NAME: ____________________________________

Calculus I - Spring 2014

Midterm Exam II, April 21, 2014

In all non-multiple choice problems you are required to show all your work and provide the necessary explanations everywhere to get full credit. In all multiple choice problems you don't have to show your work.

1. If $f(x) = x + \cos x$, then $(f^{-1})'(1)$ is

Ⓐ $\dfrac{1}{1+\cos 1}$

Ⓑ 0

Ⓒ $-\sin 1$

Ⓓ 1 $\longleftarrow$ Correct

Ⓔ $1 + \cos 1$

Solution 1: We have

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))}$$

Since

$$f(0) = 0 + \cos 0 = 0 + 1 = 1$$

it follows that $f^{-1}(1) = 0$. Hence

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)}$$

But

$$f'(x) = (x + \cos x)' = x' + (\cos x)' = 1 - \sin x$$

therefore

$$f'(0) = 1 - \sin 0 = 1 - 0 = 1$$

This yields

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1} = 1$$

Solution 2: One can see that $y = f^{-1}(x)$ satisfies the equation $x = y + \cos y$. To find y' we differentiate both sides:

$$x' = (y + \cos y)' \quad \Longrightarrow \quad 1 = y' + (\cos y)' \quad \Longrightarrow \quad 1 = y' - \sin y \cdot y' \quad \Longrightarrow \quad 1 = y'(1 - \sin y)$$

so

$$y' = \frac{1}{1 - \sin y}$$

Note that if $x = 1$, then $y = 0$ (solution of $1 = y + \cos y$). Therefore

$$(f^{-1})'(1) = y'(1) = \frac{1}{1 - \sin 0} = \frac{1}{1 - 0} = \frac{1}{1} = 1$$

Info: The average in the class for this problem was 58.1%.

2. Let $f(x) = 1 - 2\sqrt[3]{x}$, then $f^{-1}(x)$ is

Ⓐ $\dfrac{1 - x^3}{8}$

Ⓑ $\dfrac{(1 - x)^3}{8}$ $\longleftarrow$ Correct

Ⓒ $\dfrac{(x - 1)^3}{8}$

Ⓓ $\dfrac{x^3 - 1}{8}$

Ⓔ None of the above

Solution: We have:

Step 1: Replace $f(x)$ by y:

$$y = 1 - 2\sqrt[3]{x}$$

Step 2: Solve for x:

$$y = 1 - 2\sqrt[3]{x} \quad \Longrightarrow \quad y + 2\sqrt[3]{x} = 1 \quad \Longrightarrow \quad 2\sqrt[3]{x} = 1 - y \quad \Longrightarrow \quad \sqrt[3]{x} = \frac{1 - y}{2}$$

therefore

$$(\sqrt[3]{x})^3 = \left(\frac{1 - y}{2}\right)^3 \quad \Longrightarrow \quad x = \left(\frac{1 - y}{2}\right)^3 = \frac{(1 - y)^3}{2^3} = \frac{(1 - y)^3}{8}$$

Step 3: Replace x by $f^{-1}(x)$ and y by x:

$$f^{-1}(x) = \frac{(1 - x)^3}{8}$$

Info: The average in the class for this problem was 66.2%.

3. Compute the linearization of $f(x) = \sqrt{x}e^{x-1}$ at $a = 1$.

Ⓐ $L(x) = \frac{3}{2}x + \frac{1}{2}$

Ⓑ $L(x) = -\frac{3}{2}x + \frac{1}{2}$

Ⓒ $L(x) = -\frac{3}{2}x - \frac{1}{2}$

Ⓓ $L(x) = \frac{3}{2}x - \frac{1}{2}$ ⟵ Correct

Ⓔ None of the above

Solution (short version): Since

$$f'(x) = \left(x^{1/2}\right)' e^{x-1} + x^{1/2}\left(e^{x-1}\right)' = \frac{1}{2}x^{-1/2}e^{x-1} + x^{1/2}e^{x-1}$$

it follows that

$$f'(1) = \frac{1}{2} \cdot 1^{-1/2} \cdot e^{1-1} + 1^{1/2} \cdot e^{1-1} = \frac{3}{2}$$

