is called the subset of \( V \) spanned (or generated) by \( v_1, \ldots, v_p \).

EXAMPLE 1: The vectors
\[
e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
span \( \mathbb{R}^2 \), since any vector
\[
u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]
from \( \mathbb{R}^2 \) can be written as a linear combination of \( e_1, e_2 \):
\[
u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 + 0 \cdot c_2 \\ 0 \cdot c_1 + 1 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 \\ 0 \cdot c_1 \end{bmatrix} + \begin{bmatrix} 0 \cdot c_2 \\ 1 \cdot c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 e_1 + c_2 e_2
\]

REMARK: In the same way one can show that the vectors
\[
e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
\]
span \( \mathbb{R}^n \).

EXAMPLE 2: The vectors
\[
1, \quad t, \quad t^2, \quad \ldots, \quad t^n
\]
span \( \mathbb{P}_n \) (the vector space of all polynomials of degree at most \( n \)).

EXAMPLE 3: Let \( V \) be the set of all solutions of the differential equation
\[
\frac{d^2x}{dt^2} - x = 0
\]  
(1)

Let \( x_1(t) \) be the function whose value at any time \( t \) is \( e^t \) and let \( x_2(t) \) be the function whose value at any time \( t \) is \( e^{-t} \). The functions \( x_1(t) \) and \( x_2(t) \) are in \( V \) since they satisfy the differential equation. Moreover, these functions also span \( V \) since every solution \( x(t) \) of differential equation (1) can be written in the form
\[
x(t) = c_1 e^t + c_2 e^{-t}
\]
so that
\[
x(t) = c_1 x_1(t) + c_2 x_2(t)
\]
DEFINITION: Vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) are said to be linearly \textbf{dependent} if there exist scalars \( c_1, \ldots, c_p \), not all zero, such that
\[
c_1 \mathbf{v}_1 + \ldots + c_p \mathbf{v}_p = \mathbf{0}
\]
Vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) are said to be linearly \textbf{independent} if the vector equation
\[
c_1 \mathbf{v}_1 + \ldots + c_p \mathbf{v}_p = \mathbf{0}
\]
has only the trivial solution.

THEOREM 1: Vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) are linearly dependent if and only if at least one of these vectors is a linear combination of the others. (see Appendix I)

COROLLARY: Two vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly dependent if and only if one of these vectors is a constant multiple of the other. (see Appendix II, Part I)

REMARK: Any set of vectors that contains the zero vector is linearly dependent. (Appendix II, Part II)

EXAMPLE 4: Show that the vectors
\[
\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}
\]
are linearly dependent. Then show that the vectors
\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}
\]
are linearly independent.

Solution: To show that the vectors \( \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \) are linearly dependent, we find \( c_1, c_2, c_3 \), not all zero, such that
\[
c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 = \mathbf{0}
\]
that is,
\[
c_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} c_1 - 2c_2 + 3c_3 \\ c_1 - 2c_2 + 5c_3 \\ 2c_1 - 4c_2 + 6c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
This can be rewritten as the following system of equations:
\[
\begin{align*}
&c_1 - 2c_2 + 3c_3 = 0 \\
c_1 - 2c_2 + 5c_3 = 0 &\implies c_1 - 2c_2 + 3c_3 = 0 \\
2c_1 - 4c_2 + 6c_3 = 0 &\implies c_2 + 2c_3 = 0
\end{align*}
\]
For example, if \( c_3 = -1 \), then \( c_1 = 7 \) and \( c_2 = 2 \), that is, \( 7\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3 = \mathbf{0} \).

We now show that
\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0}
\]
has only the trivial solution (which means that the vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are linearly independent). We have
\[
\begin{align*}
&c_1 - 2c_2 + 3c_3 = 0 \\
c_1 - 2c_2 + 5c_3 = 0 &\implies c_2 + 2c_3 = 0 \\
2c_1 - 4c_2 + 7c_3 = 0 &\implies c_3 = 0
\end{align*}
\]

