

On hyperbolicity in the renormalization of near-critical area-preserving maps

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Abstract. We consider MacKay’s renormalization operator for pairs of area-preserving maps, near the fixed point obtained in [17]. Of particular interest is the restriction \mathfrak{R}_0 of this operator to pairs that commute and have a zero Calabi invariant. We prove that a suitable extension of \mathfrak{R}_0^3 is hyperbolic at the fixed point, with a single expanding direction. The pairs in this direction are presumably commuting, but we currently have no proof for this. Our analysis yields rigorous bounds on various “universal” quantities, including the expanding eigenvalue.

1. Introduction

We consider the operator $\langle F, G \rangle \mapsto \langle G, FG \rangle$ acting equivalence classes of pairs of area-preserving maps of the plane. A pair (F', G') belongs to the class $\langle F, G \rangle$ if $F' = \Lambda^{-1}F\Lambda$ and $G' = \Lambda^{-1}G\Lambda$, for some map $\Lambda : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose derivative has a constant determinant. For a concrete analysis, it is convenient to represent each equivalence class by a suitable member $P = (F, G)$. Restricting the similarity transformations Λ to a specific subgroup, we end up with the following operator \mathfrak{R} ,

$$\mathfrak{R}(P) = (\Lambda^{-1}G\Lambda, \Lambda^{-1}FG\Lambda), \quad \Lambda \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \lambda x + \varsigma \\ \mu z + \varpi(x) \end{bmatrix}. \quad (1.1)$$

Here λ, μ, ς are real numbers and ϖ is a polynomial of degree ≤ 3 . These quantities can be chosen to depend suitably on the pair P .

This operator \mathfrak{R} has been studied first in connection with the breakup of golden invariant circles [3,4,5]. Denote by \mathfrak{R}_0 the restriction of \mathfrak{R} to pairs that commute and have a zero Calabi invariant. The conjecture is that \mathfrak{R}_0 has a “critical” hyperbolic fixed point. More details will be given below. The existence of the fixed point was proved in [17]. Here we address the question of hyperbolicity. For technical reasons, we consider the third iterate of \mathfrak{R}_0 . In fact, we first extend \mathfrak{R}_0^3 to a dynamical system on suitable manifold of map-pairs. Our main result is that the chosen extension \mathfrak{M}' is hyperbolic near the critical fixed point P_* , with the derivative of \mathfrak{M}' at P_* having a simple eigenvalue δ^3 ,

$$\delta = 1.62795006498458161676240425734986 \dots, \quad (1.2)$$

and no other spectrum outside the open unit disk.

This number δ is a quantity that is observed numerically during the breakup of golden invariant circles in one-parameter families of maps of the two-dimensional cylinder. As a concrete example, consider Chirikov’s “standard family”, given by

$$\begin{bmatrix} x \\ z \end{bmatrix} \xrightarrow{F_\sigma} \begin{bmatrix} x-1 \\ z \end{bmatrix}, \quad \begin{bmatrix} x \\ z \end{bmatrix} \xrightarrow{G_\sigma} \begin{bmatrix} x+w \\ w \end{bmatrix}, \quad w = z - \sigma \sin(2\pi x). \quad (1.3)$$

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Notice that F_σ and G_σ commute. The orbits of F_σ define a cylinder, and G_σ defines a map of this cylinder. For $\sigma = 0$, this map has a smooth invariant circle for any given rotation number $\gamma \in \mathbb{R}$. By KAM theory, if γ is sufficiently irrational, then the invariant circle persists under small changes of σ . Here we are interested in the “most irrational” rotation number, namely the inverse golden mean $\gamma = \frac{1}{2}\sqrt{5} - \frac{1}{2}$. Numerically, the corresponding invariant circle is observed to persist as σ is increased, up to some value σ_∞ where the circle breaks up. As σ is increased further, one immediately passes through an infinite number of bifurcations involving periodic orbits whose rotation numbers are the continued fraction approximants $r_1 = \frac{1}{2}$, $r_2 = \frac{2}{3}$, $r_3 = \frac{3}{5}$, $r_4 = \frac{5}{8}$, \dots of the number γ . Namely, at values σ_n that approach σ_∞ from above as $n \rightarrow \infty$, the r_n -periodic orbits for $P_\sigma = (F_\sigma, G_\sigma)$ change stability, from elliptic to hyperbolic. By a $\frac{p}{q}$ -periodic point for a commuting pair (F, G) we mean a fixed point for $F^p G^q$. Interestingly, the ratio $(\sigma_n - \sigma_{n-1})/(\sigma_{n+1} - \sigma_n)$ is observed to have a limit as $n \rightarrow \infty$, namely the number δ . Even more interestingly, the value of δ is “universal”, in the sense that a large class of cylinder-map families exhibit the same phenomenon, with the exact same value of δ . Two other universal numbers λ_* and μ_* describe the accumulation of r_n -periodic orbits at the critical golden torus, for the pair P_σ with $\sigma = \sigma_\infty$.

The standard framework (coming from statistical mechanics) for studying this type of universal behavior involves renormalization. The renormalization operator \mathfrak{R} defined in (1.1) was proposed and studied first in [3,4,5]. What makes it relevant to the problem at hand is the fact that if ω is a r_n -periodic point for P , then $\Lambda^{-1}\omega$ is a r_{n-1} -periodic point for $\mathfrak{R}(P)$. The standard explanation of the above-mentioned observations involves the existence of a fixed point P_* for \mathfrak{R}_0 , with the property that the linearization of \mathfrak{R}_0 at P_* has an eigenvalue δ and no other spectrum in the open unit disk. Furthermore, the scaling Λ_* associated with P_* should be conjugate to $\text{diag}(\lambda_*, \mu_*)$.

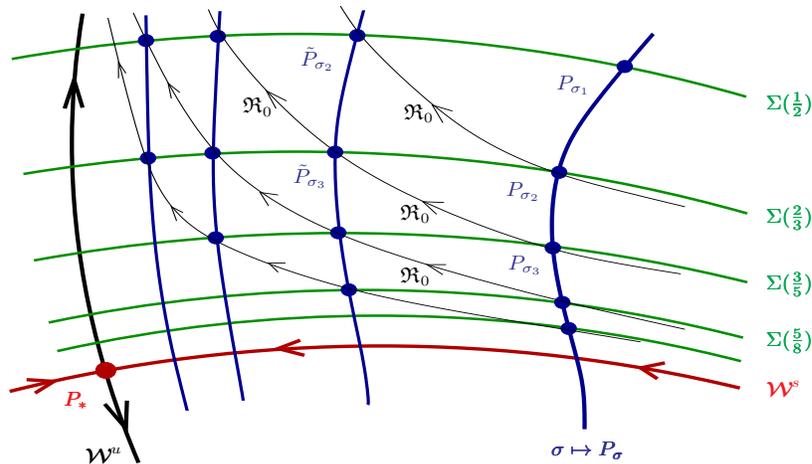


Figure 1. Expected renormalization picture. Here $\tilde{P} = \mathfrak{R}_0(P)$.

Figure 1 depicts the expected action of \mathfrak{R}_0 , assuming that (some extension of) this operator acts differentiably on a manifold of map-pairs: A family $\sigma \mapsto P_\sigma$ intersects the

“critical surface” \mathcal{W}^s at the parameter value $\sigma = \sigma_\infty$ where the golden torus breaks up. \mathcal{W}^s is the codimension-1 stable manifold of \mathfrak{R}_0 at the critical fixed point P_* . The “bifurcation surfaces” $\Sigma(r_n)$ accumulate at \mathcal{W}^s with an asymptotic rate $\propto \delta^{-n}$.

We start our analysis by discussing the fixed point problem for \mathfrak{R} . A proof for the existence of a fixed point P_* whose scaling Λ_* is consistent with numerical observations was given already in [17]. But our analysis of the operator \mathfrak{M}' requires domains that are different from the ones considered in [17], so we first need prove a similar result:

Theorem 1.1. *There exist two commuting area-preserving maps $F_* : D_F \rightarrow \mathbb{R}$ and $G_* : D_G \rightarrow \mathbb{R}$, such that the pair $P_* = (F_*, G_*)$ is a fixed point of \mathfrak{R} with scaling*

$$\Lambda_* \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} \lambda_* x \\ \mu_* z \end{bmatrix}, \quad \begin{aligned} \lambda_* &= -0.706795669179637278164917314160335\dots, \\ \mu_* &= -0.326063396625001485308122063586433\dots \end{aligned} \quad (1.4)$$

The domains D_F and D_G are open and connected and satisfy the relations (1.5) below. The maps F_* and G_* are real analytic and satisfy a twist condition. Furthermore, they are reversible with respect to the reflection $\mathcal{S} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} -x \\ z \end{bmatrix}$, in the sense that $\mathcal{S}F_*\mathcal{S} = F_*^{-1}$ and $\mathcal{S}G_*\mathcal{S} = G_*^{-1}$.

Our proof of this theorem involves estimates that have been carried out by a computer. As a by-product, we obtain highly accurate bounds on the various quantities involved [23]. In particular, we determine enclosures $\lambda_\diamond \ni \lambda_*$ and $\mu_\diamond \ni \mu_*$ that are open intervals of diameter less than 2^{-253} . Denote by Λ_\diamond the set of all 2×2 matrices $\Lambda = \text{diag}(\lambda, \mu)$ with $\lambda \in \lambda_\diamond$ and $\mu \in \mu_\diamond$. Then the domains D_F and D_G mentioned in Theorem 1.1 satisfy the relations

$$\Lambda_\diamond D_F \subset D_G, \quad \Lambda_\diamond D_G \subset D_G, \quad G\Lambda_\diamond D_G \subset D_F, \quad (1.5)$$

for $G = G_*$. The conditions (1.5) ensure that, if two continuous maps F and G are well-defined on D_F and D_G , respectively, then so are the renormalized maps $\Lambda^{-1}G\Lambda$ and $\Lambda^{-1}FG\Lambda$, respectively, if Λ is sufficiently close to Λ_* . Additional conditions with stronger implication are discussed in [10].

Since our proof of Theorem 1.1 is very indirect, let us describe some of the direct information that we have about the fixed point P_* . As in [17], the map G_* is obtained as a fixed point of the operator \mathfrak{R} defined by the equation $\mathfrak{R}(G) = \Lambda^{-1}FG_0\Lambda$, where $G_0 = \Lambda^{-1}FG\Lambda$ and $F = \Lambda^{-1}G\Lambda$. The map G is normalized by fixing the value $G(0)$, and the scaling $\Lambda = \text{diag}(\lambda, \mu)$ is then determined by the condition $G_0(0) = G(0)$. To be more precise, we solve the equation $\mathfrak{R}(G) = G$ for maps by solving the corresponding equation $\mathcal{N}(g) = g$ for generating functions.

A map G is obtained from its generating function g via the equation

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x \\ -g_1(x, y) \end{bmatrix} \xrightarrow{G} \begin{bmatrix} y \\ g_2(x, y) \end{bmatrix} = \begin{bmatrix} y \\ w \end{bmatrix}, \quad (1.6)$$

where $g_j = \partial_j g$. Assuming that g satisfies a twist condition $g_{1,2} > 0$, the corresponding map G is uniquely defined. Since the one-form $w dy - z dx$ is the differential of g and thus closed, this map G is area-preserving. Domains will be specified later. Concerning planar

domains, recall that our maps G are in fact lifts of cylinder-maps. Say $P = (F, G)$ is a commuting pair, with F defining a cylinder Z_F . Then the one-form dg may not be closed on Z_F . In that case g defines a multi-valued function on $Z_F \times Z_F$. The values at any given point differ by integer multiples of a fixed constant, namely the Calabi invariant of the cylinder map [11]. An explicit expression for the Calabi invariant $C(P)$ of a commuting pair P will be given in Subsection 3.1. If $C(P)$ is nonzero then P has a “net motion” in the non-periodic direction of the cylinder. This is a trivial obstruction to the existence of homotopically nontrivial invariant circles; so we are mostly interested in cylinder maps with zero Calabi invariant. This includes e.g. all reversible pairs.