Keeping in mind that $f(1) = \sqrt{1} \cdot e^{1-1} = 1$, we get

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{3}{2}(x-1) = \frac{3}{2}x - \frac{1}{2}$$

Solution (full version): The derivative of $f(x) = \sqrt{x}e^{x-1}$ is

$$\begin{aligned} f'(x) = \left(x^{1/2}e^{x-1}\right)' = \left(x^{1/2}\right)' e^{x-1} + x^{1/2}\left(e^{x-1}\right)' &= \frac{1}{2}x^{1/2-1}e^{x-1} + x^{1/2}e^{x-1} \cdot (x-1)' \\ &= \frac{1}{2}x^{-1/2}e^{x-1} + x^{1/2}e^{x-1} \cdot (x' - 1') = \frac{1}{2}x^{-1/2}e^{x-1} + x^{1/2}e^{x-1} \cdot (1-0) \\ &= \frac{1}{2}x^{-1/2}e^{x-1} + x^{1/2}e^{x-1} \end{aligned}$$

therefore

$$f'(1) = \frac{1}{2} \cdot 1^{-1/2} \cdot e^{1-1} + 1^{1/2} \cdot e^{1-1} = \frac{1}{2} \cdot 1 \cdot e^0 + 1 \cdot e^0 = \frac{1}{2} \cdot 1 \cdot 1 + 1 \cdot 1 = \frac{1}{2} + 1 = \frac{3}{2}$$

Since

$$f(1) = \sqrt{1} \cdot e^{1-1} = 1 \cdot e^0 = 1 \cdot 1 = 1$$

we get

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{3}{2}(x-1) = 1 + \frac{3}{2}x - \frac{3}{2} = \frac{3}{2}x - \frac{1}{2}$$

Info: The average in the class for this problem was 63.5%.

4. If $f(x) = (2-x)^x$, then $f'(x)$ is

Ⓐ $x(2-x)^{x-1}$

Ⓑ $(2-x)^x \ln(2-x) - x(2-x)^{x-1}$ $\longleftarrow$ Correct

Ⓒ $(2-x)^x \ln(2-x) + x(2-x)^{x-1}$

Ⓓ $-x(2-x)^{x-1}$

Ⓔ $(2-x)^x \ln(2-x) - x(2-x)^{x+1}$

Solution (short version): If we logarithm both sides of $f(x) = (2-x)^x$, we get

$$\ln f(x) = x \ln(2-x)$$

therefore

$$\frac{1}{f(x)} \cdot f'(x) = x' \ln(2-x) + x(\ln(2-x))' = \ln(2-x) + x \cdot \frac{1}{2-x} \cdot (2-x)' = \ln(2-x) - \frac{x}{2-x}$$

hence

$$f'(x) = f(x)\left(\ln(2-x) - \frac{x}{2-x}\right) = (2-x)^x \left(\ln(2-x) - \frac{x}{2-x}\right)$$

$$= (2-x)^x \ln(2-x) + x(2-x)^{x-1}$$

Solution (full version): We logarithm and then differentiate both sides of $f(x) = (2-x)^x$. We have

$$f(x) = (2-x)^x \quad \Longrightarrow \quad \ln f(x) = \ln(2-x)^x = x\ln(2-x) \quad \Longrightarrow \quad [\ln f(x)]' = [x \ln(2-x)]'$$

therefore

$$\frac{1}{f(x)} \cdot f'(x) = x' \ln(2-x) + x(\ln(2-x))' = 1 \cdot \ln(2-x) + x \cdot \frac{1}{2-x} \cdot (2-x)'$$

$$= \ln(2-x) + x \cdot \frac{1}{2-x} \cdot (2' - x') = \ln(2-x) + x \cdot \frac{1}{2-x} \cdot (0-1)$$

$$= \ln(2-x) + x \cdot \frac{1}{2-x} \cdot (-1) = \ln(2-x) - x \cdot \frac{1}{2-x}$$

hence

$$f'(x) = f(x)\left(\ln(2-x) - x \cdot \frac{1}{2-x}\right)$$

$$= (2-x)^x \left(\ln(2-x) - x \cdot \frac{1}{2-x}\right)$$

$$= (2-x)^x \ln(2-x) + (2-x)^x \cdot x \cdot \frac{1}{2-x}$$

$$= (2-x)^x \ln(2-x) + x(2-x)^{x-1}$$

Info: The average in the class for this problem was 66.2%.