REMARK: In Section 3.7 we will show how to prove linear independence or dependence of the vectors above in a different way (see Appendix IV).
EXAMPLE 5: The vectors (polynomials)

\[ 1 - 3t + 5t^2, \quad -3 + 5t - 7t^2, \quad -4 + 5t - 6t^2, \quad 1 - t^2 \]

are linearly dependent. In order to show it (see also Appendix V), we find \(c_1, c_2, c_3, c_4\), not all zero, such that

\[ c_1(1 - 3t + 5t^2) + c_2(-3 + 5t - 7t^2) + c_3(-4 + 5t - 6t^2) + c_4(1 - t^2) = 0 \]

We have

\[
\begin{align*}
    c_1 - 3c_1 t + 5c_1 t^2 - 3c_2 + 5c_2 t - 7c_2 t^2 - 4c_3 + 5c_3 t - 6c_3 t^2 + c_4 - c_4 t^2 &= 0 \\
    c_1 - 3c_2 - 4c_3 + c_4 - 3c_1 t + 5c_2 t + 5c_3 t + 5c_1 t^2 - 7c_2 t^2 - 6c_3 t^2 - c_4 t^2 &= 0 \\
    (c_1 - 3c_2 - 4c_3 + c_4) + (-3c_1 + 5c_2 + 5c_3) t + (5c_1 - 7c_2 - 6c_3 - c_4)t^2 &= 0
\end{align*}
\]

Note that a polynomial is identically equal to zero if, and only if, all its coefficients are equal to zero:

\[
\begin{align*}
    \begin{cases}
        c_1 - 3c_2 - 4c_3 + c_4 = 0 \\
        -3c_1 + 5c_2 + 5c_3 = 0 \\
        5c_1 - 7c_2 - 6c_3 - c_4 = 0
    \end{cases} \quad \Rightarrow \quad \begin{cases}
        c_1 - 3c_2 - 4c_3 + c_4 = 0 \\
        -4c_2 - 7c_3 + 3c_4 = 0 \\
        8c_2 + 14c_3 - 6c_4 = 0
    \end{cases} \quad \Rightarrow \quad \begin{cases}
        c_1 - 3c_2 - 4c_3 + c_4 = 0 \\
        4c_2 + 7c_3 - 3c_4 = 0
    \end{cases}
\end{align*}
\]

so

\[
\begin{align*}
    &\begin{cases}
        c_1 = 3c_2 + 4c_3 - c_4 \\
        c_2 = \frac{7}{4}c_3 + \frac{3}{4}c_4
    \end{cases} \\
    \Rightarrow \quad &\begin{cases}
        c_2 = 3 \\
        c_1 = 5
    \end{cases}
\end{align*}
\]

Plugging in any two nonzero values for \(c_3\) and \(c_4\) into the second equation, we get the corresponding value of \(c_2\). Similarly, plugging in these values into the first equation, we get the corresponding value of \(c_1\). Since \(c_3\) and \(c_4\) are nonzero, the equation

\[ c_1(1 - 3t + 5t^2) + c_2(-3 + 5t - 7t^2) + c_3(-4 + 5t - 6t^2) + c_4(1 - t^2) = 0 \]

has a nontrivial solution. Therefore the vectors

\[ 1 - 3t + 5t^2, \quad -3 + 5t - 7t^2, \quad -4 + 5t - 6t^2, \quad 1 - t^2 \]

are linearly dependent. For example, setting \(c_4 = 4, \ c_3 = 0\) in the last system, we get

\[ c_2 = 3 \quad \text{and} \quad c_1 = 5 \]

therefore

\[ 5 \cdot (1 - 3t + 5t^2) + 3 \cdot (-3 + 5t - 7t^2) + 0 \cdot (4 + 5t - 6t^2) + 4 \cdot (1 - t^2) = 0 \]

This, in particular, implies

\[
\begin{align*}
    1 - 3t + 5t^2 &= \frac{3}{5}(-3 + 5t - 7t^2) - \frac{4}{5}(1 - t^2) \\
    -3 + 5t - 7t^2 &= -\frac{5}{3}(1 - 3t + 5t^2) - \frac{4}{3}(1 - t^2) \\
    1 - t^2 &= -\frac{5}{4}(1 - 3t + 5t^2) - \frac{3}{4}(-3 + 5t - 7t^2)
\end{align*}
\]