Notice that the operator \mathfrak{R} preserves reversibility. It is not hard to show that, if G is a reversible fixed point of \mathfrak{R} , then $G_0^{-1}G$ is the identity map near the origin [17]. So modulo domain questions, a reversible fixed point of \mathfrak{R} yields a fixed point for \mathfrak{R} . Notice also that, if (F_*, G_*) is a reversible fixed point of \mathfrak{R} , then F_* and G_* commute.

Theorem 1.1 is proved by finding a fixed point g_* for \mathcal{N} , defining the associated map G_* via (1.6), and setting $F_* = \Lambda_*^{-1}G_*\Lambda_*$. The resulting domains \mathbb{D}_G and \mathbb{D}_F for G_* and F_* will be described in Section 4. The domain of F_*G_* is defined as usual as $\mathbb{D}_{FG} = G_*^{-1}(\mathbb{D}_F \cap G_*\mathbb{D}_G)$. Similarly for the domain \mathbb{D}_{GF} of G_*F_* . When saying that F_* and G_* commute, we mean that F_*G_* and G_*F_* agree on the intersection $\mathbb{D}_{FG} \cap \mathbb{D}_{GF}$. This intersection is shown to be non-empty.

Whenever two injective maps F and G commute, we can define an extension \overline{G} of G in such a way that $\overline{G}F$ is defined on the domain of FG and satisfies $\overline{G}F = FG$ on this domain. The map F can be extended similarly. This procedure is used to extend F_* and G_* to the larger domains $D_F = \mathbb{D}_F \cup \mathbb{D}_G$ and $D_G = \mathbb{D}_G \cup \Lambda_\diamond\mathbb{D}_G$. These domains are then shown to satisfy the relations (1.5).

A related fixed point problem $\mathfrak{M}(P) = P$ was considered by Stirnemann in [12], where \mathfrak{M} is a “palindromic” modification of the third iterate of \mathfrak{R} ,

$$\begin{aligned} \mathfrak{R}^3(P) &= (\Lambda^{-3}GFG\Lambda^3, \Lambda^{-3}FGGFG\Lambda^3), \\ \mathfrak{M}(P) &= (\Lambda^{-3}GFG\Lambda^3, \Lambda^{-3}GFGFG\Lambda^3). \end{aligned} \tag{1.7}$$

A useful feature of \mathfrak{M} is that it preserves reversibility with respect to \mathcal{S} . It was shown in [12] that the corresponding fixed point equation for a reduced generating function has a solution. What was left open is the question of whether the corresponding maps F and G commute (which, modulo domain questions, would yield a fixed point for \mathfrak{R}^3), and whether they are area-preserving.

We have tried to work with the operator \mathfrak{R} directly, but we were unable to find appropriate function spaces for such an analysis. The problem is that \mathfrak{R} is defined only for map-pairs whose generating functions have relatively large and/or unwieldy domains. This led us to consider \mathfrak{R}^3 and \mathfrak{M} . Fortunately, we were able to find domains that work simultaneously for both \mathfrak{R} and \mathfrak{M} . As a result, we have a fixed point P_* for \mathfrak{M} that we already know is a commuting fixed point for \mathfrak{R} . And since we are interested only in commuting pairs, there is no loss in working with \mathfrak{M} instead of \mathfrak{R}^3 . For commuting pairs with zero Calabi invariant, Theorem 1.2 below implies that \mathfrak{R}^3 has at most one unstable direction at P_* , and that all other directions are exponentially attracting.

If \mathfrak{B} is a set of pairs of area-preserving maps, and if Ω a subset of \mathbb{R}^2 , we say that Ω is a *commutator domain* for \mathfrak{B} if the composed maps FG and GF are defined on Ω , for every pair $P = (F, G)$ belonging to \mathfrak{B} . For such a domain Ω we define $\mathfrak{B}^c(\Omega)$ to be the set of all pairs $P \in \mathfrak{B}$ with the property that FG and GF agree when restricted to Ω , and that $C(P) = 0$.

Theorem 1.2. *There exists a Banach manifold \mathfrak{B} of pairs $P = (F, G)$ of real-analytic area-preserving maps $F : \mathbb{D}_F \rightarrow \mathbb{R}^2$ and $G : \mathbb{D}_G \rightarrow \mathbb{R}^2$, a non-empty open commutator domain Ω for \mathfrak{B} , an open set $\mathfrak{B}_0 \subset \mathfrak{B}$ containing the pair P_* described in Theorem 1.1, and a differentiable map $\mathfrak{M}' : \mathfrak{B}_0 \rightarrow \mathfrak{B}$ with the following properties. The derivative of \mathfrak{M}' at P_* has an eigenvalue δ^3 satisfying (1.2), and no other spectrum outside the open unit disk. If $P \in \mathfrak{B}_0$ belongs to $\mathfrak{B}^c(\Omega)$ then so does $\mathfrak{M}'(P)$, and $\mathfrak{M}'(P) = \mathfrak{M}(P)$. Furthermore, the local unstable manifold of \mathfrak{M}' is included in $\mathfrak{B}^c(\Omega)$, unless P_* is an exponentially attracting fixed point for the restriction of \mathfrak{M} to $\mathfrak{B}^c(\Omega)$.*

Our proof of Theorem 1.2 involves estimates that have been carried out by a computer.

The manifold \mathfrak{B} is the image under $(f, g) \mapsto (F, G)$ of an open set B in a Banach space \mathcal{B}' of pairs of analytic generating functions. Here $g \mapsto G$, and similarly $f \mapsto F$, is the map defined by (1.6). This space \mathcal{B}' includes all pairs of polynomials in two variables, modulo constant functions.

Concerning the last statement of Theorem 1.2, it is clear numerically that P_* does not attract all nearby pairs in $\mathfrak{B}^c(\Omega)$. We believe that this alternative can be eliminated within the framework of this paper, requiring only technical changes. A possible approach is described at the end of Section 8.

A shortcoming of the map-pair approach is that sets of commuting pairs like $\mathfrak{B}^c(\Omega)$ are most likely not manifolds. For the renormalization of critical circle maps, an approach that avoids this problem has been introduced and used successfully in [13]. For the problem considered here, commuting pairs can be avoided by studying Hamiltonian flows on $\mathbb{T}^2 \times \mathbb{R}^2$ instead of area-preserving maps.

An analogue of Theorem 1.1 for a renormalization operator acting on Hamiltonians was proved in [14], and the scaling of the fixed point Hamiltonian H_* is compatible with (1.4). Unfortunately, the renormalization operator for Hamiltonians (or vector fields) is substantially more involved than the operator \mathfrak{R} . So a Hamiltonian analogue of Theorem 1.2 seems currently out of reach. On the other hand, it was shown in [15] that every Hamiltonian on the strong local stable manifold at H_* has a golden invariant torus that is “critical”, in the sense that the flow on this torus is not differentiably conjugate to a linear flow. A similar argument should work for the operator \mathfrak{R}^3 , but we have not considered this problem here.

In order to connect these results to the breakup of golden invariant tori, one would also have to prove that, in some open neighborhood of the fixed point P_* , all commuting zero-Calabi pairs on one side of the local stable manifold of \mathfrak{M}' have a smooth golden invariant torus, while the pairs on the other side have no golden invariant torus. Two methods that have been developed and studied in connection with this problem are Greene’s criterion [2,8,9] and the obstruction method [7,16]. It should be possible to make some progress on these questions by using the methods developed in this paper.

The remaining part of this paper is organized as follows. In Section 3 we give a formal definition of the operator \mathcal{M}' that implements the above-mentioned operator \mathfrak{M}' in terms of generating functions. Some facts that have guided our choices are described in Section 2. Sections 4 and 5 are concerned with the proof of Theorem 1.1, while Sections 6,7, and 8 are devoted to the proof of Theorem 1.1. To be more precise, both proofs are reduced to the task of proving several technical lemmas. Our proof of these lemmas is computer-assisted, using some estimates that are given in Section 9. In Section 10 we describe how the remaining estimates are translated into a sequence of trivial computations (with rigorous error estimates) that can be, and have been, carried out by a computer. The complete details of this last part, written in the programming language Ada [19], can be found in [23].

2. Irrelevant eigenvalues

In order to determine a proper choice of scaling $\Lambda = \Lambda_P$ and a proper modification of the operator \mathfrak{M} , it is useful to consider what happens if the scaling is fixed. Much of the discussion in this section is purely formal; but part of it will be made rigorous later.

Let $P_* = (F_*, G_*)$ be the fixed point of \mathfrak{R} described in Theorem 1.1. Denote by \mathfrak{R}_* the operator \mathfrak{R} with a fixed scaling $\Lambda = \Lambda_*$. That is,

$$\mathfrak{R}_*(P) = \Lambda_*^{-1}(G, FG)\Lambda_*, \quad P = (F, G), \quad (2.1)$$

where we have used the notation $V(F, G)U = (VFU, VGU)$ for maps V, F, G, U of the plane. As with other renormalization operators in dynamics, fixing the scaling introduces “irrelevant” eigenvalues at the fixed point. These eigenvalues, and the corresponding eigenvectors, can be computed explicitly: Consider changes of coordinates $U_\varepsilon = I + \varepsilon T + \mathcal{O}(\varepsilon^2)$ with constant determinant $\det(DU_\varepsilon)$. Then

$$U_\varepsilon^{-1}P_*U_\varepsilon = P_* + \varepsilon P_T + \mathcal{O}(\varepsilon^2), \quad (2.2)$$

where $P_T = (DP_*)DT - TP_*$. Notice that the map $T \mapsto P_T$ is linear. Applying \mathfrak{R}_* to the left hand side of (2.2) we get a similar conjugacy, but with U_ε replaced by $\Lambda_*^{-1}U_\varepsilon\Lambda_*$. This shows that

$$D\mathfrak{R}_*(P_*)P_T = P_{\Lambda_*^{-1}T\Lambda_*}. \quad (2.3)$$

In particular, if $\Lambda_*^{-1}T\Lambda_* = \kappa T$ then P_T is an eigenvector of $D\mathfrak{R}_*(P_*)$ with eigenvalue κ . Restricting to coordinate changes that are analytic near the origin, these eigenvalues are all of the form $\kappa = \lambda_*^{m-1}\mu_*^n$ or $\kappa = \lambda_*^m\mu_*^{n-1}$, with m and n nonnegative integers.

The non-contracting eigenvalues κ and the corresponding functions T are listed in Table 1. In this table, the eigenvalue with label $0N$ is the N -th largest eigenvalue in modulus in the even subspace (defined later for generating functions). Similarly, $1N$ labels the N -th largest |eigenvalue| in the odd subspace. The same labels are also used in our programs [23].

label	$T(x, z)$	κ	value
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01	(0, 1)	μ^{-1}	-3.06688825...
12	(0, x)	$\lambda\mu^{-1}$	2.16766333...
04	(0, x^2)	$\lambda^2\mu^{-1}$	-1.53209505...
15	(1, 0)	λ^{-1}	-1.41483606...
16	(0, x^3)	$\lambda^3\mu^{-1}$	1.08287815...
05	(x , 0)	1	1.00000000...
06	(0, z)	1	1.00000000...

Table 1. Eigenvalues of $D\mathfrak{R}_*(P_*)$ related to coordinate changes.

The goal is to choose the scaling $\Lambda = \Lambda_P$ in the definition (1.1) of \mathfrak{R} in such a way that $D\mathfrak{R}(P_*)P_T = 0$ for the maps T listed in Table 1. Then we can expect $D\mathfrak{R}(P_*)$ to have a simple eigenvalue $\delta > 1$ and no other spectrum outside the open unit disk.