5. Find the slope of the tangent line at the point $(1, 1)$ on the graph of $e^{x-y} = 2x^2 - y^2$.

Ⓐ 0

Ⓑ -1

Ⓒ 1

Ⓓ 2

Ⓔ 3 $\longleftarrow$ Correct

Solution 1: Differentiating both sides of $e^{x-y} = 2x^2 - y^2$, we get

$$e^{x-y}(x-y)' = (2x^2)' - (y^2)' \quad \Longrightarrow \quad e^{x-y}(1-y') = 4x - 2y \cdot y'$$

therefore

$$e^{1-1}(1-y') = 4 \cdot 1 - 2 \cdot 1 \cdot y' \quad \Longrightarrow \quad 1 - y' = 4 - 2y' \quad \Longrightarrow \quad 2y' - y' = 4 - 1 \quad \Longrightarrow \quad y' = 3$$

Solution 2 (short version): Differentiating both sides of $e^{x-y} = 2x^2 - y^2$, we get

$$e^{x-y}(x-y)' = (2x^2)' - (y^2)' \quad \Longrightarrow \quad e^{x-y} - e^{x-y}y' = 4x - 2y \cdot y'$$

therefore

$$2y \cdot y' - e^{x-y}y' = 4x - e^{x-y} \quad \Longrightarrow \quad y'(2y - e^{x-y}) = 4x - e^{x-y} \quad \Longrightarrow \quad y' = \frac{4x - e^{x-y}}{2y - e^{x-y}}$$

hence

$$m = y'(1) = \frac{4 \cdot 1 - e^{1-1}}{2 \cdot 1 - e^{1-1}} = 3$$

Solution 2 (full version): We first find $\frac{dy}{dx}$. We have

$$e^{x-y} = 2x^2 - y^2 \quad \Longrightarrow \quad (e^{x-y})' = (2x^2 - y^2)' \quad \Longrightarrow \quad e^{x-y}(x-y)' = (2x^2)' - (y^2)'$$

hence

$$e^{x-y}(x' - y') = 2(x^2)' - 2y \cdot y' \quad \Longrightarrow \quad e^{x-y}(1 - y') = 2 \cdot 2x - 2y \cdot y' \quad \Longrightarrow \quad e^{x-y} - e^{x-y}y' = 4x - 2y \cdot y'$$

therefore

$$2y \cdot y' - e^{x-y}y' = 4x - e^{x-y} \quad \Longrightarrow \quad y'(2y - e^{x-y}) = 4x - e^{x-y} \quad \Longrightarrow \quad y' = \frac{4x - e^{x-y}}{2y - e^{x-y}}$$

It follows that the slope of the tangent line at the point $(1, 1)$ on the graph of $e^{x-y} = 2x^2 - y^2$ is

$$m = y'(1) = \frac{4 \cdot 1 - e^{1-1}}{2 \cdot 1 - e^{1-1}} = \frac{4 - e^0}{2 - e^0} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$$

Info: The average in the class for this problem was 44.6%.

6. If $f(x) = 2^{1+\arctan x}$, then $f'(x)$ is

Ⓐ $\dfrac{2^{1+\arctan x}}{(1-x^2)\ln 2}$

Ⓑ $\dfrac{2^{1+\arctan x}}{1+x^2}$

Ⓒ $\dfrac{2^{1+\arctan x}\ln 2}{1+x^2}$ $\longleftarrow$ Correct

Ⓓ $\dfrac{2^{1+\arctan x}}{(1+x^2)\ln 2}$

Ⓔ $\dfrac{2^{1+\arctan x}\ln 2}{\sqrt{1-x^2}}$

Solution (short version): We have

$$f'(x) = 2^{1+\arctan x}\ln 2 \cdot (1+\arctan x)' = \frac{2^{1+\arctan x}\ln 2}{1+x^2}$$

Solution (full version): We have

$$\begin{aligned} f'(x) = \left(2^{1+\arctan x}\right)' &= 2^{1+\arctan x}\ln 2 \cdot (1+\arctan x)' \\ &= 2^{1+\arctan x}\ln 2 \cdot (1' + (\arctan x)') \\ &= 2^{1+\arctan x}\ln 2 \cdot \left(0 + \frac{1}{1+x^2}\right) \\ &= 2^{1+\arctan x}\ln 2 \cdot \frac{1}{1+x^2} \\ &= \frac{2^{1+\arctan x}\ln 2}{1+x^2} \end{aligned}$$

Info: The average in the class for this problem was 72.8%.