One can check that the polynomial \(4 + 5t - 6t^2\) can’t be written as a linear combination of the others.
EXAMPLE 6: The vectors (matrices)
\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\]
are linearly dependent. In order to show it (see also Appendix VI), we find \(c_1, c_2, c_3, c_4\), not all zero, such that
\[
c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
which gives
\[
\begin{bmatrix} c_1 & 0 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} c_2 & c_2 \\ c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_3 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_4 \\ c_4 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
hence
\[
\begin{bmatrix} c_1 + c_2 + 0 + 0 & 0 + c_2 + c_3 + c_4 \\ 0 + c_2 + c_3 + c_4 & c_1 + 0 + 0 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} c_1 + c_2 \\ c_2 + c_3 + c_4 \\ c_1 + c_4 \\ c_2 + c_3 + c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
therefore
\[
\begin{cases}
  c_1 + c_2 = 0 \\
  c_2 + c_3 + c_4 = 0 \\
  c_2 + c_3 + c_4 = 0 \\
  c_1 + c_4 = 0
\end{cases}
\quad \Rightarrow \quad \begin{cases}
  c_1 + c_2 = 0 \\
  c_2 + c_3 + c_4 = 0 \\
  c_1 + c_4 = 0 \\
  -c_2 + c_4 = 0
\end{cases}
\quad \Rightarrow \quad \begin{cases}
  c_1 + c_2 = 0 \\
  c_2 + c_3 + c_4 = 0 \\
  c_3 + 2c_4 = 0
\end{cases}
\]
so
\[
\begin{cases}
  c_1 = -c_2 \\
  c_2 = -c_3 - c_4 \\
  c_4 = -2c_4
\end{cases}
\]
Plugging in any nonzero value for \(c_4\) into the third equation, we get the corresponding value of \(c_3\). Similarly, plugging in these values into the second equation, we get the corresponding value of \(c_2\). Finally, plugging in these values into the first equation, we get the corresponding value of \(c_1\). Since \(c_4\) is nonzero, the equation
\[
c_1m_1 + c_2m_2 + c_3m_3 + c_4m_4 = 0
\]
has a nontrivial solution. Therefore the vectors \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\) are linearly dependent. For example, setting \(c_4 = 1\), we get \(c_3 = -2\), \(c_2 = 1\), \(c_1 = -1\), therefore
\[
(-1) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
This, in particular, implies
\[
\begin{align*}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} &= 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} &= 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} &= -\frac{1}{2} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} &= 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{align*}
\]
REMARC: In Section 3.7 we will show how to prove linear dependence of the vectors from Examples 5, 6 in a different way (see Appendix VII).
EXAMPLE 7: The vectors
\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \]
are linearly independent (see Appendix VIII).

EXAMPLE 8: The vectors
\[ 1, \quad t, \quad t^2, \quad \ldots, \quad t^n \]
are linearly independent, since the expression
\[ c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n \]
is equal to zero for any \( t \) if and only if \( c_0 = c_1 = c_2 = \ldots = c_n = 0 \).

DEFINITION: The \textbf{dimension} of a vector space \( V \), denoted by \( \dim V \), is the fewest number of linearly independent vectors which span \( V \). \( V \) is said to be a \textbf{finite dimensional} space if its dimension is finite. On the other hand, \( V \) is said to be an \textbf{infinite dimensional} space if no set of finitely many elements span \( V \).

THEOREM 2: If \( n \) linearly independent vectors span \( V \), then \( \dim V = n \).

EXAMPLE 9: The dimension of \( \mathbb{R}^n \) is \( n \), since the vectors
\[ e_1, \quad e_2, \quad \ldots, \quad e_n \]
span \( \mathbb{R}^n \) and are linearly independent. The dimension of \( \mathbb{R} \) is one, since the number 1 spans \( \mathbb{R} \) and \( \{1\} \) is linearly independent.

EXAMPLE 10: The dimension of \( V = \{0\} \) is zero, since the set \( \{0\} \) is linearly dependent and therefore \( V \) does not have a basis.

EXAMPLE 11: The dimension of \( \mathbb{P}_n \) is \( n + 1 \), since the vectors
\[ 1, \quad t, \quad t^2, \quad \ldots, \quad t^n \]
span \( \mathbb{P}_n \) and are linearly independent. The dimension of \( \mathbb{P} \) is \( \infty \).

