NAME: ____________________________________

Calculus I - Spring 2014

Midterm Exam I, March 5, 2014

In all non-multiple choice problems you are required to show all your work and provide the necessary explanations everywhere to get full credit. In all multiple choice problems you don't have to show your work.

1. Find the domain of $f(x) = \sqrt{2-x} + \sqrt{x+5}$.

Ⓐ $(-\infty, -5]$

Ⓑ $(-\infty, 2]$

Ⓒ $[-5, 2]$ $\longleftarrow$ Correct

Ⓓ $[2, 5]$

Ⓔ $[-2, 5]$

Solution: The domain of $\sqrt{2-x}$ is all real numbers such that $2 - x \geq 0$, that is $(-\infty, 2]$. The domain of $\sqrt{x+5}$ is all real numbers such that $x+5 \geq 0$, that is $[-5, \infty)$. Therefore the domain of $f(x) = \sqrt{2-x} + \sqrt{x+5}$ is

$$(-\infty, 2] \cap [-5, \infty)$$

which is $[-5, 2]$.

Info: The average in the class for this problem was 75.6%.

2. The function $f(x) = \sin^2 x$ is

Ⓐ even $\longleftarrow$ Correct

Ⓑ odd

Ⓒ neither even nor odd

Solution: To make the solution clearer, let us write $f(x)$ as $(\sin x)^2$. We have

$$f(-x) = (\sin(-x))^2 = (-\sin x)^2 = (\sin x)^2 = \sin^2 x = f(x)$$

So, $f(-x) = f(x)$, therefore f is an even function.

Info: The average in the class for this problem was 58.5%.

3. Find $\lim_{x\to\infty}(\sqrt{x^2+4x}-x)$.

Ⓐ 0

Ⓑ 1

Ⓒ 2 $\longleftarrow$ Correct

Ⓓ 3

Ⓔ None of the above

Solution: We have

$$\lim_{x\to\infty}(\sqrt{x^2+4x}-x) = \lim_{x\to\infty}\frac{\sqrt{x^2+4x}-x}{1} = \lim_{x\to\infty}\frac{(\sqrt{x^2+4x}-x)(\sqrt{x^2+4x}+x)}{1\cdot(\sqrt{x^2+4x}+x)}$$

$$= \lim_{x\to\infty}\frac{(\sqrt{x^2+4x})^2-x^2}{\sqrt{x^2+4x}+x}$$

$$= \lim_{x\to\infty}\frac{x^2+4x-x^2}{\sqrt{x^2+4x}+x}$$

$$= \lim_{x\to\infty}\frac{4x}{\sqrt{x^2+4x}+x}$$

$$= \lim_{x\to\infty}\frac{\frac{4x}{\sqrt{x^2}}}{\frac{\sqrt{x^2+4x}+x}{\sqrt{x^2}}}$$

$$= \lim_{x\to\infty}\frac{\frac{4x}{x}}{\frac{\sqrt{x^2+4x}}{\sqrt{x^2}}+\frac{x}{\sqrt{x^2}}}$$

$$= \lim_{x\to\infty}\frac{4}{\sqrt{\frac{x^2+4x}{x^2}}+\frac{x}{x}}$$

$$= \lim_{x\to\infty}\frac{4}{\sqrt{\frac{x^2}{x^2}+\frac{4x}{x^2}}+1}$$

$$= \lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{4}{x}}+1}$$

$$= \frac{4}{\sqrt{1+0}+1} = \frac{4}{\sqrt{1}+1} = \frac{4}{1+1} = \frac{4}{2} = 2$$

Info: The average in the class for this problem was 82.9%.

4. Find $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$.

Ⓐ 0

Ⓑ 1

Ⓒ 2

Ⓓ 3 $\longleftarrow$ Correct

Ⓔ None of the above

Solution: We have

$$\begin{aligned}
\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \lim_{x \to 2} \frac{x^3 - 2^3}{x^2 - 2^2} &= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)} \\
&= \lim_{x \to 2} \frac{x^2 + 2x + 4}{x + 2} \\
&= \frac{2^2 + 2 \cdot 2 + 4}{2 + 2} \\
&= \frac{12}{4} \\
&= 3
\end{aligned}$$

Info: The average in the class for this problem was 52.4%.

