Calculus II - Spring 2014

Quiz #4, April 16, 2014

In the following problems you are required to show all your work and provide the necessary explanations everywhere to get full credit.

1. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\sum_{k=1}^{\infty} \frac{k+1}{2k+5}$

Solution: The series $\sum_{k=1}^{\infty} \frac{k+1}{2k+5}$ diverges by the Divergence Test, since

$$\lim_{k\to\infty} \frac{k+1}{2k+5} = \lim_{k\to\infty} \frac{(k+1)'}{(2k+5)'} = \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$$

(b) $\sum_{k=3}^{\infty} \frac{1+2^{k-1}}{3^{2k+1}}$

Solution: We have

$$\begin{aligned}
\sum_{k=3}^{\infty} \frac{1+2^{k-1}}{3^{2k+1}} = \sum_{k=3}^{\infty} \frac{1}{3^{2k+1}} + \sum_{k=3}^{\infty} \frac{2^{k-1}}{3^{2k+1}} &= \sum_{k=1}^{\infty} \frac{1}{3^{2(k+2)+1}} + \sum_{k=1}^{\infty} \frac{2^{(k+2)-1}}{3^{2(k+2)+1}} \\
&= \sum_{k=1}^{\infty} \frac{1}{3^{2k+5}} + \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^{2k+5}} \\
&= \sum_{k=1}^{\infty} \frac{1}{3^{2k} \cdot 3^5} + \sum_{k=1}^{\infty} \frac{2^{k+1}}{3^{2k} \cdot 3^5} \\
&= \sum_{k=1}^{\infty} \frac{1}{9^k \cdot 3^5} + \sum_{k=1}^{\infty} \frac{2^{k+1}}{9^k \cdot 3^5} \\
&= \sum_{k=1}^{\infty} \frac{1}{9^{k-1} \cdot 9 \cdot 3^5} + \sum_{k=1}^{\infty} \frac{2^{k-1} \cdot 2^2}{9^{k-1} \cdot 9 \cdot 3^5} \\
&= \sum_{k=1}^{\infty} \frac{1}{9 \cdot 3^5} \left(\frac{1}{9}\right)^{k-1} + \sum_{k=1}^{\infty} \frac{2^2}{9 \cdot 3^5} \left(\frac{2}{9}\right)^{k-1} \\
&= \frac{\frac{1}{9 \cdot 3^5}}{1 - \frac{1}{9}} + \frac{\frac{2^2}{9 \cdot 3^5}}{1 - \frac{2}{9}} = \frac{13}{4536}
\end{aligned}$$

2. Determine whether the series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+3}}$

Solution 1: Put $a_n = \dfrac{1}{\sqrt[3]{2n+3}}$, $b_n = \dfrac{1}{\sqrt[3]{n}}$. Then

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{\sqrt[3]{2n+3}}}{\dfrac{1}{\sqrt[3]{n}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[3]{n}}{\sqrt[3]{2n+3}}$$

$$= \lim_{n\to\infty} \sqrt[3]{\frac{n}{2n+3}}$$

$$= \sqrt[3]{\lim_{n\to\infty} \frac{n}{2n+3}}$$

$$= \sqrt[3]{\lim_{n\to\infty} \frac{n'}{(2n+3)'}} = \sqrt[3]{\lim_{n\to\infty} \frac{1}{2}} = \sqrt[3]{\frac{1}{2}}$$

Since c is a finite positive number and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ diverges by the p-test with $p = 1/3$, it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+3}}$ also diverges by the Limit Comparison Test.

Solution 2: One can show (see below) that

$$\frac{1}{\sqrt[3]{2n+3}} > \frac{1}{2\sqrt[3]{n}} \quad \text{for} \quad n \geq 1 \tag{$*$}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} \left(\text{and therefore } \sum_{n=1}^{\infty} \frac{1}{2\sqrt[3]{n}} \right) \text{ diverges by the } p\text{-test with } p = 1/3$$

it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{2n+3}}$ also diverges by the Comparison Test. To prove $(*)$ we note that

$$\frac{1}{\sqrt[3]{2n+3}} > \frac{1}{2\sqrt[3]{n}} \iff 2\sqrt[3]{n} > \sqrt[3]{2n+3}$$

$$\iff 8n > 2n+3$$

$$\iff 6n > 3$$

which is true, since $n \geq 1$.

Solution 3 (version 1): The function $f(x) = \dfrac{1}{\sqrt[3]{2x+3}}$ is continuous, positive and decreasing on $[1, \infty)$, therefore we can apply the Integral Test. We have

$$\int_1^\infty \frac{1}{\sqrt[3]{2x+3}}dx = \lim_{t\to\infty}\int_1^t (2x+3)^{-1/3}dx = \begin{bmatrix} 2x+3=u \\ d(2x+3)=du \\ 2dx=du \\ dx=\frac{1}{2}du \end{bmatrix} = \lim_{t\to\infty}\frac{1}{2}\int_{2\cdot 1+3}^{2t+3} u^{-1/3}du = \frac{1}{2}\int_5^\infty u^{-1/3}du$$

The integral $\int_5^\infty u^{-1/3}du$ diverges by the p-test, since $p = 1/3 \le 1$. Therefore the series $\sum_{n=1}^\infty \frac{1}{\sqrt[3]{2n+3}}$ also diverges.