EXAMPLE 12: The dimension of the vector space of all \( 3 \times 2 \) matrices is 6 (see Appendix IX). In general, the dimension of the vector space of all \( n \times m \) matrices is \( nm \).

EXAMPLE 13: Let \( u_1, u_2, u_3 \) and \( v_1, v_2, v_3 \) be the same as in Example 4. Then
\[ \dim (\text{Span} \{v_1, v_2, v_3\}) = 3 \]
\[ \dim (\text{Span} \{v_1, v_2\}) = 2 \]
\[ \dim (\text{Span} \{v_1\}) = 1 \]
since \( v_1, v_2, v_3 \) are linearly independent. However, \( \dim (\text{Span} \{ u_1, u_2, u_3 \}) = 2 \), since \( u_1, u_2, u_3 \) are linearly dependent (because they are not multiples of each other) and \( \text{Span} \{ u_1, u_2, u_3 \} = \text{Span} \{ u_1, u_2 \} \) (because \( u_3 \) is a linear combination of \( u_1 \) and \( u_2 \)).

**EXAMPLE 14:** The dimension of the vector space of all solutions of the differential equation \( \frac{d^2 x}{dt^2} - x = 0 \) is 2, since every solution \( x(t) \) of the differential equation can be written in the form

\[
x(t) = c_1 e^t + c_2 e^{-t}
\]

and the functions \( e^t \) and \( e^{-t} \) are linearly independent (they are not constant multiples of each other).

**DEFINITION:** If a set of linearly independent vectors span a vector space \( V \), then this set of vectors is said to be a **basis** for \( V \). A basis may also be called a **coordinate system**.

**EXAMPLE 15:** The vectors (functions) \( e^t \) and \( e^{-t} \) form a basis of the vector space of all solutions of the differential equation \( \frac{d^2 x}{dt^2} - x = 0 \), since they span this vector space and are linearly independent.

**EXAMPLE 16:** The number 1 is a basis for \( \mathbb{R} \). The vectors

\[
e_1, \ e_2, \ldots , \ e_n
\]

are a basis (so-called **standard basis**) for \( \mathbb{R}^n \). If

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]

then

\[
x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n
\]

and relative to this basis the \( x_i \) are called “components” or “coordinates.”

Similarly, the vectors

\[
1, \ t, \ t^2, \ldots , \ t^n
\]

are a so-called **standard basis** for \( \mathbb{P}_n \).

**COROLLARY:** In a finite dimensional vector space, each basis has the same number of vectors, and this number is the dimension of the space.

**THEOREM 3:** Any \( n \) linearly independent vectors in an \( n \) dimensional space \( V \) must also span \( V \). That is to say, any \( n \) linearly independent vectors in an \( n \) dimensional space \( V \) are a basis for \( V \).

**EXAMPLE 17:** Let \( v_1, v_2, v_3 \) be the same as in Example 4. By Theorem 3 above, these vectors form a basis for \( \mathbb{R}^3 \), since these three vectors are linearly independent and the dimension of \( \mathbb{R}^3 \) is also three.
Appendix I

Here we show that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ are linearly dependent if and only if at least one of these vectors is a linear combination of the others.

Suppose vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ are linearly dependent. We show that at least one of these vectors is a linear combination of the others. In fact, by definition of dependence, since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ are linearly dependent, there exist scalars $c_1, c_2, c_3, \ldots, c_p$, not all zero, such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \ldots + c_p \mathbf{v}_p = \mathbf{0}$$

If, for example, $c_1 \neq 0$, then

$$c_1 \mathbf{v}_1 = -c_2 \mathbf{v}_2 - c_3 \mathbf{v}_3 - \ldots - c_p \mathbf{v}_p$$

$$\mathbf{v}_1 = -\frac{c_2}{c_1} \mathbf{v}_2 - \frac{c_3}{c_1} \mathbf{v}_3 - \ldots - \frac{c_p}{c_1} \mathbf{v}_p$$

$$= \tilde{c}_2 \mathbf{v}_2 + \tilde{c}_3 \mathbf{v}_3 + \ldots + \tilde{c}_p \mathbf{v}_p$$

where

$$\tilde{c}_2 = -\frac{c_2}{c_1}, \quad \tilde{c}_3 = -\frac{c_3}{c_1}, \ldots, \quad \tilde{c}_p = -\frac{c_p}{c_1}$$

So, $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$.

