On approximation of real numbers by algebraic numbers of bounded degree

BY

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Theorem 1 (Dirichlet, 1842). For any real irrational number ξ there exist infinitely many rational numbers p/q such that

$$\left|\xi - \frac{p}{q}\right| < q^{-2}.$$

$$\Downarrow$$

$$|q\xi - p| < q^{-1}.$$

Theorem 2. For any real irrational number ξ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of the first degree such that

$$|P(\xi)| \ll \lceil P \rceil^{-1},$$

where $\lceil P \rceil$ denotes the height of the polynomial P, that is the maximum of absolute values of its coefficients, $\ll$ is the Vinogradov symbol.

Theorem 3. For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P \rceil^{-n},$$

where $\mathbb{A}_n$ is the set of real algebraic numbers of degree $\leq n$.

$$\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll\ ? & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}$$

Conjecture (Wirsing, 1961). For any real number $\xi \notin \mathbb{A}_n$, there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-n-1+\epsilon}, \quad \epsilon > 0,$$

where $H(\alpha)$ is the height of α.

Further W. M. Schmidt conjectured that the exponent

$$-n - 1 + \epsilon$$

can be replaced even by

$$-n - 1.$$

$$\begin{array}{ccc}
\left|\xi - \dfrac{p}{q}\right| < q^{-2} & \longrightarrow & |q\xi - p| < q^{-1} \\
 & & \downarrow \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \longleftarrow & |P(\xi)| \ll \lceil P \rceil^{-n}
\end{array}$$

At the moment this Conjecture is proved only for

$n = 1 \quad \Rightarrow \quad |\xi - \alpha| \ll H(\alpha)^{-2}$ (Dirichlet, 1842)

$n = 2 \quad \Rightarrow \quad |\xi - \alpha| \ll H(\alpha)^{-3}$ (Davenport – Schmidt, 1967)

$n > 2 \quad \Rightarrow$???

Consider the polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0 = a_n (x-\alpha_1)\cdot \ldots \cdot (x-\alpha_n).$$

Without loss of generality we can assume that α_1 is the root of $P(x)$ closest to ξ. We have

$$|\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}.$$

By Theorem 3 there are infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that

$$|P(\xi)| \ll \lceil P\rceil^{-n}.$$

Let $n = 1$. Then

$$|P'(\xi)| = |a_1| \asymp \lceil P\rceil \quad \Rightarrow \quad \boxed{|\xi - \alpha_1| \ll \frac{\lceil P\rceil^{-1}}{\lceil P\rceil} = \lceil P\rceil^{-2}}$$

Let $n = 2$. Then for some $\delta \leq 1$

$$|P'(\xi)| = |2a_2\xi + a_1| \asymp \lceil P\rceil^{\delta} \quad \Rightarrow \quad \boxed{|\xi - \alpha_1| \ll \frac{\lceil P\rceil^{-2}}{\lceil P\rceil^{\delta}} = \lceil P\rceil^{-2-\delta} \ ???}$$

<u>QUESTION</u>: Can one prove that the following is impossible:

All polynomials with $|P(\xi)| \ll \lceil P\rceil^{-n}$ have

a "small" derivative $|P'(\xi)| \asymp \lceil P\rceil^{\delta},\ \delta < 1.$

<u>ANSWER</u>: Yes, for n=1 (Dirichlet, 1842)

n=2 (Davenport – Schmidt, 1967)

THEOREM 4 (Wirsing, 1961). For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\gamma_n}, \quad \lim_{n\to\infty} \gamma_n = 2.$$

By Dirichlet's Box principle there are infinitely polynomials

$$P(x) = a_n(x-\alpha_1)\cdot \ldots \cdot (x-\alpha_n)$$

such that

$$|P(\xi)| \ll \lceil P \rceil^{-n}, \tag{$*$}$$

therefore

$$|\xi-\alpha_1|\cdot \ldots \cdot |\xi-\alpha_n| \ll \lceil P \rceil^{-n} a_n^{-1}.$$

Even if $a_n = \lceil P \rceil$, we can only prove that

$$|\xi-\alpha_1|\cdot \ldots \cdot |\xi-\alpha_n| \ll \lceil P \rceil^{-n-1} \ll H(\alpha_1)^{-n-1},$$

$$\Downarrow$$

$$\boxed{|\xi-\alpha_1| \ll H(\alpha_1)^{-\frac{n+1}{n}} \text{ ???}}$$

It is also clear, that the worse case for us is when

$$|\xi-\alpha_1| = \ldots = |\xi-\alpha_n|. \tag{$**$}$$

QUESTION: Can one prove that for infinitely many polynomials $P \in \mathbb{Z}[x]$ with $(*)$ the situation $(**)$ is impossible?