7. Find the differential of $f(x) = \sqrt{1-2x}$.

Ⓐ $-\dfrac{dx}{\sqrt{1-2x}}$ ⟵ Correct

Ⓑ $-\dfrac{dx}{2\sqrt{1-2x}}$

Ⓒ $\dfrac{dx}{\sqrt{1-2x}}$

Ⓓ $\dfrac{dx}{2\sqrt{1-2x}}$

Ⓔ None of the above

Solution (short version): Since

$$f'(x) = \frac{1}{2}(1-2x)^{1/2-1} \cdot (1-2x)' = \frac{1}{2}(1-2x)^{-1/2} \cdot (-2) = -\frac{1}{\sqrt{1-2x}}$$

we have

$$dy = f'(x)dx = -\frac{dx}{\sqrt{1-2x}}$$

Solution (full version): We have

$$\begin{aligned} f'(x) = \left((1-2x)^{1/2}\right)' &= \frac{1}{2}(1-2x)^{1/2-1} \cdot (1-2x)' \\ &= \frac{1}{2}(1-2x)^{-1/2} \cdot (1' - (2x)') \\ &= \frac{1}{2}(1-2x)^{-1/2} \cdot (1' - 2(x)') \\ &= \frac{1}{2}(1-2x)^{-1/2} \cdot (0 - 2 \cdot 1) \\ &= \frac{1}{2}(1-2x)^{-1/2} \cdot (-2) \\ &= -\frac{1}{\sqrt{1-2x}} \end{aligned}$$

therefore the differential dy of the function $f(x) = \sqrt{1-2x}$ is

$$dy = f'(x)dx = -\frac{dx}{\sqrt{1-2x}}$$

Info: The average in the class for this problem was 64.9%.

8. Let $f(x) = \arcsin x$, then $f^{-1}(x)$ is

Ⓐ $\sin x,\ 0 \le x \le 1$

Ⓑ $\dfrac{1}{\sin x},\ -1 \le x \le 1$

Ⓒ $\sin x,\ -1 \le x \le 1$

Ⓓ $\sin x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ $\longleftarrow$ Correct

Ⓔ $\sin x,\ 0 \le x \le \pi$

Solution: Since $\arcsin x$ is the inverse of the restricted sine function

$$\sin x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

it follows that the inverse of $\arcsin x$ is the restricted sine.

Info: The average in the class for this problem was 62.2%.

9. Find $\lim_{x \to \sqrt{3}^+} 5^{1/(3-x^2)}$.

Ⓐ 0 $\longleftarrow$ Correct

Ⓑ Does not exist and neither ∞ nor $-\infty$

Ⓒ ∞

Ⓓ $-\infty$

Ⓔ None of the above

Solution: Note that $3 - x^2 \to 0^-$ as $x \to \sqrt{3}^+$. Therefore $1/(3-x^2) \to -\infty$ as $x \to \sqrt{3}^+$. Hence

$$\lim_{x \to \sqrt{3}^+} 5^{1/(3-x^2)} = \left[5^{-\infty} = \frac{1}{5^\infty} = \frac{1}{\infty}\right] = 0$$

Info: The average in the class for this problem was 51.4%.

10. Find $\lim_{x \to \sqrt{3}^+} 5^{1/(x^2-3)}$.

Ⓐ $\infty \longleftarrow$ Correct

Ⓑ $-\infty$

Ⓒ Does not exist and neither ∞ nor $-\infty$

Ⓓ 0

Ⓔ None of the above

Solution: Note that $x^2 - 3 \to 0^+$ as $x \to \sqrt{3}^+$. Therefore $1/(x^2-3) \to \infty$ as $x \to \sqrt{3}^+$. Hence

$$\lim_{x \to \sqrt{3}^+} 5^{1/(x^2-3)} = [5^\infty] = \infty$$

Info: The average in the class for this problem was 73%.