5. Find $\lim\limits_{x\to 1} \dfrac{\dfrac{1}{x+2} - \dfrac{1}{3}}{x-1}$.

Ⓐ $-\dfrac{1}{9}$ $\longleftarrow$ Correct

Ⓑ $\dfrac{1}{9}$

Ⓒ $\dfrac{1}{3}$

Ⓓ $-\dfrac{1}{3}$

Ⓔ None of the above

Solution: We have

$$\begin{aligned}
\lim_{x\to 1} \frac{\dfrac{1}{x+2} - \dfrac{1}{3}}{x-1} = \lim_{x\to 1} \frac{3(x+2)\left(\dfrac{1}{x+2} - \dfrac{1}{3}\right)}{3(x+2)(x-1)} &= \lim_{x\to 1} \frac{3(x+2)\cdot\dfrac{1}{x+2} - 3(x+2)\cdot\dfrac{1}{3}}{3(x+2)(x-1)} \\
&= \lim_{x\to 1} \frac{3\cdot 1 - (x+2)\cdot 1}{3(x+2)(x-1)} \\
&= \lim_{x\to 1} \frac{3-x-2}{3(x+2)(x-1)} \\
&= \lim_{x\to 1} \frac{1-x}{3(x+2)(x-1)} \\
&= \lim_{x\to 1} \frac{-(x-1)}{3(x+2)(x-1)} \\
&= \lim_{x\to 1} \frac{-1}{3(x+2)} \\
&= \frac{-1}{3(1+2)} \\
&= -\frac{1}{9}
\end{aligned}$$

Info: The average in the class for this problem was 81.7%.

6. Find $\lim_{x \to 0} \frac{\sin 3x}{4x}$.

Ⓐ Does not exist

Ⓑ $\frac{3}{4}$ $\longleftarrow$ Correct

Ⓒ $\frac{4}{3}$

Ⓓ 0

Ⓔ None of the above

Solution: We have

$$\lim_{x \to 0} \frac{\sin 3x}{4x} = \lim_{x \to 0} \frac{\frac{\sin 3x}{3x} \cdot 3x}{4x} = \left[\lim_{u \to 0} \frac{\sin u}{u} = 1 \right] = \lim_{x \to 0} \frac{1 \cdot 3x}{4x} = \lim_{x \to 0} \frac{3}{4} = \frac{3}{4}$$

Info: The average in the class for this problem was 57.3%.

7. Find $\lim_{x \to -1^-} \frac{1}{1+x}$.

Ⓐ Does not exist and neither ∞ nor $-\infty$

Ⓑ $-\infty$ $\longleftarrow$ Correct

Ⓒ ∞

Ⓓ 1

Ⓔ None of the above

Solution: Note that $1 + x \to 0^-$ as $x \to -1^-$. Therefore

$$\lim_{x \to -1^-} \frac{1}{1+x} = \left[\frac{1}{0^-} \right] = -\infty$$

Info: The average in the class for this problem was 51.2%.

8. Find $\lim\limits_{x\to -1^-} \dfrac{1}{1+x^2}$.

Ⓐ ∞

Ⓑ $-\infty$

Ⓒ Does not exist and neither ∞ nor $-\infty$

Ⓓ 1

Ⓔ None of the above $\longleftarrow$ Correct

Solution: We have

$$\lim_{x\to -1^-} \frac{1}{1+x^2} = \frac{1}{1+(-1)^2} = \frac{1}{1+1} = \frac{1}{2}$$

Info: The average in the class for this problem was 59.8%.

9. Find $\lim\limits_{x\to -1} \dfrac{1}{1+x}$.

Ⓐ 1

Ⓑ ∞

Ⓒ $-\infty$

Ⓓ Does not exist and neither ∞ nor $-\infty$ $\longleftarrow$ Correct

Ⓔ None of the above

Solution: By Problem 7, $\lim\limits_{x\to -1^-} \dfrac{1}{1+x} = -\infty$ On the other hand, since $1+x \to 0^+$ as $x \to -1^+$, it follows that

$$\lim_{x\to -1^+} \frac{1}{1+x} = \left[\frac{1}{0^+}\right] = \infty$$

So

$$\lim_{x\to -1^-} \frac{1}{1+x} \neq \lim_{x\to -1^+} \frac{1}{1+x}$$

therefore $\lim\limits_{x\to -1} \dfrac{1}{1+x}$ does not exist and neither ∞ nor $-\infty$.

Info: The average in the class for this problem was 67.1%.