Similarly, if $c_2 \neq 0$, then

$$c_2 \mathbf{v}_2 = -c_1 \mathbf{v}_1 - c_3 \mathbf{v}_3 - \ldots - c_p \mathbf{v}_p$$

$$\mathbf{v}_2 = -\frac{c_1}{c_2} \mathbf{v}_1 - \frac{c_3}{c_2} \mathbf{v}_3 - \ldots - \frac{c_p}{c_2} \mathbf{v}_p$$

$$= \tilde{c}_1 \mathbf{v}_1 + \tilde{c}_3 \mathbf{v}_3 + \ldots + \tilde{c}_p \mathbf{v}_p$$

where

$$\tilde{c}_1 = -\frac{c_1}{c_2}, \quad \tilde{c}_3 = -\frac{c_3}{c_2}, \ldots, \quad \tilde{c}_p = -\frac{c_p}{c_2}$$

So, $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3, \ldots, \mathbf{v}_p$. In the same way we prove that if any of the remaining numbers $c_3, \ldots, c_p$ is $\neq 0$, then the corresponding vector is a linear combination of the others.

Now suppose at least one of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ is a linear combination of the others. We show that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ are linearly dependent. In fact, if, for example, $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$, then

$$\mathbf{v}_1 = c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + \ldots + c_p \mathbf{v}_p$$

therefore

$$\mathbf{v}_1 - c_2 \mathbf{v}_2 - c_3 \mathbf{v}_3 - \ldots - c_p \mathbf{v}_p = \mathbf{0}$$

$$1 \cdot \mathbf{v}_1 + (-c_2) \mathbf{v}_2 + (-c_3) \mathbf{v}_3 + \ldots + (-c_p) \mathbf{v}_p = \mathbf{0}$$

So, vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ are linearly dependent, since

$$\tilde{c}_1 \mathbf{v}_1 + \tilde{c}_2 \mathbf{v}_2 + \tilde{c}_3 \mathbf{v}_3 + \ldots + \tilde{c}_p \mathbf{v}_p = \mathbf{0}$$

where

$$\tilde{c}_1 = 1, \quad \tilde{c}_2 = -c_2, \quad \tilde{c}_3 = -c_3, \ldots, \quad \tilde{c}_p = -c_p$$

and since $\tilde{c}_1 = 1 \neq 0$, not all scalars $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \ldots, \tilde{c}_p$ are equal to zero. In the same way we prove that if any of the remaining vectors $\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ is a linear combination of the others, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p$ are linearly dependent by definition of dependence.
Appendix II

PART I: Here we show that two vectors $v_1$ and $v_2$ are linearly dependent if and only if one of these vectors is a constant multiple of the other.

In fact, by Theorem 1, vectors $v_1, v_2, v_3, \ldots, v_p$ are linearly dependent if and only if at least one of these vectors is a linear combination of the others. That is, either

$$v_1 = c_2v_2 + c_3v_3 + \ldots + c_pv_p$$

or

$$v_2 = c_1v_1 + c_3v_3 + \ldots + c_pv_p$$

or

$$v_3 = c_1v_1 + c_2v_2 + \ldots + c_pv_p$$

etc. In particular, vectors $v_1$ and $v_2$ are linearly dependent if and only if either

$$v_1 = c_2v_2$$

or

$$v_2 = c_1v_1$$

This means one of the vectors $v_1, v_2$ is a constant multiple of the other.

PART II: Here we show that any set of vectors that contains the zero vector is linearly dependent.

Consider the following set of vectors $0, v_1, v_2, \ldots, v_p$. We have

$$1 \cdot 0 + 0 \cdot v_1 + 0 \cdot v_2 + \ldots + 0 \cdot v_p = 0$$

hence

$$c_00 + c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$$

where

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = 0, \ldots, \quad c_p = 0$$

and since $c_0 = 1 \neq 0$, not all scalars $c_0, c_1, c_2, \ldots, c_p$ are equal to zero. Therefore vectors $0, v_1, v_2, \ldots, v_p$ are linearly dependent by definition of dependence.

