Calculus II - Fall 2013

Quiz #3, October 31, 2013

In the following problems you are required to show all your work and provide the necessary explanations everywhere to get full credit.

A tank has the shape of an inverted circular cone with height 5 m and base radius 2 m. It is filled with water to a height of 4 m. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of water is 1000 kg/m^3.)

Solution: Let's measure depths from the top of the tank by introducing a vertical coordinate line. The water extends from a depth of 1 m to a depth of 5 m and so we divide the interval $[1, 5]$ into n subintervals with endpoints $x_0, x_1, \ldots, x_n$ and choose x_i^* in the ith subinterval. This divides the water into n layers. The ith layer is approximated by a circular cylinder with radius r_i and height Δx. We can compute r_i from similar triangles as follows:

$$\frac{r_i}{5 - x_i^*} = \frac{2}{5} \quad \Longrightarrow \quad r_i = \frac{2}{5}\left(5 - x_i^*\right)$$

Thus an approximation to the volume of the ith layer of water is

$$V_i \approx \pi r_i^2 \Delta x = \frac{4\pi}{25}\left(5 - x_i^*\right)^2 \Delta x$$

and so its mass is

$$m_i = \text{density} \times \text{volume} \approx 1000 \cdot \frac{4\pi}{25}\left(5 - x_i^*\right)^2 \Delta x = 160\pi \left(5 - x_i^*\right)^2 \Delta x$$

The force required to raise this layer must overcome the force of gravity and so

$$F_i = m_i g \approx 160\pi g \left(5 - x_i^*\right)^2 \Delta x$$

Each particle in the layer must travel a distance of approximately x_i^*. The work W_i done to raise this layer to the top is approximately the product of the force F_i and the distance x_i^*:

$$W_i \approx F_i x_i^* \approx 160\pi g x_i^* \left(5 - x_i^*\right)^2 \Delta x$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \to \infty$:

$$\begin{aligned} W &= \lim_{n\to\infty} \sum_{i=1}^{n} 160\pi g x_i^* \left(5 - x_i^*\right)^2 \Delta x = \int_1^5 160\pi g x \left(5 - x\right)^2 dx \\ &= 160\pi g \int_1^5 \left(25x - 10x^2 + x^3\right) dx = 160\pi g \left[\frac{25x^2}{2} - \frac{10x^3}{3} + \frac{x^4}{4}\right]_1^5 \\ &= 160\pi g \cdot \frac{128}{3} = \pi g \frac{20480}{3} \approx 70059 \text{ J} \end{aligned}$$

Other answers: $W = 160\pi g \int_0^4 (5 - x)x^2 dx$ or $W = 160\pi g \int_0^4 (4 - x)^2(1 + x)dx$.