The above-mentioned expectation assumes that \mathfrak{R} is restricted to commuting pairs with zero Calabi invariant. If we allow “general” pairs, then $D\mathfrak{R}(P_*)$ has other expanding eigenvalues, as was already described in [5]. They can again be computed explicitly: If $\Theta = FG(GF)^{-1}$ is the commutator for a pair $P = (F, G)$, then the corresponding commutator $\tilde{\Theta}$ for the renormalized pair $\tilde{P} = \mathfrak{R}(P)$ is given by

$$\begin{aligned}\tilde{\Theta} &= \Lambda^{-1}GFG(FGG)^{-1}\Lambda = \Lambda^{-1}GFGG^{-1}G^{-1}F^{-1}\Lambda \\ &= \Lambda^{-1}GFG^{-1}F^{-1}\Lambda = \Lambda^{-1}GF(FG)^{-1}\Lambda = \Lambda^{-1}\Theta^{-1}\Lambda.\end{aligned}\tag{2.4}$$

The derivative of the map $\Theta \mapsto \tilde{\Theta}$ at the trivial commutator $\Theta_* = \text{I}$ is given by $\theta \mapsto -\Lambda_*^{-1}\theta\Lambda_*$. To see how this relates to the spectrum of $D\mathfrak{R}(P_*)$, consider a one-parameter family of pairs $P_* + \varepsilon P' + \mathcal{O}(\varepsilon^2)$ and the corresponding family of commutators $\text{I} + \varepsilon\theta + \mathcal{O}(\varepsilon^2)$. If P' is an eigenvector of $D\mathfrak{R}(P_*)$ with eigenvalue κ , then we must have $\Lambda_*^{-1}\theta\Lambda_* = \kappa\theta$. So either $\theta = 0$, or else κ has to be an eigenvalue of the linear operator $\theta \mapsto -\Lambda_*^{-1}\theta\Lambda_*$. When restricted to functions θ that are analytic near the origin, the eigenvalues of this operator are of the form $\kappa = -\lambda_*^{m-1}\mu_*^n$ or $\kappa = -\lambda_*^m\mu_*^{n-1}$, with m and n nonnegative integers. Numerically, these values do indeed appear as eigenvalues of $D\mathfrak{R}(P_*)$.

Consider now the operator \mathfrak{M} defined by (1.7), but with a fixed scaling $\Lambda = \Lambda_*$. This operator will be denoted by \mathfrak{M}_* . When restricted to commuting directions, $D\mathfrak{M}_*(P_*)$ has the same eigenvalues and eigenvectors as $D\mathfrak{R}^3(P_*)$. But other eigenvalues and eigenvectors need not agree. In particular, consider the commutator $\tilde{\Theta}$ for the pair $\tilde{P} = \mathfrak{M}(P)$. A computation similar to (2.4) yields

$$\tilde{\Theta} = \tilde{F}\Lambda^{-3}\Theta^{-1}\Lambda^3\tilde{F}^{-1}.\tag{2.5}$$

To first order in $P - P_*$, the map $\Theta \mapsto \tilde{\Theta}$ is given by $\theta \mapsto -\mathbb{A}^{-1}\theta\mathbb{A}$, where $\mathbb{A} = \Lambda_*^3F_*^{-1}$. As we will show, the scaling map \mathbb{A} has a fixed point $\bar{\omega}$, which belongs to the domain of Θ for the maps considered here. Numerically, the derivative $D\mathbb{A}(\bar{\omega})$ has two eigenvalues α^3 and β^3 , where

$$\begin{aligned}\alpha &= -0.590942551826517690952558\dots, \\ \beta &= -0.389987480001625160585061\dots.\end{aligned}\tag{2.6}$$

Thus, the operator $\theta \mapsto -\mathbb{A}^{-1}\theta\mathbb{A}$ has eigenvalues κ^3 with $\kappa = -\alpha^{m-1}\beta^n$ or $\kappa = \alpha^m\beta^{n-1}$. Notice that $\alpha\beta = \lambda_*\mu_*$ since F_* is area-preserving. For reasons analogous to those described after (2.4), any eigenvalue of $\mathfrak{M}_*(P_*)$ corresponding to a non-commuting direction has to take one of these values κ^3 .

The largest (in modulus) five eigenvalues^{1/3} of this type are listed in Table 2. In addition, we list as first entry the eigenvalue that corresponds to a commuting direction with nonzero Calabi invariant. The coordinates (a, b) used in the the second column of this table are those that diagonalize \mathbb{A} to $\text{diag}(\alpha, \beta)$.

label	$\theta(a, b)$	κ	value
11		$-\alpha^{-1}\beta^{-1}$	4.33914408...
02	(0, 1)	$-\beta^{-1}$	2.56418488...
13	(1, 0)	$-\alpha^{-1}$	1.69221186...
14	(0, a)	$-\alpha\beta^{-1}$	-1.51528595...
07	(a , 0)	-1	-1.00000000...
08	(0, a^2)	$-\alpha^2\beta^{-1}$	0.89544695...

Table 2. Third roots of eigenvalues of $\mathcal{M}_*(p_*)$ for non-commuting directions.

We note that, for reversible pairs, a relation analogous to (2.5) holds if the commutator Θ is replaced by the map $\Xi = FG\mathcal{S}$. This should not be surprising, since $\Theta = \Xi^2$ in this case. Restricting to commuting pairs, this implies that the fixed point of $\Xi_*\mathbb{A}$ plays a role similar to the fixed point ω of \mathbb{A} . Thus, one would expect that $\Xi_*(\omega) = \omega$. (Notice that Ξ_* is an involution and thus has a curve of fixed points.) This is indeed observed numerically.

3. A formal definition of the operator \mathfrak{M}'

The main goal in this section is to describe the operator \mathcal{M}' that represents \mathfrak{M}' in terms of generating functions. We also compute the Calabi invariant $C(P)$. The descriptions are formal in the sense that we do not specify any domains at this point.

3.1. Generating functions for composed maps

Consider an area-preserving map G defined by a generating function g via (1.6). The composition of G with a map F generated by f can be represented as

$$\begin{bmatrix} x \\ -g_1(x, \mathcal{V}) \end{bmatrix} \xrightarrow{G} \begin{bmatrix} \mathcal{V} \\ g_2(x, \mathcal{V}) \end{bmatrix} = \begin{bmatrix} \mathcal{V} \\ -f_1(\mathcal{V}, y) \end{bmatrix} \xrightarrow{F} \begin{bmatrix} y \\ f_2(\mathcal{V}, y) \end{bmatrix}. \quad (3.1)$$

So the generating function $f \circ g$ for the composed map FG is

$$(f \circ g)(x, y) = g(x, \mathcal{V}) + f(\mathcal{V}, y), \quad g_2(x, \mathcal{V}) + f_1(\mathcal{V}, y) = 0, \quad (3.2)$$

where $\mathcal{V} = \mathcal{V}(x, y)$ is determined by the second equation in (3.2), assuming that this equation has a unique solution. \mathcal{V} will be referred to as the *midpoint function* for $f \circ g$.

Denote by \mathcal{V} and \mathcal{W} the midpoint functions for $f \circ g$ and $g \circ f$, respectively. Let $\mathbf{x}(x, y) = x$ and $\mathbf{y}(x, y) = y$. We define the commutator for the pair $p = (f, g)$ as

$$\begin{aligned} \mathcal{C}(p) &= f \circ g - g \circ f \\ &= g(\mathbf{x}, \mathcal{V}) + f(\mathcal{V}, \mathbf{y}) - f(\mathbf{x}, \mathcal{W}) - g(\mathcal{W}, \mathbf{y}). \end{aligned} \quad (3.3)$$

Assume now that $P = (F, G)$ is a commuting pair defining a cylinder map. Our goal is to compute the Calabi invariant for this map. To this end, we need to choose a

differentiable curve γ in the plane, from a point ω to its image under F . When regarded as a curve on the cylinder defined by F , this curve γ is closed. By definition, the Calabi invariant is the area of the region between the closed curves $G \circ \gamma$ and γ . In the planar lift of the cylinder, this region is bounded by two other curves: a curve σ connecting ω to $G(\omega)$, and its image under the map F .

To be more explicit, let (x, y) be some point in the domain of both $f \circ g$ and $g \circ f$. Define $v = \mathcal{V}(x, y)$ and $w = \mathcal{W}(x, y)$. Let $z = -f_1(x, w) = -g_1(x, v)$ and $w = f_2(v, y) = g_2(w, y)$. Consider a differentiable planar curve γ from $\omega = \begin{bmatrix} x \\ z \end{bmatrix}$ to $F(\omega)$. Then $G \circ \gamma$ passes from $G(\omega)$ to $\omega' = \begin{bmatrix} y \\ w \end{bmatrix}$. Pick another planar curve σ from ω to $G(\omega)$. Then $F \circ \sigma$ passes from $F(\omega)$ to ω' . Consider the one-form $\theta = zdx$, using the notation (1.6). Its pushforward under the map G is $G^*\theta = wdy$, and $G^*\theta - \theta = dg$. Similarly for the map F . The area defining the Calabi invariant for P can now be written as

$$\begin{aligned} C(P) &= \int_{G \circ \gamma - \gamma} \theta - \int_{F \circ \sigma - \sigma} \theta = \int_{\gamma} [G^*\theta - \theta] - \int_{\gamma} [F^*\theta - \theta] = \int_{\gamma} dg - \int_{\sigma} df \\ &= g(x, v) - g(w, y) - f(x, w) + f(v, y). \end{aligned} \quad (3.4)$$

So $C(P)$ is just the commutator $\mathcal{C}(p)$, evaluated at (x, y) . Its value is independent of the point (x, y) : having assumed that FG and GF agree, the generating functions $f \circ g$ and $g \circ f$ have the same derivatives, so $\mathcal{C}(p)$ is a constant in this case.

Notice also that $C(G, FG) = C(G, F) = -C(F, G)$. Thus, given that the Calabi invariant is an area, we have $C(\mathfrak{R}(P)) = -\det(D\Lambda)^{-1}C(P)$. The eigenvalue $-(\alpha\beta)^{-1}$ in the first row of Table 2 is precisely this factor $-\det(D\Lambda)^{-1}$.

3.2. Definition of the operator \mathfrak{M}

Consider the operator \mathfrak{R} with scaling maps Λ of the form $\Lambda(x, z) = (\lambda x, \mu z)$. Formally, the corresponding operator for pairs of generating functions $p = (f, g)$ is given by

$$\mathcal{R}(p) = (\lambda\mu)^{-1}(g, f \circ g) \circ \ell \stackrel{\text{def}}{=} ((\lambda\mu)^{-1}g \circ \ell, (\lambda\mu)^{-1}(f \circ g) \circ \ell), \quad (3.5)$$

where $\ell(x, y) = (\lambda x, \lambda y)$, with λ and μ to be determined. Similarly, the operator \mathfrak{M}_* is represented in terms of generating functions by

$$\mathcal{M}_*(p) = (\lambda_*\mu_*)^{-3}(g \circ f \circ g, g \circ f \circ g \circ f \circ g) \circ \ell_*^3, \quad (3.6)$$

where $\ell_*(x, y) = (\lambda_*x, \lambda_*y)$. We will prove later that this operator \mathcal{M}_* is well-defined in an open neighborhood of p_* in a space $\mathcal{B}_p = \mathcal{B}_f \times \mathcal{B}_g$ of pairs $p = (f, g)$.

As we have seen in Section 2, the coordinate-related eigenvalues of modulus ≥ 1 arise from coordinate transformations of the form

$$U(t) \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} t_5 + t_6 x \\ t_7 z + \tau'(x) \end{bmatrix}, \quad \tau(x) = \sum_{n=1}^4 t_n x^n. \quad (3.7)$$

Here τ' denotes the derivative of τ , and t is a vector in \mathbb{R}^7 close to the origin. If F is an area-preserving map with generating function f , then $U(t)^{-1}FU(t)$ is the area-preserving map with generating function $\mathcal{U}(t, f)$, where

$$\mathcal{U}(t, f)(x, y) = (1 + t_7)f(x + t_5 + t_6x, y + t_5 + t_6y) + \tau(x) - \tau(y). \quad (3.8)$$

For pairs $p = (f, g)$ we define $\mathcal{U}(t, p) = (\mathcal{U}(t, f), \mathcal{U}(t, g))$. Conditioned on the choice of a continuous linear map $\mathcal{T} : \mathcal{B}_p \rightarrow \mathbb{R}^7$, define now

$$\mathcal{M}(p) = \mathcal{U}(t, \mathcal{M}_*(p)), \quad t = \mathcal{T}(p - p_*), \quad (3.9)$$

for all p close to p_* . Here p_* is the fixed point of \mathcal{M}_* obtained from the fixed point g_* of \mathcal{N} . The derivative of \mathcal{M} at p_* is given by the equation

$$D\mathcal{M}(p_*)p = D\mathcal{M}_*(p_*)p + D_1\mathcal{U}(0, p_*)\mathcal{T}p. \quad (3.10)$$

Let E_1, E_2, \dots, E_7 be the seven coordinate-related eigenvectors of $D\mathcal{M}_*(p_*)$ for the eigenvalues listed in Table 1. Then the equation (3.10) may be written in the form

$$D\mathcal{M}(p_*)p = D\mathcal{M}_*(p_*)p - \sum_{k=1}^7 [\Phi_{1,k}(f) + \Phi_{2,k}(g)]E_k, \quad (3.11)$$

where $\Phi_{1,k} : \mathcal{B}_f \rightarrow \mathbb{R}$ and $\Phi_{2,k} : \mathcal{B}_g \rightarrow \mathbb{R}$ are continuous linear functionals. In fact, it is these functionals that we choose (as described below); the map \mathcal{T} is implicitly defined by this choice.