REMARK: In particular, the set $\{0\}$ is linearly dependent.
Appendix III

We already showed that the vectors

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

span \( \mathbb{R}^2 \), since any vector \( u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \) from \( \mathbb{R}^2 \) can be written as a linear combination of \( e_1, e_2 \):

\[
    u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 + 0 \cdot c_2 \\ 0 \cdot c_1 + 1 \cdot c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 \\ 0 \cdot c_1 \end{bmatrix} + \begin{bmatrix} 0 \cdot c_2 \\ 1 \cdot c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_1 e_1 + c_2 e_2
\]

Note that the vectors

\[ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

also span \( \mathbb{R}^2 \).

**Reason 1:** Since for any \( u = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \) from \( \mathbb{R}^2 \) we have \( u = c_1 e_1 + c_2 e_2 \), it follows that

\[
    u = c_1 e_1 + c_2 e_2 + 0 \cdot x
\]

so any \( u \) can be written as a linear combination of \( e_1, e_2 \) and \( x \). So, the set of all linear combinations of \( e_1, e_2, x \) (that is, span of \( e_1, e_2, x \)) can't be smaller than \( \mathbb{R}^2 \). On the other hand, the set of all linear combinations of \( e_1, e_2, x \) can't be bigger than \( \mathbb{R}^2 \), since \( \mathbb{R}^2 \) is a vector space and therefore closed under addition and multiplication by a scalar.

**Reason 2:** Note that vectors \( e_1, e_2 \) and \( x \) are linearly dependent, since one of them is a linear combination of the others. For example,

\[
    x = 1 \cdot e_1 + 1 \cdot e_2
\]

because

\[
    \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Therefore any linear combination of \( e_1, e_2, x \) can be written as a linear combination of \( e_1, e_2 \), since

\[
    c_1 e_1 + c_2 e_2 + c_3 x = c_1 e_1 + c_2 e_2 + c_3 (1 \cdot e_1 + 1 \cdot e_2)
    = c_1 e_1 + c_2 e_2 + c_3 e_1 + c_3 e_2
    = (c_1 + c_3) e_1 + (c_2 + c_3) e_2
    = \tilde{c}_1 e_1 + \tilde{c}_2 e_2
\]

where

\[ \tilde{c}_1 = c_1 + c_3 \quad \text{and} \quad \tilde{c}_2 = c_2 + c_3 \]

From this it is easy to conclude that the set of all linear combinations of \( e_1, e_2, x \) (that is, span of \( e_1, e_2, x \)) is equal to the set of all linear combinations of \( e_1, e_2 \) (that is, span of \( e_1, e_2 \)).

**Remark:** In general, if vectors \( v_1, \ldots, v_p \) span a vector space \( V \) and are linearly dependent, then one can remove at least one vector from the set \( v_1, \ldots, v_p \) and the resulting smaller set of vectors will span \( V \) again. On the other hand, if vectors \( v_1, \ldots, v_p \) span a vector space \( V \) and are linearly independent, then any smaller subset of these vectors does not span \( V \) any more (see Theorem 2).

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Appendix IV

Let
\[ u_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \]
and
\[ v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \]

Here we give two other ways of proving that \( u_1, u_2, u_3 \) are linearly dependent and \( v_1, v_2, v_3 \) are linearly independent.

Solution 2 (Section 3.7 is required): We have
\[
\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\]
Since the number of pivots (two) is lesser than the number of columns (three), the vectors \( u_1, u_2, u_3 \) are linearly dependent. Similarly,
\[
\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\]
Since the number of pivots (three) is equal to the number of columns (three), the vectors \( v_1, v_2, v_3 \) are linearly independent.

Solution 3 (Sections 3.5, 3.7 are required): We have
\[
\det[\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 6 \end{vmatrix} = 0
\]
Since the determinant is equal to zero, the vectors \( u_1, u_2, u_3 \) are linearly dependent by Lemma 3 from Section 3.7. Similarly,
\[
\det[\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}] = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & 5 \\ 2 & -4 & 7 \end{vmatrix} = 1
\]
Since the determinant is not equal to zero, the vectors \( v_1, v_2, v_3 \) are linearly independent by Lemma 3 from Section 3.7.
Appendix V