10. Find $\lim_{x\to -1} \dfrac{1}{(1+x)^2}$.

Ⓐ Does not exist and neither ∞ nor $-\infty$

Ⓑ 1

Ⓒ $-\infty$

Ⓓ ∞ $\longleftarrow$ Correct

Ⓔ None of the above

Solution: Since we square $1+x$, it follows that $(1+x)^2$ approaches 0 from the right as $x \to -1$. Therefore

$$\lim_{x\to -1} \frac{1}{(1+x)^2} = \left[\frac{1}{0^+}\right] = \infty$$

Info: The average in the class for this problem was 75.6%.

11. If $f(x) = \sqrt{2x}$, which one of the following limits correspond to $f'(a)$?

Ⓐ $\lim_{h\to 0} \dfrac{\sqrt{2a+2h} - \sqrt{2a}}{h}$ $\longleftarrow$ Correct

Ⓑ $\lim_{h\to 0} \dfrac{\sqrt{2a+h} - \sqrt{2a}}{h}$

Ⓒ $\lim_{h\to 0} \dfrac{\sqrt{2a-h} - \sqrt{2a}}{h}$

Ⓓ $\lim_{x\to 0} \dfrac{\sqrt{2x} - \sqrt{2a}}{x-a}$

Ⓔ $\lim_{a\to 0} \dfrac{\sqrt{2x} - \sqrt{2a}}{x-a}$

Solution: By the definition of the derivative,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\sqrt{2(a+h)} - \sqrt{2a}}{h} = \lim_{h\to 0} \frac{\sqrt{2a+2h} - \sqrt{2a}}{h}$$

Info: The average in the class for this problem was 75.6%.

12. Let $f(x)$ be a function which is differentiable for all x values. Which of these is the derivative of $g(x) = \dfrac{1}{f(\sqrt{x})}$?

Ⓐ $\dfrac{f'(\sqrt{x})}{2(f(\sqrt{x}))^2}$

Ⓑ $\dfrac{f'(\sqrt{x})}{\sqrt{x}(f(\sqrt{x}))^2}$

Ⓒ $-\dfrac{f'(\sqrt{x})}{\sqrt{x}(f(\sqrt{x}))^2}$

Ⓓ $\dfrac{f'(\sqrt{x})}{2\sqrt{x}(f(\sqrt{x}))^2}$

Ⓔ $-\dfrac{f'(\sqrt{x})}{2\sqrt{x}(f(\sqrt{x}))^2}$ $\longleftarrow$ Correct

Solution: We can find $g'(x)$ in two different ways. Either

$$\begin{aligned} g'(x) = \left[\frac{1}{f(\sqrt{x})}\right]' = \left[(f(\sqrt{x}))^{-1}\right]' &= (-1)(f(\sqrt{x}))^{-1-1} \cdot [f(\sqrt{x})]' \\ &= -(f(\sqrt{x}))^{-2} \cdot [f(\sqrt{x})]' \\ &= -(f(\sqrt{x}))^{-2} \cdot f'(\sqrt{x}) \cdot (\sqrt{x})' \\ &= -(f(\sqrt{x}))^{-2} \cdot f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \\ &= -\frac{f'(\sqrt{x})}{2\sqrt{x}(f(\sqrt{x}))^2} \end{aligned}$$

or

$$\begin{aligned} g'(x) = \left[\frac{1}{f(\sqrt{x})}\right]' = \frac{1' \cdot f(\sqrt{x}) - 1 \cdot [f(\sqrt{x})]'}{(f(\sqrt{x}))^2} &= \frac{0 \cdot f(\sqrt{x}) - 1 \cdot f'(\sqrt{x}) \cdot (\sqrt{x})'}{(f(\sqrt{x}))^2} \\ &= \frac{-f'(\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}}}{(f(\sqrt{x}))^2} \\ &= -\frac{f'(\sqrt{x})}{2\sqrt{x}(f(\sqrt{x}))^2} \end{aligned}$$

Info: The average in the class for this problem was 56.1%.

13. Let $f(x) = \cos x$, then the <u>second</u> derivative of f is

Ⓐ $\sin x$

Ⓑ $-\sin x$

Ⓒ $\cos x$

Ⓓ $-\cos x \longleftarrow$ Correct

Ⓔ None of the above

Solution: We have

$$f'(x) = (\cos x)' = -\sin x$$

and

$$f''(x) = (f'(x))' = (-\sin x)' = -(\sin x)' = -\cos x$$

Info: The average in the class for this problem was 90.2%.