Recall that $D\mathcal{M}(p_*)$ also has six undesired eigenvalues in non-commuting directions, namely the cubes of the values listed in Table 2. In order to “eliminate” these eigenvalues, we compute polynomial approximations E_8, E_9, \dots, E_{13} for the corresponding eigenvectors, and define

$$\mathcal{M}'(p) = \mathcal{M}(p) - \sum_{k=8}^{13} \Phi_{3,k}(\mathcal{C}(p))E_k, \quad (3.12)$$

after choosing suitable linear functionals $\Phi_{3,k}$ for $k = 8, 9, \dots, 13$. Clearly, if p is a commuting pair in the sense that $\mathcal{C}(p) = f \circ g - g \circ f$ vanishes, then $\mathcal{M}'(p) = \mathcal{M}(p)$.

All of our functionals $\Phi_{j,k}$ are of course chosen in such a way that $D\mathcal{M}'(p_*)E_k \approx 0$ for $k = 1, 2, \dots, 13$. For simplicity, they are taken to be of the form

$$\Phi_{j,k}(h) = \sum_{n=1}^{N_{j,k}} \Phi_{j,k,n} h(\xi_{j,k,n}), \quad (3.13)$$

with $\Phi_{j,k,n} \in \mathbb{R}$, and with $\xi_{j,k,n} \in \mathbb{R}^2$ well inside the domain of the functions h being considered. The precise values of these quantities can be found in [23]. By definition, the choices made in defining \mathcal{M}' are “suitable” if $D\mathcal{M}'(p_*)$ has a single expanding eigenvalue and no other eigenvalues outside the open unit disk. We note that the last eigenvalue in Table 2 is not expanding; but we choose to eliminate it as well, since it is uncomfortably close to 1.

4. Construction of the fixed point

As mentioned in the introduction, the first component G_* of the fixed point $P_* = (F_*, G_*)$ described in Theorem 1.1 is constructed as a fixed point of the operator \mathfrak{N} defined by $\mathfrak{N}(G) = \Lambda^{-2}GFG\Lambda^2$, where $F = \Lambda^{-1}G\Lambda$. The scaling $\Lambda = \text{diag}(\lambda, \mu)$ is determined by requiring that $G_0 = \Lambda^{-1}FG\Lambda$ maps the origin $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the point $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. The corresponding condition for generating functions is that $g_{01}(0, -1) = 0$ and $g_{02}(0, -1) = -1$.

Applying the identity (3.2) twice, we see that the generating function for GFG is given by

$$(g \circ f \circ g)(x, y) = g(x, \mathcal{V}_0) + f(\mathcal{V}_0, \mathcal{W}_0) + g(\mathcal{W}_0, y), \quad (4.1)$$

with the numbers $\mathcal{V}_0 = \mathcal{V}_0(x, y)$ and $\mathcal{W}_0 = \mathcal{W}_0(x, y)$ being determined by the two equations $g_2(x, \mathcal{V}_0) + f_1(\mathcal{V}_0, \mathcal{W}_0) = 0$ and $f_2(\mathcal{V}_0, \mathcal{W}_0) + g_1(\mathcal{W}_0, y) = 0$. These equations simplify if we assume that f and g generate maps F and G that are reversible. To be more precise, notice that the generating function for G^{-1} is $(x, y) \mapsto -g(y, x)$. This can be seen directly from (1.6). Thus, the generating function $\mathbb{S}g$ for the reversed map $\mathbb{S}G = \mathbb{S}G^{-1}\mathbb{S}$ is given by

$$\mathbb{S}g = g \circ S, \quad S(x, y) = (-y, -x). \quad (4.2)$$

Using that $\mathbb{S}g_1 = -(\mathbb{S}g)_2$ and $\mathbb{S}f_2 = -(\mathbb{S}f)_1$, the equations determining the midpoint functions \mathcal{V}_0 and \mathcal{W}_0 can be written as

$$\begin{aligned} 0 &= g_2(\mathbf{x}, \mathcal{V}_0) + f_1(\mathcal{V}_0, \mathcal{W}_0), \\ 0 &= (\mathbb{S}g)_2(\mathbf{x}, -\mathbb{S}\mathcal{W}_0) + (\mathbb{S}f)_1(-\mathbb{S}\mathcal{W}_0, -\mathbb{S}\mathcal{V}_0). \end{aligned} \quad (4.3)$$

Assume now that f and g are reversible, in the sense that $\mathbb{S}f = f$ and $\mathbb{S}g = g$. Then (4.3) reduces to the single equation $g_2(x, \mathcal{V}_0) + f_1(\mathcal{V}_0, -\mathbb{S}\mathcal{V}_0) = 0$ if we choose $\mathcal{W}_0 = -\mathbb{S}\mathcal{V}_0$. This relation between \mathcal{V}_0 and \mathcal{W}_0 also implies that $g \circ f \circ g$ is reversible. The representation of \mathfrak{N} in terms of generating functions is now given by

$$\mathcal{N}(g) = (\lambda\mu)^{-2}(g \circ f \circ g) \circ \ell^2, \quad f = (\lambda\mu)^{-1}g \circ \ell. \quad (4.4)$$

If we express our generating functions in the variables $t = x + y$ and $s = x - y$, then f and g are reversible if and only if they are even functions of t . As was found in [17], the generating functions arising in the analysis of \mathcal{N} are well approximated by polynomials in the variables u and v ,

$$u = [t^2 - t_0^2] + b[s - s_0], \quad v = s - s_0, \quad t = x + y, \quad s = x - y, \quad (4.5)$$

if $t_0, s_0, b \in \mathbb{R}$ are chosen appropriately. This suggested the following choice of function spaces. Given a pair $\rho = (\rho_u, \rho_v)$ of positive real numbers, denote by D_ρ the set of all points $(u, v) \in \mathbb{C}^2$ such that $|u| < \rho_u$ and $|v| < \rho_v$. Define \mathcal{A}_ρ to be the space of all analytic functions $\phi : D_\rho \rightarrow \mathbb{C}$, that extend continuously to the boundary of D_ρ and have a finite norm

$$\|\phi\|_\rho = \sum_{m,n} |\phi_{m,n}| \rho_u^m \rho_v^n, \quad \phi(u, v) = \sum_{m,n} \phi_{m,n} u^m v^n. \quad (4.6)$$

Clearly \mathcal{A}_ρ is a Banach algebra, in the sense that $\|\phi\psi\|_\rho \leq \|\phi\|_\rho\|\psi\|_\rho$ for all $\phi, \psi \in \mathcal{A}_\rho$.

Consider now a fixed change of variables $\varphi(x, y) = (u, v)$ of the form (4.5). Let $\varrho = (t_0, s_0, b, \rho)$ and $\mathcal{D}_\varrho = \varphi^{-1}D_\rho$. Then every function $f : \mathcal{D}_\varrho \rightarrow \mathbb{C}$ can be written as

$$f = [\phi + \mathbf{t}\psi] \circ \varphi, \quad \mathbf{t}(x, y) = x + y, \quad (4.7)$$

where ϕ and ψ are functions on D_ρ .

Definition 4.1. We define \mathcal{B}_ϱ to be Banach space of all functions (4.7) with ϕ and ψ belonging to \mathcal{A}_ρ , equipped with the norm

$$\|f\|_\varrho = \|\phi\|_\rho + \rho_t\|\psi\|_\rho, \quad \rho_t = (t_0^2 + \rho_u + |b|\rho_v)^{1/2}. \quad (4.8)$$

The Taylor coefficient $\phi_{0,0}$ of ϕ will be referred to as the constant term of f . We will call f even if $\psi = 0$, or odd if $\phi = 0$. The subspace of even and odd functions in \mathcal{B}_ϱ will be denoted by \mathcal{B}_ϱ^0 and \mathcal{B}_ϱ^1 , respectively.

Clearly, \mathcal{B}_ϱ^0 is isometrically isomorphic to \mathcal{A}_ρ . It is not hard to see that both \mathcal{B}_ϱ^0 and \mathcal{B}_ϱ are Banach algebras.

Remark. As defined above, \mathcal{B}_ϱ is a Banach space over \mathbb{C} . When discussing maps associated with generating functions in \mathcal{B}_ϱ , we only consider the corresponding Banach space over \mathbb{R} , where a function is assumed to take real values for real arguments. Since it should be clear from the context which number field is being used, we will denote both spaces by \mathcal{B}_ϱ . Similarly for the spaces \mathcal{A}_ρ . If we wish to stress that a function takes real values for real arguments, we will refer to this function as being *real*.

For the domain of the function g in (4.4) we use parameter values

$$\varrho_g(r) : \quad t_0 = \frac{17367}{32768}, \quad s_0 = \frac{78643}{65536}, \quad b = \frac{314573}{65536}, \quad \rho_u = r, \quad \rho_v = \frac{25887}{32768}r, \quad (4.9)$$

with r to be specified. Here, the subscript g in ϱ_g is just a symbol and does not refer to a specific function g . This is part of a convention that is used throughout this paper: any subscript in non-slant font is an abstract symbol and not an argument. The domain \mathcal{D}_ϱ and space \mathcal{B}_ϱ with $\varrho = \varrho_g(r)$ will be denoted by $\mathcal{D}_g(r)$ and $\mathcal{B}_g(r)$, respectively. For the function $f = (\lambda\mu)^{-1}g \circ \ell$ we use the domain $\mathcal{D}_f(r)$ and space $\mathcal{B}_f(r)$, defined by the parameter values

$$\varrho_f(r) : \quad t_0 = 0, \quad s_0 = -\frac{7}{4}, \quad b = -7, \quad \rho_u = \frac{3}{2}r, \quad \rho_v = \frac{14746}{16384}r. \quad (4.10)$$

In addition, define real domains $\mathbb{D}_g(r) = \mathbb{R}^2 \cap \mathcal{D}_g(r)$ and $\mathbb{D}_f(r) = \mathbb{R}^2 \cap \mathcal{D}_f(r)$. For the values of r considered here, $\mathbb{D}_g(r)$ is a simply connected region $\eta_r^-(x) < y < \eta_r^+(x)$ between the graphs of two continuous functions $y = \eta_r^\pm(x)$, defined over an interval $\xi_r^- < x < \xi_r^+$. The values ξ_r^\pm and the functions η_r^\pm can be computed explicitly. Similarly for the domain $\mathbb{D}_f(r)$.

Theorem 4.2. The operator \mathcal{N} is well-defined and analytic on an open set in $\mathcal{B}_g^0(\frac{9}{8})$. This open set contains a unique real fixed point g_* of \mathcal{N} . The associated scaling constants

λ_* and μ_* satisfy the bounds (1.4). The corresponding function f_* belongs to $\mathcal{B}_f^0(\frac{9}{8})$. On $\mathbb{D}_g(\frac{31}{32})$ and $\mathbb{D}_f(\frac{31}{32})$, the functions $f = f_*$ and $g = g_*$ satisfy the twist condition $g_{1,2}, f_{1,2} > 1$ and the bounds $g_{1,1}, g_{2,2}, f_{1,1}, f_{2,2} < -1$.