1. Find $\lim\limits_{x\to 0}\left(\dfrac{1}{\ln(x+1)}-\dfrac{1}{x}\right)$.

Solution: We have

$$\lim_{x\to 0}\left(\frac{1}{\ln(x+1)}-\frac{1}{x}\right)=[\infty-\infty]=\lim_{x\to 0}\left(\frac{x\cdot 1}{x\cdot\ln(x+1)}-\frac{1\cdot\ln(x+1)}{x\cdot\ln(x+1)}\right)$$

$$=\lim_{x\to 0}\frac{x-\ln(x+1)}{x\ln(x+1)}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{(x-\ln(x+1))'}{(x\ln(x+1))'}$$

$$=\lim_{x\to 0}\frac{x'-(\ln(x+1))'}{x'\cdot\ln(x+1)+x\cdot(\ln(x+1))'}=\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{\ln(x+1)+\frac{x}{x+1}}$$

Now we can continue in two different ways. Either

$$\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{\ln(x+1)+\frac{x}{x+1}}=\lim_{x\to 0}\frac{(x+1)\left(1-\frac{1}{x+1}\right)}{(x+1)\left(\ln(x+1)+\frac{x}{x+1}\right)}$$

$$=\lim_{x\to 0}\frac{(x+1)\cdot 1-(x+1)\cdot\frac{1}{x+1}}{(x+1)\ln(x+1)+(x+1)\cdot\frac{x}{x+1}}=\lim_{x\to 0}\frac{x+1-1}{(x+1)\ln(x+1)+x}$$

$$=\lim_{x\to 0}\frac{x}{(x+1)\ln(x+1)+x}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{x'}{((x+1)\ln(x+1)+x)'}$$

$$=\lim_{x\to 0}\frac{1}{(x+1)'\ln(x+1)+(x+1)(\ln(x+1))'+x'}=\lim_{x\to 0}\frac{1}{1\cdot\ln(x+1)+(x+1)\cdot\frac{1}{x+1}+1}$$

$$=\lim_{x\to 0}\frac{1}{\ln(x+1)+1+1}=\lim_{x\to 0}\frac{1}{\ln(x+1)+2}=\frac{1}{\ln(0+1)+2}=\frac{1}{\ln 1+2}=\frac{1}{0+2}=\frac{1}{2}$$

In short,

$$\lim_{x\to 0}\left(\frac{1}{\ln(x+1)}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\ln(x+1)}{x\ln(x+1)}=\lim_{x\to 0}\frac{(x-\ln(x+1))'}{(x\ln(x+1))'}=\lim_{x\to 0}\frac{1-\frac{1}{x+1}}{\ln(x+1)+\frac{x}{x+1}}$$

$$=\lim_{x\to 0}\frac{x}{(x+1)\ln(x+1)+x}=\lim_{x\to 0}\frac{x'}{((x+1)\ln(x+1)+x)'}$$

$$=\lim_{x\to 0}\frac{1}{\ln(x+1)+2}=\frac{1}{\ln(0+1)+2}=\frac{1}{2}$$

Or

$$\lim_{x\to 0}\frac{1-\dfrac{1}{x+1}}{\ln(x+1)+\dfrac{x}{x+1}}=\left[\frac{0}{0}\right]=\lim_{x\to 0}\frac{\left(1-\dfrac{1}{x+1}\right)'}{\left(\ln(x+1)+\dfrac{x}{x+1}\right)'}=\lim_{x\to 0}\frac{1'-\left(\dfrac{1}{x+1}\right)'}{(\ln(x+1))'+\left(\dfrac{x}{x+1}\right)'}$$

$$=\lim_{x\to 0}\frac{0-\dfrac{1'\cdot(x+1)-1\cdot(x+1)'}{(x+1)^2}}{\dfrac{1}{x+1}(x+1)'+\dfrac{x'(x+1)-x(x+1)'}{(x+1)^2}}=\lim_{x\to 0}\frac{-\dfrac{0\cdot(x+1)-1\cdot 1}{(x+1)^2}}{\dfrac{1}{x+1}\cdot 1+\dfrac{1\cdot(x+1)-x\cdot 1}{(x+1)^2}}$$