14. Let $f(x) = (1+3x)^8$, then $f'(x)$ is

Ⓐ $24(1+3x)^7 \longleftarrow$ Correct

Ⓑ $8(1+3x)^7$

Ⓒ $3(1+3x)^7$

Ⓓ $(1+3x)^7$

Ⓔ $8(1+3x)^9$

Solution: We have

$$\begin{aligned} f'(x) = \left[(1+3x)^8\right]' &= 8(1+3x)^{8-1} \cdot (1+3x)' \\ &= 8(1+3x)^7 \cdot (1' + (3x)') \\ &= 8(1+3x)^7 \cdot (1' + 3(x)') \\ &= 8(1+3x)^7 \cdot (0 + 3 \cdot 1) \\ &= 8(1+3x)^7 \cdot 3 \\ &= 24(1+3x)^7 \end{aligned}$$

Info: The average in the class for this problem was 89%.

15. Suppose that $s(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the particle's instantaneous velocity at time $t = 2$ s.

Ⓐ 1 m/s

Ⓑ 0 m/s

Ⓒ −1 m/s

Ⓓ −2 m/s

Ⓔ −3 m/s $\longleftarrow$ Correct

Solution: Since the instantaneous velocity $v(t)$ is $s'(t)$, we have

$$\begin{aligned} s'(t) &= (1 + 5t - 2t^2)' \\ &= 1' + (5t)' - (2t^2)' \\ &= 1' + 5(t)' - 2(t^2)' \\ &= 0 + 5 \cdot 1 - 2 \cdot 2t \\ &= 5 - 4t \end{aligned}$$

therefore the particle's instantaneous velocity at time $t = 2$ s is

$$s'(2) = 5 - 4 \cdot 2 = 5 - 8 = -3 \text{ m/s}$$

Info: The average in the class for this problem was 96.3%.

1. Let $f(x) = \dfrac{x+1}{\sqrt{3x^2-1}}$.

(a) Find all horizontal asymptotes of f.

Solution: We have

$$\lim_{x\to\infty} \frac{x+1}{\sqrt{3x^2-1}} = \lim_{x\to\infty} \frac{\frac{x+1}{\sqrt{x^2}}}{\frac{\sqrt{3x^2-1}}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{\frac{x+1}{x}}{\sqrt{\frac{3x^2-1}{x^2}}} = \lim_{x\to\infty} \frac{\frac{x}{x}+\frac{1}{x}}{\sqrt{\frac{3x^2}{x^2}-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{1}{x}}{\sqrt{3-\frac{1}{x^2}}} = \frac{1+0}{\sqrt{3-0}} = \frac{1}{\sqrt{3}}$$

and

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{3x^2-1}} = \lim_{x\to-\infty} \frac{\frac{x+1}{\sqrt{x^2}}}{\frac{\sqrt{3x^2-1}}{\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{\frac{x+1}{-x}}{\sqrt{\frac{3x^2-1}{x^2}}} = \lim_{x\to-\infty} \frac{-\frac{x}{x}+-\frac{1}{x}}{\sqrt{\frac{3x^2}{x^2}-\frac{1}{x^2}}}$$

$$= \lim_{x\to-\infty} \frac{-1-\frac{1}{x}}{\sqrt{3-\frac{1}{x^2}}} = \frac{-1-0}{\sqrt{3-0}} = -\frac{1}{\sqrt{3}}$$

Therefore the function $f(x) = \dfrac{x+1}{\sqrt{3x^2-1}}$ has two horizontal asymptotes $y = \dfrac{1}{\sqrt{3}}$ and $y = -\dfrac{1}{\sqrt{3}}$.

(b) Find all vertical asymptotes of f.

Solution: The vertical asymptotes of f are $x = \dfrac{1}{\sqrt{3}}$ and $x = -\dfrac{1}{\sqrt{3}}$, since $\sqrt{3x^2-1} = 0$ and $x+1 \neq 0$ at $x = \pm\dfrac{1}{\sqrt{3}}$.

Info: The average in the class for this problem was 55%.