Solution 3 (version 2): The function $f(x) = \dfrac{1}{\sqrt[3]{2x+3}}$ is continuous, positive and decreasing on $[1, \infty)$, therefore we can apply the Integral Test. We have

$$\int_1^\infty \frac{1}{\sqrt[3]{2x+3}}dx = \lim_{t\to\infty}\int_1^t (2x+3)^{-1/3}dx = \begin{bmatrix} 2x+3=u \\ d(2x+3)=du \\ 2dx=du \\ dx=\frac{1}{2}du \end{bmatrix} = \lim_{t\to\infty}\frac{1}{2}\int_{2\cdot 1+3}^{2t+3} u^{-1/3}du$$

$$= \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{u^{-1/3+1}}{-1/3+1}\right]_5^{2t+3} = \lim_{t\to\infty}\left[\frac{1}{2}\cdot\frac{u^{2/3}}{2/3}\right]_5^{2t+3} = \lim_{t\to\infty}\left[\frac{3}{4}u^{2/3}\right]_5^{2t+3} = \lim_{t\to\infty}\left[\frac{3}{4}(2t+3)^{2/3} - \frac{3}{4}\cdot 5^{2/3}\right] = \infty$$

Since this integral diverges, the series $\sum_{n=1}^\infty \frac{1}{\sqrt[3]{2n+3}}$ also diverges.

Solution 3 (version 3): The function $f(x) = \dfrac{1}{\sqrt[3]{2x+3}}$ is continuous, positive and decreasing on $[1, \infty)$, therefore we can apply the Integral Test. First we note that

$$\int \frac{1}{\sqrt[3]{2x+3}}dx = \int (2x+3)^{-1/3}dx = \begin{bmatrix} 2x+3=u \\ d(2x+3)=du \\ 2dx=du \\ dx=\frac{1}{2}du \end{bmatrix} = \frac{1}{2}\int u^{-1/3}du$$

$$= \frac{1}{2}\cdot\frac{u^{-1/3+1}}{-1/3+1} + C = \frac{1}{2}\cdot\frac{u^{2/3}}{2/3} + C = \frac{3}{4}u^{2/3} + C = \frac{3}{4}(2x+3)^{2/3} + C$$

therefore

$$\int_1^\infty \frac{1}{\sqrt[3]{2x+3}}dx = \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt[3]{2x+3}}dx = \lim_{t\to\infty}\left[\frac{3}{4}(2x+3)^{2/3}\right]_1^t = \lim_{t\to\infty}\left[\frac{3}{4}(2t+3)^{2/3} - \frac{3}{4}(2\cdot 1+3)^{2/3}\right] = \infty$$

Since this integral diverges, the series $\sum_{n=1}^\infty \frac{1}{\sqrt[3]{2n+3}}$ also diverges.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^5 n}$

Solution (version 1): The function $f(x) = \dfrac{1}{x \ln^5 x}$ is continuous, positive and decreasing on $[2, \infty)$, therefore we can apply the Integral Test. We have

$$\int_2^{\infty} \frac{1}{x \ln^5 x} dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \ln^5 x} dx = \begin{bmatrix} \ln x = u \\ d(\ln x) = du \\ \frac{1}{x} dx = du \end{bmatrix} = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u^5} du = \int_{\ln 2}^{\infty} \frac{1}{u^5} du$$

The integral $\displaystyle\int_{\ln 2}^{\infty} \frac{1}{u^5} du$ converges by the p-test, since $p = 5 > 1$. Therefore the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^5 n}$ also converges.

Solution (version 2): The function $f(x) = \dfrac{1}{x \ln^5 x}$ is continuous, positive and decreasing on $[2, \infty)$, therefore we can apply the Integral Test. We have

$$\int_2^{\infty} \frac{1}{x \ln^5 x} dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \ln^5 x} dx = \begin{bmatrix} \ln x = u \\ d(\ln x) = du \\ \frac{1}{x} dx = du \end{bmatrix} = \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u^5} du = \lim_{t \to \infty} \left[\frac{u^{-5+1}}{-5+1} \right]_{\ln 2}^{\ln t}$$

$$= \lim_{t \to \infty} \left[-\frac{u^{-4}}{4} \right]_{\ln 2}^{\ln t} = \lim_{t \to \infty} \left[-\frac{1}{4u^4} \right]_{\ln 2}^{\ln t} = \lim_{t \to \infty} \left[-\frac{1}{4 \ln^4 t} + \frac{1}{4 \ln^4 2} \right] = \frac{1}{4 \ln^4 2}$$

Since this integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^5 n}$ also converges.

Solution (version 3): The function $f(x) = \dfrac{1}{x \ln^5 x}$ is continuous, positive and decreasing on $[2, \infty)$, therefore we can apply the Integral Test. First we note that

$$\int \frac{1}{x \ln^5 x} dx = \begin{bmatrix} \ln x = u \\ d(\ln x) = du \\ \frac{1}{x} dx = du \end{bmatrix} = \int \frac{1}{u^5} du = \frac{u^{-5+1}}{-5+1} + C = -\frac{u^{-4}}{4} + C = -\frac{1}{4u^4} + C = -\frac{1}{4 \ln^4 x} + C$$

therefore

$$\int_2^{\infty} \frac{1}{x \ln^5 x} dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \ln^5 x} dx = \lim_{t \to \infty} \left[-\frac{1}{4 \ln^4 x} \right]_2^t = \lim_{t \to \infty} \left[-\frac{1}{4 \ln^4 t} + \frac{1}{4 \ln^4 2} \right] = \frac{1}{4 \ln^4 2}$$

Since this integral converges, the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln^5 n}$ also converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 1 + \sin n}$

Solution: Since $\dfrac{1}{n^2 + 1 + \sin n} \leq \dfrac{1}{n^2 + 1 + (-1)} = \dfrac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test with $p = 2$, it follows that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 1 + \sin n}$ also converges by the Comparison Test.