$$=\lim_{x\to 0}\frac{-\dfrac{0-1}{(x+1)^2}}{\dfrac{1}{x+1}+\dfrac{x+1-x}{(x+1)^2}}=\lim_{x\to 0}\frac{\dfrac{1}{(x+1)^2}}{\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}}=\lim_{x\to 0}\frac{(x+1)^2\cdot\dfrac{1}{(x+1)^2}}{(x+1)^2\left(\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}\right)}$$

$$=\lim_{x\to 0}\frac{1}{(x+1)^2\cdot\dfrac{1}{x+1}+(x+1)^2\cdot\dfrac{1}{(x+1)^2}}=\lim_{x\to 0}\frac{1}{(x+1)+1}=\lim_{x\to 0}\frac{1}{x+2}=\frac{1}{0+2}=\frac{1}{2}$$

In short,

$$\lim_{x\to 0}\left(\frac{1}{\ln(x+1)}-\frac{1}{x}\right)=\lim_{x\to 0}\frac{x-\ln(x+1)}{x\ln(x+1)}=\lim_{x\to 0}\frac{(x-\ln(x+1))'}{(x\ln(x+1))'}=\lim_{x\to 0}\frac{1-\dfrac{1}{x+1}}{\ln(x+1)+\dfrac{x}{x+1}}$$

$$=\lim_{x\to 0}\frac{\left(1-\dfrac{1}{x+1}\right)'}{\left(\ln(x+1)+\dfrac{x}{x+1}\right)'}=\lim_{x\to 0}\frac{\dfrac{1}{(x+1)^2}}{\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}}=\lim_{x\to 0}\frac{1}{x+2}=\frac{1}{2}$$

Info: The average in the class for this problem was 49.2%.

2. Let $f(x) = xe^x$.

(a) Find the x-intercept of f.

Solution: If $y = 0$, then

$$xe^x = 0 \quad \Longrightarrow \quad x = 0 \quad \Longrightarrow \quad (0,0)$$

Info: The average in the class for this problem was 81.6%.

(b) Find the y-intercept of f.

Solution: If $x = 0$, then

$$y = 0 \cdot e^0 = 0 \quad \Longrightarrow \quad (0,0)$$

Info: The average in the class for this problem was 86.7%.

(c) Is the function f even, odd, or neither? **Justify!**

Solution: The function $f(x) = xe^x$ is neither even nor odd, since

$$f(-1) \neq \pm f(1)$$

because

$$f(-1) = (-1)e^{-1} = -e^{-1} \quad \text{and} \quad f(1) = 1 \cdot e^1 = e$$

Info: The average in the class for this problem was 77.4%.

(d) Find the horizontal asymptote of f.

Solution: We first note that since

$$\lim_{x\to\infty} xe^x = [\infty \cdot \infty] = \infty$$

the graph of f does not approach a horizontal asymptote when $x \to \infty$. We now show that the graph of f does approach a horizontal asymptote when $x \to -\infty$. Indeed, by L'Hospital's Rule we have

$$\lim_{x\to-\infty} xe^x = \left\{ \lim_{x\to-\infty} \frac{xe^x}{1} = \lim_{x\to-\infty} \frac{xe^x \cdot e^{-x}}{1 \cdot e^{-x}} \right\} = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to-\infty} \frac{x'}{(e^{-x})'}$$

$$= \lim_{x\to-\infty} \frac{1}{e^{-x}(-x)'} = \lim_{x\to-\infty} \frac{1}{e^{-x}(-1)} = -\lim_{x\to-\infty} \frac{1}{e^{-x}} = \left[\frac{1}{e^{-(-\infty)}} = \frac{1}{e^{\infty}} = \frac{1}{\infty}\right] = 0$$

In short,

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{x'}{(e^{-x})'} = -\lim_{x\to-\infty} \frac{1}{e^{-x}} = 0$$

It follows that $y = 0$ is the horizontal asymptote.

Info: The average in the class for this problem was 24.4%.

(e) Does f have any vertical asymptote? **Justify!**

Solution: No, since f is continuous everywhere.