Here we provide another way of showing that the vectors (polynomials)

\[ 1 - 3t + 5t^2, \ -3 + 5t - 7t^2, \ -4 + 5t - 6t^2, \ 1 - t^2 \]

are linearly dependent. Indeed, consider the coefficients of each polynomial as the coordinates of a vector in \( \mathbb{R}^3 \):

\[
\mathbf{u}_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}
\]

We find \( c_1, c_2, c_3, c_4, \) not all zero, such that

\[ c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 = 0 \]

that is,

\[
c_1 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix} + c_3 \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \iff \quad \begin{bmatrix} c_1 - 3c_2 - 4c_3 + c_4 \\ -3c_1 + 5c_2 + 5c_3 \\ 5c_1 - 7c_2 - 6c_3 - c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

This can be rewritten as the following system of equations:

\[
\begin{align*}
c_1 - 3c_2 - 4c_3 + c_4 &= 0 \\
-3c_1 + 5c_2 + 5c_3 &= 0 \\
5c_1 - 7c_2 - 6c_3 - c_4 &= 0
\end{align*}
\]

\[
\Rightarrow \quad \begin{align*}
c_1 - 3c_2 - 4c_3 + c_4 &= 0 \\
-4c_2 - 7c_3 + 3c_4 &= 0 \\
8c_2 + 14c_3 - 6c_4 &= 0
\end{align*}
\]

so

\[
\begin{align*}
c_1 &= 3c_2 + 4c_3 - c_4 \\
c_2 &= \frac{7}{4}c_3 + \frac{3}{4}c_4
\end{align*}
\]

Plugging in any two nonzero values for \( c_3 \) and \( c_4 \) into the second equation, we get the corresponding value of \( c_2 \). Similarly, plugging in these values into the first equation, we get the corresponding value of \( c_1 \). Since \( c_3 \) and \( c_4 \) are nonzero, the equation

\[ c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 + c_4 \mathbf{u}_4 = 0 \]

has a nontrivial solution. Therefore the vectors

\[ 1 - 3t + 5t^2, \ -3 + 5t - 7t^2, \ -4 + 5t - 6t^2, \ 1 - t^2 \]

are linearly dependent. For example, setting \( c_4 = 4, \ c_3 = 0, \) we get \( c_2 = 3, \ c_1 = 5, \) therefore

\[
5 \cdot \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + 3 \cdot \begin{bmatrix} -3 \\ 5 \\ -7 \end{bmatrix} + 0 \cdot \begin{bmatrix} -4 \\ 5 \\ -6 \end{bmatrix} + 4 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

hence

\[ 5 \cdot (1 - 3t + 5t^2) + 3 \cdot (-3 + 5t - 7t^2) + 0 \cdot (4 + 5t - 6t^2) + 4 \cdot (1 - t^2) = 0 \]
Here we provide another way of showing that the vectors (matrices)

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\]

are linearly dependent. Indeed, consider the components of each matrix as the coordinates of a vector in \( \mathbb{R}^4 \):

\[
\begin{align*}
\mathbf{m}_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, & \mathbf{m}_2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, & \mathbf{m}_3 &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, & \mathbf{m}_4 &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}
\end{align*}
\]

We find \( c_1, c_2, c_3, c_4 \), not all zero, such that

\[c_1 \mathbf{m}_1 + c_2 \mathbf{m}_2 + c_3 \mathbf{m}_3 + c_4 \mathbf{m}_4 = 0\]

that is,

\[
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} c_2 + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} c_3 + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} c_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]

This can be rewritten as the following system of equations:

\[
\begin{align*}
c_1 + c_2 &= 0 \\
c_2 + c_3 + c_4 &= 0 \\
c_2 + c_3 + c_4 &= 0 \\
c_1 + c_4 &= 0
\end{align*}
\]

\[
\begin{align*}
c_1 + c_2 &= 0 \\
c_2 + c_3 + c_4 &= 0 \\
c_2 + c_3 + c_4 &= 0 \\
-c_2 + c_4 &= 0
\end{align*}
\]

so

\[
\begin{align*}
c_1 &= -c_2 \\
c_2 &= -c_3 - c_4 \\
c_3 &= -2c_4
\end{align*}
\]