Concerning our proof of this theorem: The existence of the fixed point $g_* \in \mathcal{B}_g^0(\frac{9}{8})$ with $f_* \in \mathcal{B}_f^0(\frac{9}{8})$, and the bounds on λ_* and μ_* , are proved the same way as the corresponding parts of Theorem 4.1 in [17]. In particular, the fixed point problem for \mathcal{N} is reduced to a fixed point problem for a quasi-Newton map associated with \mathcal{N} . The second derivative inequalities are then easy to verify, since our construction of f_* and g_* yields highly accurate polynomial approximations with rigorous error bounds. For details we refer to [17,23]. The task of controlling $g \circ f \circ g$ also appears in connection with the operator \mathcal{M}_* and will be discussed in Sections 6,9, and 10.

5. Map domains

When considering map-pairs (F, G) associated with pairs (f, g) of generating functions, the domains of f and g are always taken to be $\mathbb{D}_g = \mathbb{D}_g(r)$ and $\mathbb{D}_f = \mathbb{D}_f(r)$, with $r = \frac{15}{16}$.

In the remaining part of this section, f, g, F, G stand for f_*, g_*, F_*, G_* . The inequalities in Theorem 4.2 imply that $f \circ g$ and $g \circ f$ are real analytic and satisfies a twist condition on their domains $\mathbb{D}_{f \circ g}$ and $\mathbb{D}_{g \circ f}$, respectively. By definition, $\mathbb{D}_{f \circ g}$ is the set of all points $(x, y) \in \mathbb{R}^2$ for which there exists a unique $v \in \mathbb{R}$ such that $(x, v) \in \mathbb{D}_g$ and $(v, y) \in \mathbb{D}_f$ and $g_2(x, v) + f_1(v, y) = 0$. Similarly for $\mathbb{D}_{g \circ f}$. The domain of the map G is defined by the equation

$$\mathbb{D}_G = \Gamma_g \mathbb{D}_g, \quad \Gamma_g \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -g_1(x, y) \end{bmatrix}. \quad (5.1)$$

Notice that, by the twist property of g , \mathbb{D}_G is a simply connected region $\zeta^-(x) < z < \zeta^+(x)$ between the graphs of two continuous functions $z = \zeta^\pm(x)$, defined over the interval $\xi_r^- < x < \xi_r^+$. The domain \mathbb{D}_F of F is defined analogously.

Lemma 5.1. *Every map in the set $\Lambda_\diamond^{-1} G \Lambda_\diamond$ is well-defined on \mathbb{D}_F , and every map in $\Lambda_\diamond^{-2} G F G \Lambda_\diamond^2$ is well-defined on \mathbb{D}_G . In particular, $\Lambda_\diamond \mathbb{D}_F \subset \mathbb{D}_G$ and $\Lambda_\diamond^2 \mathbb{D}_G \subset \mathbb{D}_G$.*

This lemma is proved easily by including two additional domain-checks in the proof of Theorem 4.2. Namely, using the estimates on the midpoint function \mathcal{V}_0 and \mathcal{W}_0 obtained in the proof of Theorem 4.2, we verify that the composed maps on the right hand side of (4.1) are well defined if f and g are restricted $\mathbb{D}_f(r)$ and $\mathbb{D}_g(r)$, respectively, for some $r < \frac{15}{16}$. No additional work is needed to allow arbitrary scalings $\Lambda \in \Lambda_\diamond$ in Lemma 5.1, since by definition, Λ_\diamond is the enclosure used for Λ_* in the proof of Theorem 4.2. For the complete details we refer to [23].

Notice that

$$\mathfrak{R}(F, G) = (F, G_0), \quad \mathfrak{R}(F, G_0) = (F_0, G), \quad (5.2)$$

where $G_0 = \Lambda_*^{-1} F G \Lambda_*$ and $F_0 = \Lambda_*^{-1} G_0 \Lambda_*$. Given that $\Lambda_\diamond^2 \mathbb{D}_G \subset \mathbb{D}_G$, the map G_0 is well-defined on $\Lambda_\diamond \mathbb{D}_G$, and F_0 is well-defined on \mathbb{D}_G . This has motivated our choice of domains $D_G = \mathbb{D}_G \cup \Lambda_\diamond \mathbb{D}_G$ and $D_F = \mathbb{D}_F \cup \mathbb{D}_G$.

For monodromy arguments it would be useful to know that (the union of) the domains $\mathbb{D}_{f \circ g}$ and $\mathbb{D}_{g \circ f}$ are simply connected. We saw no easy way to prove this, but the sets W and SW given below achieve the same purpose.

Lemma 5.2. *The domain $\lambda_*^{-1}\mathbb{D}_{f \circ g}$ of the generating function $g_0 = (\lambda_*\mu_*)^{-1}(f \circ g) \circ \ell_*$ of the map G_0 includes the intersection of \mathbb{D}_g with the half-plane $x + y \leq 0$. This intersection includes an open neighborhood of the point $(0, -1)$, and $g_0 = g$ on this open set. Furthermore, g_0 extends to a real analytic function on an open domain W with the property that $W \cup SW$ is simply connected.*

We will now give a sketch of our proof and refer to [17,23] for details.

The claim that $g_0 = g$ near $(0, -1)$ follows from a simple computation [17], which shows that $J = G_0^{-1}G$ satisfies $J = \Lambda_*^{-1}J^{-1}\Lambda_*$ near the origin. Using that J is analytic, and that $|\lambda|^3 < |\mu| < |\lambda|^4$ by Theorem 4.2, it follows that $J = \pm I$ near the origin. Using our computer-generated bound on J , the case $J = -I$ is easily excluded.

The remaining part of the proof is an exercise in interval arithmetic. We outline it here since it is qualitatively different from our other proofs: As mentioned earlier, our construction of f_* and g_* yields highly accurate approximations (polynomials in u and v) with rigorous error bounds. When analyzing the domain of $f \circ g$, this allows us to decide e.g. whether the function $\nu \mapsto g_2(x, \nu) + f_1(\nu, y)$ has a zero $\nu = \mathcal{V}(x, y)$ with $(x, \nu) \in \mathbb{D}_g$ and $(\nu, y) \in \mathbb{D}_f$. If (x, y) is an arbitrary (unknown) point in a given square, the answer can be either **True**, **False**, or **Uncertain**. The squares being considered are from a finite collection of closed dyadic squares that cover a “sufficiently large” region. Denote by A and B the union of squares yielding answer = **True** and answer = **Uncertain**, respectively. Then B covers the boundary of $\mathbb{D}_{f \circ g}$. By inspection (after determining the boundary of B) we find that the complement of B is the disjoint union of two connected sets, with one of them being the interior of the bounded set A .

Concerning the extension of g_0 , notice that the inequalities in Theorem 4.2 are proved on domains that are larger than \mathbb{D}_g and \mathbb{D}_f . This allows us to construct and estimate $f \circ g$ on an open domain that includes $A \cup B$. Now it is clear that there exists an open domain W with the properties described in Lemma 5.2. For the complete details of the proof we refer to [23].

The above-mentioned domains, scaled by a factor λ_* , are depicted in Figure 2. The larger contour is the boundary of $\mathbb{D}_{f \circ g}$ or (indistinguishably at the given resolution) the set B described above. The smaller contour is the boundary of $\lambda_*\mathbb{D}_g$. The straight line shown in Figure 1 is the line of fixed points for the reflection S , where $x + y = 0$. Seven of the eight isolated points that can be seen near (or on) this line are the points $\xi_{3,k,n}$. The eighth point (\bar{x}, \bar{y}) will be described below.

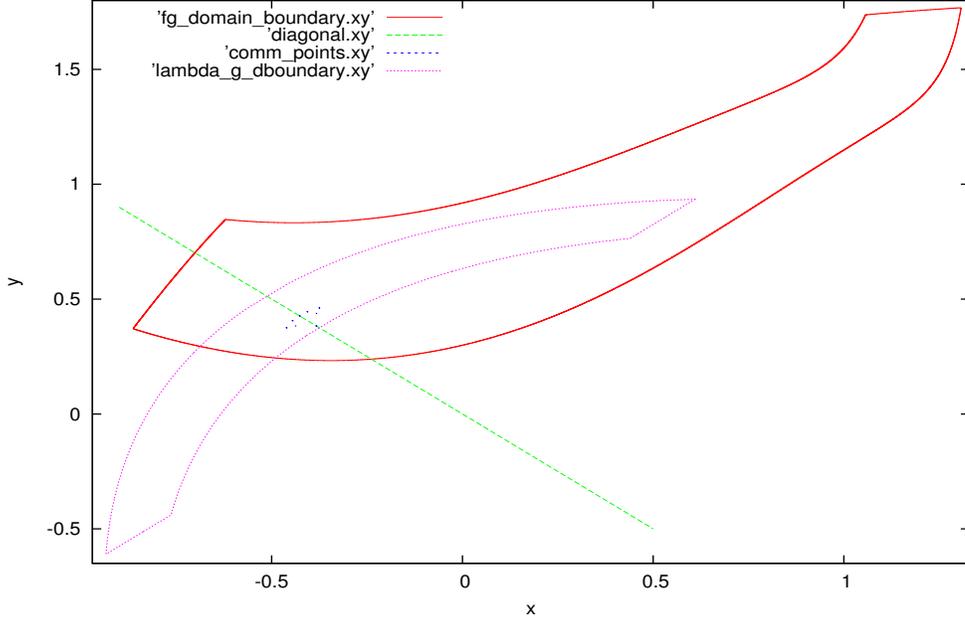


Figure 2. Domains of $f \circ g$ and $g \circ \ell_*^{-1}$, and points $\xi_{3,k,n}$.

Proof of Theorem 1.1. By Lemma 5.2, the function g_0 agrees with g on a complex open neighborhood of $\mathbb{D}_g \cap \{x + y \leq 0\}$. Since f and g are even, we have $\mathbb{S}g_0 = (\lambda\mu)^{-1}(g \circ f) \circ \ell$, and this function agrees with g on a complex open neighborhood of $\mathbb{D}_g \cap \{x + y \geq 0\}$. Thus, the functions g_0 and $\mathbb{S}g_0$ together define a (unique) real analytic extension of g to the domain $W \cup SW$. This domain includes both the domains of g_0 and $\mathbb{S}g_0$. This implies e.g. that the maps F and G commute.

By Lemma 5.1, the domain of G_0 includes $\Lambda_\diamond \mathbb{D}_G$. Furthermore, G and G_0 agree on the intersection of their domains. Thus, G and G_0 extend to a real analytic map \bar{G} on $D_G = \mathbb{D}_G \cup \Lambda_\diamond \mathbb{D}_G$. Similarly, the domain of $F_0 = \Lambda_*^{-1} G_0 \Lambda_*$ includes \mathbb{D}_G . Thus, F and F_0 extend to a real analytic map \bar{F} on $D_F = \mathbb{D}_F \cup \mathbb{D}_G$. By (5.2) the pair (\bar{F}, \bar{G}) is a fixed point of \mathfrak{R} . The bounds (1.4) follow from Theorem 4.2.

What remains to be proved is (1.5). We will use Lemma 5.1 repeatedly without mentioning it each time. From $D_F = \mathbb{D}_F \cup \mathbb{D}_G$ we get $\Lambda_\diamond D_F \subset \Lambda_\diamond \mathbb{D}_F \cup \Lambda_\diamond \mathbb{D}_G$. Since $\Lambda_\diamond \mathbb{D}_F \subset \mathbb{D}_G$, this implies the first relation $\Lambda_\diamond D_F \subset D_G$ in (1.5). Here we have used the definition $D_G = \mathbb{D}_G \cup \Lambda_\diamond \mathbb{D}_G$. From this definition we also get $\Lambda_\diamond D_G \subset \Lambda_\diamond \mathbb{D}_G \cup \Lambda_\diamond^2 \mathbb{D}_G$. Since $\Lambda_\diamond^2 \mathbb{D}_G \subset \mathbb{D}_G$, this implies the second relation $\Lambda_\diamond D_G \subset D_G$ in (1.5). Finally, by using that G maps $\Lambda_\diamond^2 \mathbb{D}_G$ into \mathbb{D}_F , and that G_0 maps $\Lambda_\diamond \mathbb{D}_G$ into \mathbb{D}_F , we obtain

$$\bar{G} \Lambda_\diamond D_G = \bar{G} (\Lambda_\diamond \mathbb{D}_G \cup \Lambda_\diamond^2 \mathbb{D}_G) \subset G_0 \Lambda_\diamond \mathbb{D}_G \cup G \Lambda_\diamond^2 \mathbb{D}_G \subset \mathbb{D}_F. \quad (5.3)$$

This implies the third inclusion relation in (1.5) and completes the proof of Theorem 1.1.

