Calculus II - Spring 2014

Quiz #3, March 26, 2014

In the following problems you are required to show all your work and provide the necessary explanations everywhere to get full credit.

1. Find the arc length of the graph of $f(x) = \ln(\sin x)$ over $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$.

Solution: We have

$$f'(x) = (\ln(\sin x))' = \frac{1}{\sin x} \cdot (\sin x)' = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x}$$

and so the arc length formula gives

$$\begin{aligned}
L = \int_{\pi/4}^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\pi/4}^{\pi/2} \sqrt{1 + \frac{\cos^2 x}{\sin^2 x}} dx &= \int_{\pi/4}^{\pi/2} \sqrt{\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}} dx \\
&= \int_{\pi/4}^{\pi/2} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx \\
&= \int_{\pi/4}^{\pi/2} \sqrt{\frac{1}{\sin^2 x}} dx \\
&= \int_{\pi/4}^{\pi/2} \frac{1}{\sin x} dx \\
&= \int_{\pi/4}^{\pi/2} \csc x dx \\
&= \ln|\csc x - \cot x| \Big]_{\pi/4}^{\pi/2} \\
&= \ln\left(\csc\left(\frac{\pi}{2}\right) - \cot\left(\frac{\pi}{2}\right)\right) - \ln\left(\csc\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4}\right)\right) \\
&= \ln(1 - 0) - \ln\left(\sqrt{2} - 1\right) \\
&= -\ln\left(\sqrt{2} - 1\right)
\end{aligned}$$

This can be rewritten as $\ln\left(\sqrt{2} + 1\right)$, since

$$-\ln\left(\sqrt{2} - 1\right) = \ln\left(\sqrt{2} - 1\right)^{-1} = \ln\left(\frac{1}{\sqrt{2} - 1}\right) = \ln\left(\frac{1 \cdot (\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}\right)$$

$$= \ln\left(\frac{\sqrt{2} + 1}{(\sqrt{2})^2 - 1^2}\right) = \ln\left(\frac{\sqrt{2} + 1}{2 - 1}\right) = \ln\left(\frac{\sqrt{2} + 1}{1}\right) = \ln\left(\sqrt{2} + 1\right)$$

2. Solve the differential equation $\frac{dy}{dx} = \frac{y}{x^2}$. Then find the solution of this equation that satisfies the initial condition $y(1) = 3$.

Solution: If $y \neq 0$, we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = \frac{dx}{x^2}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x^2}$$

$$\ln|y| = -\frac{1}{x} + C_1$$

hence

$$y = \pm e^{-1/x+C_1}$$

We can easily verify that the function $y = 0$ is also a solution of the given differential equation. So we can write the general solution in the form

$$y = \pm e^{-1/x+C_1} = \pm e^{C_1} e^{-1/x} = Ce^{-1/x}$$

where C is an arbitrary constant. If we put $x = 1$ in the general solution, we get $y(1) = Ce^{-1}$. To satisfy the initial condition $y(1) = 3$, we must have $C = 3e$. Thus the solution of the initial-value problem is

$$y = 3e^{-1/x+1}$$

Appendix

Find $\displaystyle\int \csc x dx$.

Solution 1: We have

$$\int \csc x dx = \int \frac{dx}{\sin x} = \int \frac{dx}{\sin\left(2\cdot\frac{x}{2}\right)} = \begin{bmatrix} \frac{x}{2} = u \\ d\left(\frac{x}{2}\right) = du \\ \frac{1}{2}dx = du \\ dx = 2du \end{bmatrix} = \int \frac{2du}{\sin(2u)} = \int \frac{2du}{2\sin u\cos u} = \int \frac{du}{\sin u \cos u}$$

$$= \int \frac{\sec^2 u du}{\sec^2 u \sin u \cos u} = \int \frac{\sec^2 u du}{\frac{\sin u \cos u}{\cos^2 u}} = \int \frac{\sec^2 u du}{\frac{\sin u}{\cos u}} = \int \frac{\sec^2 u du}{\tan u} = \begin{bmatrix} \tan u = v \\ d\tan u = dv \\ \sec^2 u du = dv \end{bmatrix} = \int \frac{dv}{v}$$

$$= \ln|v| + C = \ln|\tan u| + C = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

In short,

$$\int \csc x dx = \int \frac{dx}{\sin x} = \int \frac{dx}{\sin\left(2\cdot\frac{x}{2}\right)} = \int \frac{dx}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} = \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{2\sec^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right)dx}{2\tan\left(\frac{x}{2}\right)} = \begin{bmatrix} \tan\left(\frac{x}{2}\right) = v \\ d\tan\left(\frac{x}{2}\right) = dv \\ \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dv \\ \sec^2\left(\frac{x}{2}\right)dx = 2dv \end{bmatrix} = \int \frac{dv}{v} = \ln|v| + C = \ln\left|\tan\left(\frac{x}{2}\right)\right| + C$$

This can be rewritten as $\ln|\csc x - \cot x| + C$, since

$$\ln\left|\tan\left(\frac{x}{2}\right)\right| = \ln\left|\frac{1-\cos x}{\sin x}\right| = \ln\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| = \ln|\csc x - \cot x|$$

Solution 2: We have

$$\int \csc x dx = \int \frac{1}{\sin x}dx = \int \frac{\sin x}{\sin^2 x}dx = \int \frac{\sin x}{1-\cos^2 x}dx = \begin{bmatrix} \cos x = u \\ d\cos x = du \\ -\sin x dx = du \\ \sin x dx = -du \end{bmatrix} = -\int \frac{1}{1-u^2}du$$

$$= -\int \frac{1}{(1-u)(1+u)}du = -\int \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du = -\frac{1}{2}\int \frac{1}{1-u}du - \frac{1}{2}\int \frac{1}{1+u}du$$

$$= \frac{1}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| + C = \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) + C$$

This can be rewritten as $\ln|\csc x - \cot x| + C$, since

$$\frac{1}{2}\ln\left(\frac{1-\cos x}{1+\cos x}\right) = \frac{1}{2}\ln\left(\frac{(1-\cos x)^2}{(1-\cos x)(1+\cos x)}\right) = \frac{1}{2}\ln\left(\frac{(1-\cos x)^2}{1-\cos^2 x}\right) = \frac{1}{2}\ln\left(\frac{(1-\cos x)^2}{\sin^2 x}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1-\cos x}{\sin x}\right)^2 = \frac{1}{2}\cdot 2\ln\left|\frac{1-\cos x}{\sin x}\right| = \ln\left|\frac{1-\cos x}{\sin x}\right|$$

$$= \ln\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| = \ln|\csc x - \cot x|$$

Solution 3: We have

$$\int \csc x dx = \int \csc x \frac{\csc x - \cot x}{\csc x - \cot x} dx = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} dx = \begin{bmatrix} \csc x - \cot x = u \\ d(\csc x - \cot x) = du \\ (-\csc x \cot x + \csc^2 x)dx = du \end{bmatrix}$$

$$= \int \frac{1}{u} du = \ln|u| + C = \ln|\csc x - \cot x| + C$$