Plugging in any nonzero value for \( c_4 \) into the third equation, we get the corresponding value of \( c_3 \). Similarly, plugging in these values into the second equation, we get the corresponding value of \( c_2 \). Finally, plugging in these values into the first equation, we get the corresponding value of \( c_1 \). Since \( c_4 \) is nonzero, the equation

\[c_1 \mathbf{m}_1 + c_2 \mathbf{m}_2 + c_3 \mathbf{m}_3 + c_4 \mathbf{m}_4 = 0\]

has a nontrivial solution. Therefore the vectors

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\]

are linearly dependent. For example, setting \( c_4 = 1 \), we get \( c_3 = -2, \ c_2 = 1, \ c_1 = -1 \), therefore

\[
(-1) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
Appendix VII

Here we use methods from Section 3.7 to prove that

\[ p_1 = 1 - 3t + 5t^2, \quad p_2 = -3 + 5t - 7t^2, \quad p_3 = -4 + 5t - 6t^2, \quad p_4 = 1 - t^2 \]

and

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{bmatrix}
\]

are linearly independent.

Consider the coefficients of each polynomial as the coordinates of a vector in \( \mathbb{R}^3 \). We have

\[
\begin{bmatrix}
1 & -3 & -4 \\
-3 & 5 & 5 \\
5 & -7 & -6 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & -3 & -4 & 1 \\
0 & -4 & -7 & 3 \\
0 & 8 & 14 & -6 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & -3 & -4 & 1 \\
0 & 4 & 7 & -3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Echelon Form

Since the number of pivots (two) is less than the number of columns (four), the vectors are linearly dependent.

Similarly, consider the components of each matrix as the coordinates of a vector in \( \mathbb{R}^4 \). We have

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & -1 & 0 & 1 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Echelon Form

Since the number of pivots (three) is less than the number of columns (four), the vectors are linearly dependent.

Lastly, we can show that

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

are linearly independent by observing that the determinant

\[
\begin{vmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 \\
\end{vmatrix}
\]

is equal to zero, since it has two equal rows (see Section 3.5). Therefore the vectors are linearly dependent (see Lemma 3 from Section 3.7).
Appendix VIII

Here we prove that the vectors

\[ e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \]

are linearly independent.

Solution: The vectors are linearly independent, since

\[ c_1 e_1 + c_2 e_2 + \ldots + c_n e_n = 0 \]

has only the trivial solution. Indeed, this equation can be rewritten as

\[
\begin{bmatrix}
  c_1 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
+
\begin{bmatrix}
  0 \\
  1 \\
  \vdots \\
  0
\end{bmatrix}
+
\ldots
+
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  1
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

therefore

\[
\begin{bmatrix}
  c_1 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
+
\begin{bmatrix}
  0 \\
  c_2 \\
  \vdots \\
  0
\end{bmatrix}
+
\ldots
+
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  c_n
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

hence

\[
\begin{bmatrix}
  1 \cdot c_1 + 0 \cdot c_2 + \ldots + 0 \cdot c_n \\
  0 \cdot c_1 + 1 \cdot c_2 + \ldots + 0 \cdot c_n \\
  \vdots \\
  0 \cdot c_1 + 0 \cdot c_2 + \ldots + 1 \cdot c_n
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

which gives

\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  \vdots \\
  c_n
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

so, \( c_1 = c_2 = \ldots = c_n = 0 \).
Appendix IX

To show that the dimension of the vector space $V$ of all $3 \times 2$ matrices is 6, consider the following matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We first note that these matrices span $V$, because any $3 \times 2$ matrix can be written as a linear combination of $M_1, M_2, \ldots, M_6$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_{31} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & a_{32} \end{bmatrix}$$

$$= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{31} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{32} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We also note that $M_1, M_2, \ldots, M_6$ are linearly independent. Indeed, suppose

$$c_1M_1 + c_2M_2 + \ldots + c_6M_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for some scalars $c_1, c_2, \ldots, c_6$. Since

$$c_1M_1 + c_2M_2 + \ldots + c_6M_6$$

$$= c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + c_6 \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_5 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & c_6 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

therefore $c_1 = c_2 = \ldots = c_6 = 0$. This means that $M_1, M_2, \ldots, M_6$ are linearly independent. Since 6 linearly independent vectors span $V$, it follows that then $\dim V = 6$ by Theorem 2 above.