QED

The following will be needed in our discussion of the operator \mathfrak{M}' .

Lemma 5.3. *The map $\mathbb{A} = \Lambda_*^3 F^{-1}$ has a locally unique fixed point $\bar{\omega} = (\bar{y}, \bar{z})$ in $\text{range}(FG) \cap \text{range}(GF)$. The derivative $D\mathbb{A}(\bar{\omega})$ has two eigenvalues α^3 and β^3 satis-*

fying the bounds (2.6). Since $\bar{\omega}$ belongs to the range of $H = FG$, there exists a unique point (\bar{x}, \bar{y}) in the domain of h such that $h_2(\bar{x}, \bar{y}) = \bar{z}$. There exist a closed connected set $C \subset \mathbb{D}_{f \circ g} \cap \mathbb{D}_{g \circ f}$ whose interior contains (\bar{x}, \bar{y}) and all points $\xi_{3,k,n}$ that enter the definition (3.13) of the functionals $\Phi_{j,k}$ for $j = 3$. Furthermore $C = SC$.

Proving this lemma is a straightforward rigorous ‘‘computation’’. The existence of a set C with the asserted properties can be seen by inspection (using the set B described after Lemma 5.2). For the complete details of the proof we refer to [23].

We note that $\bar{x} + \bar{y} = 0$, since the maps F and G commute. The point (\bar{x}, \bar{y}) is shown in Figure 2: it is the point on the line $x + y = 0$ near the lower boundary of $\lambda_* \mathbb{D}_g$.

6. The midpoint equations for even pairs

The definition (3.6) of the operator \mathcal{M}_* involves the composed functions $g \circ f \circ g$ and $g \circ f \circ g \circ f \circ g$. If we denote these two functions by \hat{f} and \hat{g} , respectively, then

$$\hat{f}(x, y) = g(x, \mathcal{V}_0) + f(\mathcal{V}_0, \mathcal{W}_0) + g(\mathcal{W}_0, y), \quad (6.1)$$

$$\hat{g}(x, y) = g(x, \mathcal{V}_1) + f(\mathcal{V}_1, \mathcal{V}_2) + g(\mathcal{V}_2, \mathcal{W}_2) + f(\mathcal{W}_2, \mathcal{W}_1) + g(\mathcal{W}_1, y), \quad (6.2)$$

where $\mathcal{V}_j = \mathcal{V}_j(x, y)$ and $\mathcal{W}_j = \mathcal{W}_j(x, y)$ are determined by the condition that the right hand side of (6.1) and (6.2) be stationary with respect to variations of each \mathcal{V}_j and each \mathcal{W}_j . The resulting equations for the functions \mathcal{V}_j and \mathcal{W}_j will be referred to as the *midpoint equations*. For $j = 0$ they are given by (4.3), and for $j = 1, 2$ they are

$$\begin{aligned} 0 &= g_2(\mathbf{x}, \mathcal{V}_1) + f_1(\mathcal{V}_1, \mathcal{V}_2), \\ 0 &= (\mathbb{S}g)_2(\mathbf{x}, -\mathbb{S}\mathcal{W}_1) + (\mathbb{S}f)_1(-\mathbb{S}\mathcal{W}_1, -\mathbb{S}\mathcal{W}_2), \\ 0 &= f_2(\mathcal{V}_1, \mathcal{V}_2) + g_1(\mathcal{V}_2, \mathcal{W}_2), \\ 0 &= (\mathbb{S}f)_2(-\mathbb{S}\mathcal{W}_1, -\mathbb{S}\mathcal{W}_2) + (\mathbb{S}g)_1(-\mathbb{S}\mathcal{W}_2, -\mathbb{S}\mathcal{V}_2). \end{aligned} \quad (6.3)$$

For the functions \mathcal{V}_0 and \mathcal{W}_0 we use spaces $\mathcal{B}_{\mathcal{V}_f}(r) = \mathcal{B}_{\varrho'}(r)$ with domain parameter $\varrho' = \sigma(\lambda^3, \varrho_f)$, where $\sigma(a, (t_0, s_0, b, \rho_u, \rho_v)) \approx (|a|t_0, as_0, ab, |a|\rho_u, |a|\rho_v)$. The exact definition of the scaling σ can be found in [23]. For the functions \mathcal{V}_j and \mathcal{W}_j with $j = 1, 2$ we use spaces $\mathcal{B}_{\mathcal{V}_g}(r) = \mathcal{B}_{\varrho'}(r)$ with domain parameter $\varrho' = \sigma(\lambda^3, \varrho_g)$.

The midpoint equations need to be solved in two different situations. First, we need to compute and estimate the midpoint functions for the fixed point p_* , since the same functions appear in the derivative $D\mathcal{M}_*(p_*)$. In this case, the functions f and g involved are even. Then we also need to estimate \mathcal{M}_* in an open neighborhood of p_* . In this case, we will use perturbation theory, since the odd parts of f and g can be chosen arbitrarily small. This will be discussed in the next section. As it turns out, controlling odd perturbations involves the same estimates as controlling even perturbations. Thus, we assume in this section that f and g are even. In addition, we make here the ansatz $\mathcal{W}_j = -\mathbb{S}\mathcal{V}_j$. This is justified a-posteriori by the fact that it yields a solution for (4.3) and (6.3), and that these solutions are unique due to the twist properties of f_* and g_* .

The equation for the midpoint function \mathcal{V}_0 , as described after (4.3), can be written as

$$\mathcal{K}'(\mathcal{V}_0) = 0, \quad \mathcal{K}'(\mathcal{V}_0) = g_2(\mathbf{x}, \mathcal{V}_0) + f_1(\mathcal{V}_0, -\mathbb{S}\mathcal{V}_0). \quad (6.4)$$

Formally, the derivative of \mathcal{K}' is given by $D\mathcal{K}'(\mathcal{V}_0) = \mathcal{L}'_{-1}$, where

$$\mathcal{L}'_{\pm 1} = [g_{22}(\mathbf{x}, \mathcal{V}_0) + f_{11}(\mathcal{V}_0, -\mathbb{S}\mathcal{V}_0)] \pm f_{12}(\mathcal{V}_0, -\mathbb{S}\mathcal{V}_0)\mathbb{S}. \quad (6.5)$$

Here, and in what follows, if u and v are functions in one of the spaces $\mathcal{B}_\rho(r)$, then $u + v\mathbb{S}$ denotes the operator $h \mapsto uh + v\mathbb{S}h$.

Similarly, the midpoint equations (6.3) can be written as

$$\mathcal{K}''(\mathcal{V}) = 0, \quad \mathcal{K}'' \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix} = \begin{bmatrix} g_2(\mathbf{x}, \mathcal{V}_1) + f_1(\mathcal{V}_1, \mathcal{V}_2) \\ f_2(\mathcal{V}_1, \mathcal{V}_2) + g_1(\mathcal{V}_2, -\mathbb{S}\mathcal{V}_2) \end{bmatrix}. \quad (6.6)$$

Here, we have used that the second and fourth equations in (6.3) become redundant in the case considered here, where f and g are even and $\mathcal{W}_j = -\mathbb{S}\mathcal{V}_j$. Formally, the derivative of \mathcal{K}'' is given by $D\mathcal{K}''(\mathcal{V}) = \mathcal{L}''_{-1}$, where

$$\mathcal{L}''_{\pm 1} h = \begin{bmatrix} g_{22}(\mathbf{x}, \mathcal{V}_1) + f_{11}(\mathcal{V}_1, \mathcal{V}_2) & f_{12}(\mathcal{V}_1, \mathcal{V}_2) \\ f_{12}(\mathcal{V}_1, \mathcal{V}_2) & L_{\pm 1} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \quad (6.7)$$

with

$$L_{\pm 1} = [f_{22}(\mathcal{V}_1, \mathcal{V}_2) + g_{11}(\mathcal{V}_2, -\mathbb{S}\mathcal{V}_2)] \pm g_{12}(\mathcal{V}_2, -\mathbb{S}\mathcal{V}_2)\mathbb{S}. \quad (6.8)$$

The operators \mathcal{L}'_1 and \mathcal{L}''_1 will be needed in the next section.

In what follows, if \mathcal{X} and \mathcal{Y} are Banach spaces, the product space $\mathcal{X} \times \mathcal{Y}$ is assumed to be equipped with the norm $\|(x, y)\| = \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$.

Lemma 6.1. *There exist open sets $V_0 \subset \mathcal{B}_{\mathcal{V}_f}(1)$ and $V_1, V_2 \subset \mathcal{B}_{\mathcal{V}_g}(1)$ such that the following holds. For $j = 0, 1, 2$ define $W_j = -\mathbb{S}\mathcal{V}_j$. Then for every $f \in \mathcal{B}_f(1)$ and every $g \in \mathcal{B}_g(1)$, each term on the right hand side of (6.1) defines a function in $\mathcal{B}_{\mathcal{V}_f}(1)$, and each term on the right hand side of (6.2) defines a function in $\mathcal{B}_{\mathcal{V}_g}(1)$. The dependence of these functions on \mathcal{V}_j and W_j is analytic. Furthermore, there exists an open neighborhood U_f of f_* in \mathcal{B}_f^0 , and an open neighborhood U_g of g_* in \mathcal{B}_g^0 , such that the following holds. For every $f \in U_f$ and every $g \in U_g$, the linear operators $\mathcal{L}'_{\pm 1}$ and $\mathcal{L}''_{\pm 1}$ are invertible on $\mathcal{B}_{\mathcal{V}_f}(1)$ and $\mathcal{B}_{\mathcal{V}_g}(1) \times \mathcal{B}_{\mathcal{V}_g}(1)$, respectively; and the equations $\mathcal{K}'(\mathcal{V}_0) = 0$ and $\mathcal{K}''(\mathcal{V}_1, \mathcal{V}_2) = 0$ have unique solutions $\mathcal{V}_j \in V_j$. These solutions depend analytically on $f \in U_f$ and $g \in U_g$.*

Our proof of this lemma is computer-assisted and will be described in Sections 9,10.

Concerning the invertibility of the operators \mathcal{L}'_σ and \mathcal{L}''_σ for $\sigma = \pm 1$, our programs only check explicitly the case $\sigma = -1$. This suffices for the following reason. The basic identity being used is

$$(\tilde{u} - v\mathbb{S})(u + v\mathbb{S}) = (\tilde{u} + v\mathbb{S})(u - v\mathbb{S}) = u\tilde{u} - v\tilde{v}, \quad (6.9)$$

where $\tilde{u} = \mathbb{S}u$ and $\tilde{v} = \mathbb{S}v$. We invert $u - v\mathbb{S}$ by first checking that multiplication by $w = u\tilde{u} - v\tilde{v}$ is invertible, so that $(u - v\mathbb{S})^{-1} = w^{-1}(\tilde{u} + v\mathbb{S})$. Then $u + v\mathbb{S}$ is invertible as

well, and $(u + v\mathbb{S})^{-1} = w^{-1}(\check{u} - v\mathbb{S})$. This applies directly to $\mathcal{L}'_{\pm 1}$ and $L_{\pm 1}$. To compute the inverse of a 2×2 matrix of operators such as (6.7), we use the standard formula involving the inverse of the two operators in the main diagonal. It is straightforward to check that if $g_{22}(\mathbf{x}, \mathcal{V}_1) + f_{11}(\mathcal{V}_1, \mathcal{V}_2)$ and $L_{\pm 1}$ are invertible, then either both $\mathcal{L}''_{\pm 1}$ can be inverted this way, or neither.

