Calculus II - Spring 2014

Quiz #2, February 26, 2014

In the following problems you are required to show all your work and provide the necessary explanations everywhere to get full credit.

1. Determine whether the improper integral converges and, if so, evaluate it.

(a) $\int_1^\infty \sqrt{x}dx$

Solution: Note that

$$\sqrt{x} = x^{1/2} = \frac{1}{x^{-1/2}}$$

The integral $\int_1^\infty \frac{1}{x^{-1/2}}dx$ is divergent by the p-test, since $p = -1/2 \leq 1$.

(b) $\int_{-\infty}^0 x^2 e^{x^3} dx$

Solution (version 1): We have

$$\int x^2 e^{x^3} dx = \begin{bmatrix} x^3 = u \\ d(x^3) = du \\ 3x^2 dx = du \\ x^2 dx = \frac{1}{3}du \end{bmatrix} = \int e^u \frac{1}{3} du = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

Therefore

$$\int_{-\infty}^0 x^2 e^{x^3} dx = \lim_{t\to-\infty} \int_t^0 x^2 e^{x^3} dx = \lim_{t\to-\infty} \left.\frac{1}{3}e^{x^3}\right]_t^0 = \lim_{t\to-\infty}\left(\frac{1}{3}e^{0^3} - \frac{1}{3}e^{t^3}\right) = \frac{1}{3}\cdot 1 - \frac{1}{3}\cdot 0 = \frac{1}{3}$$

Solution (version 2): We have

$$\int_{-\infty}^0 x^2 e^{x^3} dx = \lim_{t\to-\infty} \int_t^0 x^2 e^{x^3} dx = \begin{bmatrix} x^3 = u \\ d(x^3) = du \\ 3x^2 dx = du \\ x^2 dx = \frac{1}{3}du \end{bmatrix} = \lim_{t\to-\infty} \int_{t^3}^{0^3} e^u \frac{1}{3} du = \lim_{t\to-\infty} \left.\frac{1}{3}e^u\right]_{t^3}^{0^3}$$

$$= \lim_{t\to-\infty}\left(\frac{1}{3}e^{0^3} - \frac{1}{3}e^{t^3}\right)$$

$$= \frac{1}{3}\cdot 1 - \frac{1}{3}\cdot 0 = \frac{1}{3}$$

(c) $\int\limits_0^3 \frac{1}{\sqrt[3]{2-x}}dx$

Solution (version 1): We first note that

$$\int\limits_0^3 \frac{1}{\sqrt[3]{2-x}}dx = \int\limits_0^2 \frac{1}{\sqrt[3]{2-x}}dx + \int\limits_2^3 \frac{1}{\sqrt[3]{2-x}}dx$$

We have

$$\int \frac{1}{\sqrt[3]{2-x}}dx = \int (2-x)^{-1/3}dx = \begin{bmatrix} 2-x=u \\ d(2-x)=du \\ -dx=du \\ dx=-du \end{bmatrix} = -\int u^{-1/3}du = -\frac{u^{-1/3+1}}{-1/3+1}+C$$

$$= -\frac{3}{2}u^{2/3}+C$$

$$= -\frac{3}{2}(2-x)^{2/3}+C$$

Therefore

$$\int\limits_0^2 \frac{1}{\sqrt[3]{2-x}}dx = \lim_{t\to 2^-}\int\limits_0^t \frac{1}{\sqrt[3]{2-x}}dx = \lim_{t\to 2^-}\left[-\frac{3}{2}(2-x)^{2/3}\right]_0^t = \lim_{t\to 2^-}\left(-\frac{3}{2}(2-t)^{2/3}+\frac{3}{2}(2-0)^{2/3}\right)$$

$$= -\frac{3}{2}\cdot 0+\frac{3}{2}\cdot 2^{2/3} = \frac{3}{2}\cdot 2^{2/3}$$

and

$$\int\limits_2^3 \frac{1}{\sqrt[3]{2-x}}dx = \lim_{t\to 2^+}\int\limits_t^3 \frac{1}{\sqrt[3]{2-x}}dx = \lim_{t\to 2^+}\left[-\frac{3}{2}(2-x)^{2/3}\right]_t^3 = \lim_{t\to 2^+}\left(-\frac{3}{2}(2-3)^{2/3}+\frac{3}{2}(2-t)^{2/3}\right)$$

$$= -\frac{3}{2}\cdot 1+\frac{3}{2}\cdot 0 = -\frac{3}{2}$$

It follows that

$$\int\limits_0^3 \frac{1}{\sqrt[3]{2-x}}dx = \int\limits_0^2 \frac{1}{\sqrt[3]{2-x}}dx + \int\limits_2^3 \frac{1}{\sqrt[3]{2-x}}dx$$

$$= \frac{3}{2}\cdot 2^{2/3}-\frac{3}{2}$$

$$= \frac{3}{2}\left(2^{2/3}-1\right)$$

Solution (version 2): We have

$$\int_0^3 \frac{1}{\sqrt[3]{2-x}}dx = \int_0^2 \frac{1}{\sqrt[3]{2-x}}dx + \int_2^3 \frac{1}{\sqrt[3]{2-x}}dx$$

$$= \lim_{t\to 2^-} \int_0^t \frac{1}{\sqrt[3]{2-x}}dx + \lim_{t\to 2^+} \int_t^3 \frac{1}{\sqrt[3]{2-x}}dx = \begin{bmatrix} 2-x=u \\ d(2-x)=du \\ -dx=du \\ dx=-du \end{bmatrix}$$

$$= -\lim_{t\to 2^-} \int_2^{2-t} u^{-1/3}du - \lim_{t\to 2^+} \int_{2-t}^{-1} u^{-1/3}du$$

$$= -\lim_{t\to 2^-} \left[\frac{3}{2}u^{2/3}\right]_2^{2-t} - \lim_{t\to 2^+} \left[\frac{3}{2}u^{2/3}\right]_{2-t}^{-1}$$

$$= -\lim_{t\to 2^-} \left(\frac{3}{2}(2-t)^{2/3} - \frac{3}{2}\cdot 2^{2/3}\right) - \lim_{t\to 2^+} \left(\frac{3}{2}(-1)^{2/3} - \frac{3}{2}(2-t)^{2/3}\right)$$

$$= -\left(\frac{3}{2}\cdot 0 - \frac{3}{2}\cdot 2^{2/3}\right) - \left(\frac{3}{2}\cdot 1 - \frac{3}{2}\cdot 0\right)$$

$$= \frac{3}{2}\cdot 2^{2/3} - \frac{3}{2}$$

$$= \frac{3}{2}\left(2^{2/3}-1\right)$$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt[5]{x^4-x-1}}dx$

Solution: We have

$$0 < \frac{1}{\sqrt[5]{x^4}} < \frac{1}{\sqrt[5]{x^4-x-1}}$$

Note that $\displaystyle\int_2^\infty \frac{1}{\sqrt[5]{x^4}}dx$ is divergent by the p-test, since $p = 4/5 \le 1$. Therefore the integral $\displaystyle\int_2^\infty \frac{1}{\sqrt[5]{x^4-x-1}}dx$ diverges.