2. Let

$$f(x) = \begin{cases} -x+5 & \text{if } x < -1 \\ \sin(x^2-1) & \text{if } -1 \le x \le 1 \\ \sqrt{x} & \text{if } x > 1 \end{cases}$$

Find all points where f is discontinuous. **Use limits to justify your answer!**

Solution: We first note that

$-x+5$ is continuous everywhere, since it is a polynomial

$\sin(x^2-1)$ is continuous everywhere, since it is a composition of a polynomial and $\sin x$

$\sqrt{x}$ is continuous at any point $x > 1$, since $(1, \infty)$ is in the domain of $\sqrt{x}$

Therefore the only potential points of discontinuities of f are $x = -1$ and $x = 1$. Let us show that f is indeed discontinuous at these points. We have

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-x+5) = -1+5 = 4$$

and

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \sin(x^2-1) = \sin((-1)^2-1) = \sin(1-1) = \sin 0 = 0$$

Since

$$\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$$

it follows that f is discontinuous at $x = -1$. Simiarly,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \sin(x^2-1) = \sin((-1)^2-1) = \sin(1-1) = \sin 0 = 0$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sqrt{x} = \sqrt{1} = 1$$

Since

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

it follows that f is discontinuous at $x = 1$.

Info: The average in the class for this problem was 69.8%.

3. Let $f(x) = \sqrt{2x+5}$. Use the **definition of the derivative** to find $f'(x)$.

Solution 1: We have

$$
\begin{aligned}
f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} &= \lim_{x\to a} \frac{\sqrt{2x+5}-\sqrt{2a+5}}{x-a} \\
&= \lim_{x\to a} \frac{(\sqrt{2x+5}-\sqrt{2a+5})(\sqrt{2x+5}+\sqrt{2a+5})}{(x-a)(\sqrt{2x+5}+\sqrt{2a+5})} \\
&= \lim_{x\to a} \frac{(\sqrt{2x+5})^2-(\sqrt{2a+5})^2}{(x-a)(\sqrt{2x+5}+\sqrt{2a+5})} \\
&= \lim_{x\to a} \frac{(2x+5)-(2a+5)}{(x-a)(\sqrt{2x+5}+\sqrt{2a+5})} \\
&= \lim_{x\to a} \frac{2x+5-2a-5}{(x-a)(\sqrt{2x+5}+\sqrt{2a+5})} \\
&= \lim_{x\to a} \frac{2x-2a}{(x-a)(\sqrt{2x+5}+\sqrt{2a+5})} \\
&= \lim_{x\to a} \frac{2(x-a)}{(x-a)(\sqrt{2x+5}+\sqrt{2a+5})} \\
&= \lim_{x\to a} \frac{2}{\sqrt{2x+5}+\sqrt{2a+5}} \\
&= \frac{2}{\sqrt{2a+5}+\sqrt{2a+5}} \\
&= \frac{2}{2\sqrt{2a+5}} \\
&= \frac{1}{\sqrt{2a+5}}
\end{aligned}
$$

Solution 2: We have

$$
\begin{aligned}
f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{2(x+h)+5}-\sqrt{2x+5}}{h} \\
&= \lim_{h\to 0}\frac{(\sqrt{2(x+h)+5}-\sqrt{2x+5})(\sqrt{2(x+h)+5}+\sqrt{2x+5})}{h(\sqrt{2(x+h)+5}+\sqrt{2x+5})} \\
&= \lim_{h\to 0}\frac{(\sqrt{2(x+h)+5})^2-(\sqrt{2x+5})^2}{h(\sqrt{2(x+h)+5}+\sqrt{2x+5})} \\
&= \lim_{h\to 0}\frac{(2(x+h)+5)-(2x+5)}{h(\sqrt{2(x+h)+5}+\sqrt{2x+5})} \\
&= \lim_{h\to 0}\frac{2x+2h+5-2x-5}{h(\sqrt{2(x+h)+5}+\sqrt{2x+5})} \\
&= \lim_{h\to 0}\frac{2h}{h(\sqrt{2(x+h)+5}+\sqrt{2x+5})} \\
&= \lim_{h\to 0}\frac{2}{\sqrt{2(x+h)+5}+\sqrt{2x+5}} \\
&= \frac{2}{\sqrt{2(x+0)+5}+\sqrt{2x+5}} \\
&= \frac{2}{\sqrt{2x+5}+\sqrt{2x+5}} \\
&= \frac{2}{2\sqrt{2x+5}} \\
&= \frac{1}{\sqrt{2x+5}}
\end{aligned}
$$

Info: The average in the class for this problem was 82.5%.