7. Domain of the operator \mathcal{M}'

Lemma 6.1 covers most of what is needed to find a domain where \mathcal{M}' is well-defined and analytic. Among the missing steps is the scaling:

Lemma 7.1. *There exists $\varepsilon > 0$ such that the following holds. Let $f \in \mathcal{B}_{V_f}(1)$ and $g \in \mathcal{B}_{V_g}(1)$. Then $f \circ \ell_*^3$ belongs to $\mathcal{B}_f(1 + \varepsilon)$ and $g \circ \ell_*^3$ belongs to $\mathcal{B}_g(1 + \varepsilon)$.*

This simple result is proved together with Lemma 6.1; see Sections 9,10.

The main problem that is not yet covered in Lemma 6.1 is the solution of the midpoint equations (4.3) and (6.3) for functions f and g that are not necessarily even. However, the non-even part can be taken arbitrarily small; so we can use simple perturbation theory, as we will now describe.

The system of equations (4.3) can be written in the form $K'(p, q) = 0$, where $p = (f, g)$ and $q = (\mathcal{V}_0, \mathcal{W}_0)$. Here p is restricted to some open neighborhood U_p of p_* in $\mathcal{B}_p(1) = \mathcal{B}_f(1) \times \mathcal{B}_g(1)$, and q is restricted to $U_q = V_0 \times W_0$, with V_0 and W_0 as specified in Lemma 6.1. Then K is analytic on $U_p \times U_q$ and vanishes at (p_*, q_*) , where q_* is the pair of midpoint functions $(\mathcal{V}_0, \mathcal{W}_0)$ for p_* . Next we consider pairs $p_* + p'$ of generating functions with p' close to zero. To simplify notation, let us write (p, q) in place of (p_*, q_*) . To first order in the perturbation p' , the equation $K'(p + p', q + q') = 0$ for the midpoint functions $q + q'$ becomes $D_1 K'(p, q)p' + D_2 K'(p, q)q' = 0$. Assume first that p' has a fixed parity: $\mathbb{S}f' = \sigma f'$ and $\mathbb{S}g' = \sigma g'$, with $\sigma = \pm 1$. Then a straightforward computation shows that the equation $D_1 K'(p, q)p' + D_2 K'(p, q)q' = 0$ becomes

$$\mathcal{L}'_{+\sigma}(\mathcal{V}'_0 + \sigma \mathbb{S}\mathcal{W}'_0) = 0, \quad \mathcal{L}'_{-\sigma}(\mathcal{V}'_0 - \sigma \mathbb{S}\mathcal{W}'_0) = h', \quad (7.1)$$

with

$$h' = -2g'_2(\mathbf{x}, \mathcal{V}_0) - 2f'_1(\mathcal{V}_0, -\mathbb{S}\mathcal{V}_0), \quad (7.2)$$

where $\mathcal{L}'_{\pm 1}$ are the operators defined in (6.5). By Lemma 6.1, these operators are invertible, and by linearity, this implies that $D_2 K'(p, q)$ is invertible. Thus, by the implicit function theorem, the equation $K'(p, q) = 0$ defines an analytic map $p \mapsto q$ in some open neighborhood of p_* , taking values in $V_0 \times W_0$.

Similarly, the system (6.3) can be written in the form $K''(p, q) = 0$, where $p = (f, g)$ and $q = (\mathcal{V}_1, \mathcal{W}_1, \mathcal{V}_2, \mathcal{W}_2)$. For fixed parity σ , the equations $D_1 K''(p, q)p' + D_2 K''(p, q)q' = 0$ become

$$\mathcal{L}''_{+\sigma} \begin{bmatrix} \mathcal{V}'_1 + \sigma \mathbb{S}\mathcal{W}'_1 \\ \mathcal{V}'_2 + \sigma \mathbb{S}\mathcal{W}'_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathcal{L}''_{-\sigma} \begin{bmatrix} \mathcal{V}'_1 - \sigma \mathbb{S}\mathcal{W}'_1 \\ \mathcal{V}'_2 - \sigma \mathbb{S}\mathcal{W}'_2 \end{bmatrix} = \begin{bmatrix} h'' \\ h'' \end{bmatrix}, \quad (7.3)$$

with

$$h'' = -2f'_2(\mathcal{V}_1, \mathcal{V}_2) - 2g'_1(\mathcal{V}_2, \mathcal{W}_2), \quad (7.4)$$

where $\mathcal{L}''_{\pm 1}$ are the operators defined in (6.7). By the same argument as above, we conclude that the equation $K''(p, q) = 0$ defines an analytic map $p \mapsto q$ in some open neighborhood of p_* , taking values in $V_1 \times W_1 \times V_2 \times W_2$.

In the following theorem we use the fact [23] that the points $\xi_{1,k,n}$ and $\xi_{2,k,n}$ in (3.13) have been chosen to lie in \mathcal{D}_f and \mathcal{D}_g , respectively. The choice of the constants $\Phi_{j,k,n}$ and polynomials E_8, E_9, \dots, E_{13} is not relevant at this point.

Theorem 7.2. *The equation (3.9) defines an analytic map $\mathcal{M} : B' \rightarrow \mathcal{B}_p(1)$, where B' is some open neighborhood of p_* in $\mathcal{B}_p(1)$. The derivative $D\mathcal{M}(p)$ is compact, for every $p \in B'$. Similarly, the equation (3.12) defines an analytic map $\mathcal{M}' : B' \rightarrow \mathcal{B}_p(1)$ with compact derivative.*

Proof. The analyticity of $\mathcal{M}_* : B \rightarrow \mathcal{B}_p(1+\epsilon)$ in some open neighborhood B of p_* in $\mathcal{B}_p(1)$ follows from the above discussion, together with Lemma 6.1 and Lemma 7.1. Furthermore, the linear map $p \mapsto \mathcal{U}(t, p)$ defined by (3.8) is continuous from $\mathcal{B}_p(1+\epsilon)$ to $\mathcal{B}_p(1+\epsilon/2)$ for $t \in \mathbb{C}^7$ sufficiently close to zero, and $\mathcal{U}(t, p)$ depends analytically on t . Thus, by the chain rule, the equation (3.9) defines an analytic map $\mathcal{M} : B' \rightarrow \mathcal{B}_p(1+\epsilon/2)$ on some open neighborhood B' of p_* in $\mathcal{B}_p(1)$. Clearly the same holds for the map \mathcal{M}' . The compactness of $D\mathcal{M}(p)$ and $D\mathcal{M}'(p)$, as linear operators on $\mathcal{B}_p(1)$, follows from the fact that the inclusion map $\mathcal{B}_p(1+\epsilon/2) \rightarrow \mathcal{B}_p(1)$ is compact. QED

8. Hyperbolicity

In order to simplify notation, we write \mathcal{B}_p instead of $\mathcal{B}_p(r)$ whenever $r = 1$. Similarly for all other spaces and domains that depend on a choice of r in (4.9) or (4.10).

Denote by \mathcal{V}_j and $\mathcal{W}_j = -\mathcal{S}\mathcal{V}_j$ the midpoint functions for the fixed point $p_* = (f_*, g_*)$ of the operator \mathcal{M}_* . For every pair $p = (f, g)$ in \mathcal{B}_p we have

$$D\mathcal{M}_*(p_*)p = (\lambda_*\mu_*)^{-3}(\hat{f}, \hat{g}) \circ \ell_*^3, \quad (8.1)$$

where \hat{f} and \hat{g} are the functions defined by (6.1) and (6.2), respectively. The derivative of \mathcal{M} at p_* is given by (3.10), and for the derivative of \mathcal{M}' we have

$$D\mathcal{M}'(p_*)p = D\mathcal{M}_*(p_*)p - \sum_{k=1}^7 [\Phi_{1,k}(f) + \Phi_{2,k}(g)]E_k - \sum_{k=8}^{13} \Phi_{3,k}(DC(p_*)p)E_k. \quad (8.2)$$

Here, DC is its derivative of the commutator (3.3). With \mathcal{V} and \mathcal{W} as defined above,

$$DC(p_*)p = g(\mathbf{x}, \mathcal{V}) + f(\mathcal{V}, \mathbf{y}) - f(\mathbf{x}, \mathcal{W}) - g(\mathcal{W}, \mathbf{y}). \quad (8.3)$$

A straightforward computation shows that $D\mathcal{M}_*(p_*)$ preserves parity, in the sense that the even subspace $\mathcal{B}_p^0 = \mathcal{B}_f^0 \times \mathcal{B}_g^0$ and the odd subspace $\mathcal{B}_p^1 = \mathcal{B}_f^1 \times \mathcal{B}_g^1$ of \mathcal{B}_p are both invariant under $D\mathcal{M}_*(p_*)$. By our choice of polynomials E_8, E_9, \dots, E_{13} and constants in (3.13), the derivatives of \mathcal{M} and \mathcal{M}' at p_* are parity-preserving as well. And of course,

these quantities have been chosen in such a way that, numerically, the derivative of \mathcal{M}' at p_* has the desired spectrum. For the precise values we refer to [23].

By Theorem 7.2 the linear operator $D\mathcal{M}'(p_*)$ is compact on \mathcal{B}_p . Since we had mentioned it only in passing, let us recall that the norm on \mathcal{B}_p is given by

$$\|p\| = \|f\|_{e_f} + \|g\|_{e_g}, \quad p = (f, g). \quad (8.4)$$

Denote by A the restriction of $D\mathcal{M}'(p_*)$ to the (invariant) even subspace \mathcal{B}_p^0 . In order to prove that A has no eigenvalues outside the open unit disk, except for a simple eigenvalue $\delta^3 > 1$, we consider a map \mathcal{F} that can be expected to be a contraction (for some norm) and whose fixed point is the normalized eigenvector \bar{p} of A for the eigenvalue δ^3 . A simple candidate for such a map is $p \mapsto \|Ap\|^{-1}Ap$. We choose a smooth modification $\mathcal{F} : p \mapsto [\varphi(Ap)]^{-1}Ap$ of this map, where φ is a continuous linear functional on \mathcal{B}_p^0 , with the property that $\varphi(p) \approx \|p\|$ for p near \bar{p} . See [23] for the precise definition of φ . The iterates of \mathcal{F} and their derivatives are given by

$$\mathcal{F}^n(p) = \frac{1}{\varphi(A^n p)} A^n p, \quad D\mathcal{F}^n(p)h = \frac{1}{\varphi(A^n p)} \left[A^n h - \frac{\varphi(A^n h)}{\varphi(A^n p)} A^n p \right]. \quad (8.5)$$

Lemma 8.1. *Let $m = 6$. There exists a pair $p_0 \in \mathcal{B}_p^0$ and a real number $r > 0$ such that the following holds. Let $B = \{p \in \mathcal{B}_p^0 : \|p - p_0\| \leq r\}$. Then the map \mathcal{F}^m is a contraction on B and thus has a unique fixed point $\bar{p} \in B$. Both \bar{p} and $\vartheta = \varphi(A\bar{p})$ are real, and $\delta = \vartheta^{1/3}$ satisfies the bound (1.2). Furthermore, $\|D\mathcal{F}^m(\bar{p})\| < \vartheta^{-m}$.*

Our proof of this lemma is computer-assisted and will be described in Sections 9,10. With the same tools we also prove

Lemma 8.2. *The fifth power of $D\mathcal{M}'(p_*)$, when restricted to the (invariant) odd subspace \mathcal{B}_p^1 , is a contraction.*

As an immediate consequence of these two lemmas we have the

Corollary 8.3. *The operator $D\mathcal{M}'(p_*)$ has no spectrum outside the open unit disk, except for a simple eigenvalue δ^3 that satisfies (1.2).*

Proof. By Lemma 8.2 it suffices to prove the claim for the operator A .

From (8.5) we see that any fixed point p of \mathcal{F}^m satisfies $\varphi(p) = 1$ and is an eigenvector of A^m with eigenvalue $\varphi(A^m p)$. By uniqueness, every eigenvector of A^m near \bar{p} is a constant multiple of \bar{p} . This applies in particular to $\bar{p} + \varepsilon A\bar{p}$ for $\varepsilon \neq 0$ close to zero, so \bar{p} is an eigenvector of A ; and the corresponding eigenvalue is ϑ .