ANSWER: For infinitely many polynomials $P \in \mathbb{Z}[x]$ with $(*)$ we have:

$|\xi-\alpha_1| \ll |\xi-\alpha_2| \ll 1,$

$|\xi-\alpha_3|, \ldots, |\xi-\alpha_n|$ are "big".

THEOREM 5 (Bernik-Tsishchanka, 1993). For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\gamma_n}, \quad \lim_{n\to\infty} \gamma_n = 3.$$

THEOREM 6. For any real number $\xi \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\gamma_n}, \quad \lim_{n\to\infty} \gamma_n = 4.$$

The following table contains the values of

$$\frac{n}{2} + \gamma_n$$

corresponding to Wirsing's Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), Theorem 6 (2001), and the Conjecture (Conj.):

n	1961	1993	2001	Conj.
3	3.28	3.5	3.73	4
4	3.82	4.12	4.45	5
5	4.35	4.71	5.14	6
6	4.87	5.28	5.76	7
7	5.39	5.84	6.36	8
8	5.9	6.39	6.93	9
9	6.41	6.93	7.50	10
10	6.92	7.47	8.06	11
15	9.44	10.09	10.77	16
20	11.95	12.67	13.40	21
50	26.98	27.84	28.70	51
100	51.99	52.92	53.84	101

OPEN QUESTION. For any real number $\xi \notin \mathbb{A}_n$, there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}-\gamma_n}, \quad \lim_{n\to\infty} \gamma_n = \infty.$$

Consider the sequence of polynomials $P_i \in \mathbb{Z}[x]$ of degree $\le n$ such that

(*i*) $|P_1(\xi)| > |P_2(\xi)| > \ldots > |P_i(\xi)| > \ldots,$

(*ii*) $\overline{P_1}| < \overline{P_2}| < \ldots < \overline{P_i}| < \ldots,$

(*iii*) for any $i > 1$ the polynomial P_i has the smallest height among other polynomials P with $|P(\xi)| < |P_{i-1}(\xi)|$.

LEMMA. For any $i \ge 1$ we have:

$$|P_i(\xi)| \ll \overline{P_{i+1}}|^{-n}.$$

THEOREM 7. Let $n = 3$. Suppose that for any $i \ge 1$ the following estimation holds:

$$|P_i(\xi)| \ll \overline{P_{i+1}}|^{-1}\overline{P_{i+2}}|^{-2}.$$

Then the Conjecture is true, that is for any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-4}.$$

THEOREM 8. For any real number $\xi \notin \mathbb{A}_3$ there exist infinitely many algebraic numbers $\alpha \in \mathbb{A}_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-3.7475}.$$

OPEN QUESTION. Let $n = 3$. Suppose that for any $i \ge 1$ the following estimation holds:

$$|P_i(\xi)| \ll \overline{P_{i+1}}|^{-2}\overline{P_{i+2}}|^{-1}.$$

Prove that the Conjecture is true.

Conjecture. For any two real numbers $\xi_1,\ \xi_2 \notin \mathbb{A}_n$ there exist infinitely many algebraic numbers $\alpha_1,\ \alpha_2$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-(n+1)/2}, \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-(n+1)/2}, \end{cases}$$

where $P(x) \in \mathbb{Z}[x],\ \ P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \le n$. The implicit constant in $\ll$ should depend on $\xi_1,\ \xi_2$, and n.

One can prove it for almost all real numbers. However, this is not true for all real numbers. In 2001 D. Roy and M. Waldschmidt constructed the following

Counter-example (Roy-Waldschmidt, 2001). For any $\epsilon > 0$ and any sufficiently large n there exist real numbers ξ_1 and ξ_2 such that

$$\max\{|\xi_1 - \alpha_1|, |\xi_2 - \alpha_2|\} > \lceil P \rceil^{-\sqrt{n^{1+6\epsilon}}}.$$

At the same time we can prove the following

Theorem 9. For any real numbers $\xi_1,\ \xi_2 \notin \mathbb{A}_n$ at least one of the following assertions is true:

(i) There exist infinitely many algebraic numbers $\alpha_1,\ \alpha_2$ of degree $\le n$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll \lceil P \rceil^{-\frac{n}{8}-\frac{3}{8}}, \\ |\xi_2 - \alpha_2| \ll \lceil P \rceil^{-\frac{n}{8}-\frac{3}{8}}. \end{cases}$$

(ii) For some $\xi \in \{\xi_1,\ \xi_2\}$ there exist infinitely many algebraic numbers α of degree $2 \le k \le \frac{n+2}{4}$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n+2}{4}-1}.$$