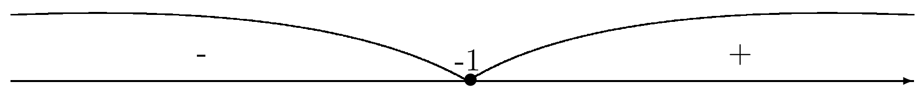
Info: The average in the class for this problem was 55.9%.

(f) Find the intervals of increase and decrease of f.

Solution: We have

$$f'(x) = (xe^x)' = x'e^x + x\,(e^x)' = 1 \cdot e^x + xe^x = (1 + x)e^x$$

Since $(1+x)e^x = 0$ at $x = -1$ and exists everywhere, this is the only critical number. We have



It follows that this function decreases on $(-\infty, -1)$ and increases on $(-1, \infty)$.

Info: The average in the class for this problem was 64.4%.

(g) Find local maximum and minimum value(s) of f (if any).

Solution: Because $f'(x)$ changes from negative to positive at -1, the First Derivative Test tells us that $f(-1) = -e^{-1}$ is the local minimum value. There are no local maximum values.

Info: The average in the class for this problem was 59.7%.

(h) Find absolute maximum and absolute minimum values of f on $[-2, 1]$.

Solution: To find absolute maximum and absolute minimum values of f on $[-2, 1]$ we evaluate f at the end points and at the critical number. We have

$$f(-2) = -2e^{-2}, \qquad f(-1) = -e^{-1}, \qquad f(1) = e$$

Obviously, $f(1) = e$ is the absolute maximum value of f on $[-2, 1]$, since $e > -2e^{-2}$ and $e > -e^{-1}$.

To show that $f(-1) = -e^{-1}$ is the absolute minimum value of f on $[-2, 1]$ we note that

$$-e^{-1} < -2e^{-2}$$

since

$$e > 2 \quad \Longrightarrow \quad e \cdot e^{-2} > 2 \cdot e^{-2} \quad \Longrightarrow \quad e^{-1} > 2e^{-2} \quad \Longrightarrow \quad -e^{-1} < -2e^{-2}$$

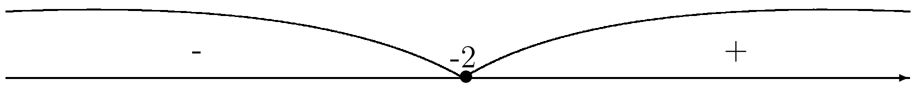
Info: The average in the class for this problem was 63.2%.

(i) Find the intervals of concavity of f.

Solution: We have

$$f''(x) = (f'(x))' = ((1+x)e^x)' = (1+x)'e^x + (1+x)(e^x)'$$
$$= 1 \cdot e^x + (1+x)e^x$$
$$= (1 + (1+x))e^x$$
$$= (2+x)e^x$$

Since $(1+x)e^x = 0$ at $x = -2$, we have



It follows that this function is concave downward on $(-\infty, -2)$ and concave upward on $(-2, \infty)$.

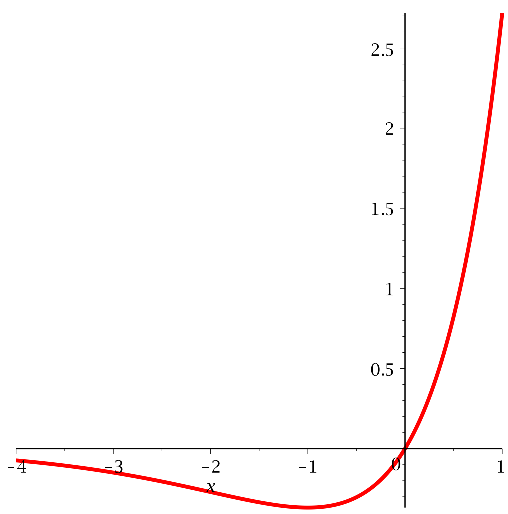
Info: The average in the class for this problem was 58.8%.

(j) Find the inflection point of f.

Solution: Because $f''(x)$ changes from negative to positive at $x = -2$, the inflection point is $(-2, -2e^{-2})$.

Info: The average in the class for this problem was 51.2%.

(k) Sketch the graph of f.



Info: The average in the class for this problem was 47%.