On approximation of quadratic irrationals by rational numbers

Let $\xi = [0, \overline{a_1, \ldots, a_n}]$ be a quadratic irrational with discriminant D and let p_i/q_i be its ith convergent. Put

$$K_i = \left| \xi - \frac{p_i}{q_i} \right| q_i^2.$$

We first suppose that $n = 1$, i.e. $\xi = [0, \overline{a_1}]$.

STATEMENT 1. *We have:*

$$\lim_{i \to \infty} K_i = \frac{1}{\sqrt{D}}.$$

Moreover, $K_1 < K_3 < K_5 < \ldots$ *and* $K_2 > K_4 > K_6 > \ldots$.

We now generalize Statement 1 to the case $n > 1$. Let r be an integer with $0 \leq r \leq n-1$. We define n-tuples $(a_1^{(r)}, \ldots, a_{n-1}^{(r)})$ in the following way:

$$(a_1^{(0)}, \ldots, a_{n-1}^{(0)}) = (a_2, a_3, \ldots, a_{n-1}, a_n)$$
$$(a_1^{(1)}, \ldots, a_{n-1}^{(1)}) = (a_1, a_2, \ldots, a_{n-2}, a_{n-1})$$
$$(a_1^{(2)}, \ldots, a_{n-1}^{(2)}) = (a_n, a_1, \ldots, a_{n-3}, a_{n-2})$$
$$(a_1^{(3)}, \ldots, a_{n-1}^{(3)}) = (a_{n-1}, a_n, \ldots, a_{n-4}, a_{n-3})$$
$$\ldots$$
$$(a_1^{(n-1)}, \ldots, a_{n-1}^{(n-1)}) = (a_3, a_4, \ldots, a_n, a_1).$$

Denote by $N(r)$ the numerator of the continued fraction $[a_1^{(r)}, \ldots, a_{n-1}^{(r)}]$.

STATEMENT 2. *If n is even, then*

$$N(0)a_1 + \sum_{r=1}^{n-1} (-1)^r N(r) a_{n-r+1} = 0.$$

STATEMENT 3. *If n is even, $i > n$ and $i \equiv 0$ or $n-1 \mod n$, then*

$$N(1) q_{2i-n} = q_i q_{i-1} - q_{i-n} q_{i-n-1}.$$

Put

$$G = \gcd(N(0), \ldots, N(n-1)), \quad F(k,r) = \frac{C(k) N(0)^k N(r)^{k+1}}{G^{2k+1} D^k \sqrt{D}},$$

where $C(k) = \dfrac{(2k)!}{k!(k+1)!}$ are the Catalan numbers.

STATEMENT 4. *For any r with $0 \leq r \leq n-1$ we have*

$$\lim_{j \to \infty} K_{nj+r} = F(0, r).$$

Moreover, a subsequence of $\{K_{nj+r}\}$ with odd indexes (if any) is increasing and a subsequence of $\{K_{nj+r}\}$ with even indexes (if any) is decreasing.

COROLLARY 5. *If n is even, then*

$$\lim_{j \to \infty} \left(K_{nj} a_1 + \sum_{r=1}^{n-1} (-1)^r K_{nj+r} a_{n-r+1} \right) = 0.$$

STATEMENT 6. *Let r be an integer as in Statement 4. Let also i be an odd integer ≥ 1 and let m be an integer ≥ 0. Suppose $i \equiv r \mod n$. Then*

$$\sum_{k=0}^{2m+1} (-1)^k \frac{F(k,r)}{q_i^{2k+2}} < \frac{p_i}{q_i} - \xi < \sum_{k=0}^{2m} (-1)^k \frac{F(k,r)}{q_i^{2k+2}}.$$

COROLLARY 7. *Let i, m and r be integers as in Statement 6. Then*

$$\left| \xi - \frac{p_i}{q_i} - \sum_{k=0}^{m} (-1)^{k+1} \frac{F(k,r)}{q_i^{2k+2}} \right| < \frac{F(m+1,r)}{q_i^{2m+4}}.$$