Consider the operator B defined by $Bp = Ap - \varphi(Ap)\bar{p}$. Using (8.5) we have

$$D\mathcal{F}(\bar{p}) = \vartheta^{-1}B, \quad B^n p = A^n p - \varphi(A^n p)\bar{p}. \quad (8.6)$$

And $\|B^m\| < 1$ by Lemma 8.1. Assume for contradiction that the eigenvalue ϑ of A is not simple. Then there exists $p \in \mathcal{B}_p$ such that $(A - \vartheta)p = \bar{p}$. A straightforward computation

shows that this p satisfies $B^n p = \vartheta^n[p - \varphi(p)\bar{p}]$ for all n . But this is incompatible with B^m being a contraction. Similarly, it is clear from the second identity in (8.6) that A cannot have an eigenvalue of modulus ≥ 1 , besides ϑ . **QED**

Our next goal is to relate the operator \mathcal{M} for pairs of generating functions to the operator \mathfrak{M} for pairs of area-preserving maps. At this point it becomes relevant that two generating functions that differ only by a constant yield the same map. We have ignored this fact up to now in order to simplify the description. Denote by \mathcal{B}'_f the subspace of \mathcal{B}_f consisting of all function $f \in \mathcal{B}_f$ whose constant term (see Definition 4.1) is zero. \mathcal{B}'_g is defined similarly, and $\mathcal{B}'_p = \mathcal{B}'_g \times \mathcal{B}'_f$. We may assume that $p_* \in \mathcal{B}'_p$. Since the constant-term issue is trivial, let us simply redefine \mathcal{M} to take values in \mathcal{B}'_p , by setting the constant terms in $\mathcal{M}(p)$ equal to zero. Similarly for the operator \mathcal{M}' .

Denote by Υ_g the map $g \mapsto G$ defined by the equation (1.6). To be more precise, we assume that g belongs to \mathcal{B}'_g and is close enough to g_* to satisfy a twist condition. Then $\Upsilon_g(g)$ is well-defined as a map from \mathbb{D}_G to \mathbb{R}^2 . Similarly, denote by Υ_f the map that associates to a generating function $f \in \mathcal{B}'_f$ close to f_* the corresponding map $F : \mathbb{D}_F \rightarrow \mathbb{R}^2$. Now we need to check that the maps $F = \Upsilon_f(f)$ and $G = \Upsilon_g(g)$ can be composed as required by (1.7).

Lemma 8.4. *There exists $r < \frac{15}{16}$ such that the following holds. Denote by \mathcal{V}_j and \mathcal{W}_j the midpoint functions for p_* . Let $f \in \mathcal{B}'_f(r)$ and $g \in \mathcal{B}'_g(r)$. Then the function \hat{f} given by (6.1) is well-defined on the domain of \mathcal{V}_0 , and the function \hat{g} given by (6.2) is well-defined on the domain of \mathcal{V}_1 .*

This lemma is proved easily by using the estimates on the midpoint function \mathcal{V}_j and \mathcal{W}_j obtained in the proof of Lemma 6.1. For the complete details we refer to [23].

Based on Lemma 5.3, Theorem 7.2, Corollary 8.3, and Lemma 8.4, we can now give a

Proof of Theorem 1.2. First, we note that Lemma 8.4 can be generalized as follows. Let B' be the domain of \mathcal{M} and \mathcal{M}' as described in Theorem 7.2. We may assume that B' is an open ball, centered at p_* . Consider the midpoint functions \mathcal{V}'_j and \mathcal{W}'_j associated with pairs $p' = (f', g')$ in B' . From the discussion preceding Theorem 7.2 we know that these midpoint functions depend analytically on p' . Thus, Lemma 8.4 remains true if we replace the functions \mathcal{V}_j and \mathcal{W}_j by \mathcal{V}'_j and \mathcal{W}'_j , respectively, assuming (which we shall) that the radius of B' is sufficiently small.

As described earlier, to every pair of generating functions $p = (f, g)$ in some open ball $B \subset \mathcal{B}'_p$ centered at p_* , we can associate two area-preserving maps $F = \Upsilon_f(f)$ and $G = \Upsilon_g(g)$. Using these two maps, define $\Upsilon_p(p) = (F, G)$.

Choose another ball $B_0 \subset B' \cap B$ centered at p_* , such that $\mathcal{M}(B_0)$ and $\mathcal{M}'(B_0)$ are included in B . Assuming (which we shall) that the radius of B_0 has been chosen sufficiently small, the above-mentioned generalization of Lemma 8.4 implies that if $p \in B_0$ and $P = \Upsilon_p(p)$, then $\mathfrak{M}(P)$ is well-defined and agrees with $\Upsilon_p(\mathcal{M}(p))$. We note that the change of coordinates Λ in the expression (1.7) for $\mathfrak{M}(P)$ depends on p and can be expressed explicitly in terms of the change of coordinates $U(t)$ given in (3.7), with $t = \mathcal{T}(p - p_*)$.

Clearly Υ_p is one-to-one on B . Equip $\mathfrak{B} = \Upsilon_p(B)$ with the topology that makes Υ_p a homeomorphism. Then \mathfrak{B} is a Banach manifold with a single chart $\Upsilon_p^{-1} : \mathfrak{B} \rightarrow B$. On the

open neighborhood $\mathfrak{B}_0 = \Upsilon_p(B_0)$ of $P_* = \Upsilon_p(p_*)$ we can define $\mathfrak{M}' = \Upsilon_p \mathcal{M}' \Upsilon_p^{-1}$. By Theorem 7.2 the operators \mathfrak{M} and \mathfrak{M}' are differentiable on \mathfrak{B}_0 , and in fact real analytic. The assertion in Theorem 1.2 concerning the spectrum of $D\mathfrak{M}'(P_*)$ follows from Corollary 8.3.

Next we consider map-pairs that commute near the fixed point $\bar{\omega}$ of $\mathbb{A} = \Lambda_*^3 F_*^{-1}$. Consider the associated point (\bar{x}, \bar{y}) and the set C described in Lemma 5.3. Pick $\epsilon > 0$ such that the closure of the square $R = \{(x, y) \in \mathbb{R}^2 : |x - \bar{x}| < \epsilon, |y - \bar{y}| < \epsilon\}$ is included in the domain of both $f \circ g$ and $g \circ f$, for every pair $p \in B$. This is possible if the radius of B has been chosen sufficiently small. By decreasing $\epsilon > 0$, if necessary, the commutators $\mathcal{C}(p)$ are uniformly bounded on R . Thus $B^c = \{p \in B : \mathcal{C}(p)(\xi) = 0 \text{ for all } \xi \in R\}$ is relatively closed in B . The image of B^c under Υ_p is the set $\mathfrak{B}^c(\Omega)$ mentioned in Theorem 1.2. Here Ω can be any sufficiently small open neighborhood of $\bar{\omega}$.

Consider now a pair $p \in B^c$. By Lemma 5.3, the commutator $\mathcal{C}(p)$ vanishes at all points $\xi_{3,k,n}$ appearing in (3.13). Thus, if $p \in B_0$ then $\mathcal{M}'(p) = \mathcal{M}(p)$. Notice also that $\mathfrak{M}(P)$ is related to $\mathfrak{M}_*(P)$ via a change of coordinates $I + \mathcal{O}(\|p - p_*\|)$. And by (2.5), the commutator for $\tilde{P} = \mathcal{M}_*(P)$ vanishes in some open neighborhood of $\tilde{F}(\Lambda_*^{-3}\bar{\omega})$. Thus, if the radius of B_0 has been chosen sufficiently small, then $\mathcal{C}(\mathcal{M}(p))$ vanishes in an open set that has a non-empty intersection with the square R . Here, we have used Lemma 5.3 and the continuity of \mathcal{M} . By analyticity, this implies that $\mathcal{M}(p)$ belongs to B^c .

To prove the last statement in Theorem 1.2, denote by \mathcal{W}^s and \mathcal{W}^u the local stable and unstable manifolds, respectively, for the operator \mathcal{M}' at p_* . Consider first the case where \mathcal{M} is not expanding on B^c . That is, there exists an open ball $B_1 \subset B_0$ centered at p_* such that $\mathcal{M}^n(B^c \cap B_1) \subset B^c \cap B_0$ for all n . Given that \mathcal{M} agrees with \mathcal{M}' on $B^c \cap B_0$, and that \mathcal{M}' is hyperbolic, this implies that $B^c \cap B_1 \subset \mathcal{W}^s$.

Next, consider the case where \mathcal{M} is expanding on B^c . Then there exists $N > 0$ such that the following holds. For every $n \geq N$ there exists a sequence of pairs $p_1, p_2, \dots \in B^c$ converging to p_* , such that the orbit of p_k contains a point p'_k at a distance between 2^{-n-1} and 2^{-n} from p_* . So some subsequence of $k \mapsto p'_k$ converges to a point $p''_n \in \mathcal{W}^u$, and $2^{-n-1} \leq \|p''_n - p_*\| \leq 2^{-n}$. Since B^c is closed in B , we have $p''_n \in B^c$. Now, given any $\xi \in R$, consider the function $h : \mathcal{W}^u \rightarrow \mathbb{R}$ defined by $h(p) = \mathcal{C}(p)(\xi)$. We have $h(p''_n) = 0$ for all $n \geq N$. Since \mathcal{W}^s is real analytic and $p''_n \rightarrow p_*$, it follows that $h = 0$. Given that $\xi \in R$ was arbitrary, we conclude that $\mathcal{W}^u \subset B^c$. **QED**

It seems surprising that we cannot exclude the possibility that p_* is an attracting fixed point for the restriction of \mathcal{M} to B^c . The underlying problem is that we know very little about how the set B^c embeds into the space \mathcal{B}_p . The most useful pieces of information are probably the scaling relation (2.5) for the commutator Θ , and the corresponding scaling of the Calabi invariant (see Subsection 3.1). One way to take advantage of these scaling properties would be the following.

Assume that the eigenvector \bar{p} of $D\mathcal{M}'(p_*)$ for the eigenvalue $\vartheta = \delta^3$ belongs to the subspace B^c . Then \bar{p} is also an eigenvector of $D\mathcal{M}(p_*)$, with eigenvalue ϑ . We claim that this implies that the local unstable manifold of \mathfrak{M} is included in B^c . Here is just a sketch of a proof: Since no power ϑ^n with $n \geq 2$ is an eigenvalue of $D\mathcal{M}(p_*)$, there exists a real analytic \mathcal{M} -invariant curve $t \mapsto p(t)$ such that $p(0) = p_*$ and $p'(0) = \bar{p}$, and $\mathcal{M}(p(\vartheta^{-1}t)) = p(t)$ for t near zero. Now consider the Taylor series of $(t, \omega) \mapsto P(t)(\omega)$ about the point $(0, \bar{\omega})$. Using the above-mentioned scaling properties, together with our

bounds on α , β , and δ , one finds that all pairs $P(t)$ for t near zero are necessarily commuting and have a zero Calabi invariant.

Within the framework developed in this paper, the “cleanest” way to prove the above assumption would be to replace the approximate eigenfunctions E_8, E_9, \dots, E_{13} in our definition (3.12) of \mathcal{M}' by the true eigenfunctions. If Corollary 8.3 remains true for this operator \mathcal{M}' , then we have $\bar{p} \in \mathcal{B}^c$, as is easy to see.

In practice, we cannot determine these eigenfunctions exactly. But it suffices to have good bounds (enclosures), as is the case for the function g_* and related quantities. Obtaining such bounds is a purely technical problem. The remaining part of our analysis should not change. Our only reason for working with approximate eigenfunctions was to avoid adding yet another level of complexity to the problem.

9. Reduction to simpler estimates

Here we give some estimates that are used in our programs. They concern mostly simple operations like the differentiation and composition of functions, and the norm and inverses of certain operators. We also describe why the remaining problem reduces to such estimates.

9.1. Derivatives

Derivatives of functions in \mathcal{B}_ρ reduce via (4.7) to derivatives of functions in \mathcal{A}_ρ . For these derivatives we use the following estimate from [17]. Given positive real numbers $\sigma < \tau$, and a non-negative integer k , define

$$W_k(\sigma, \tau) = \max_{m \geq k} W_{k,m}(\sigma, \tau), \quad W_{k,m}(\sigma, \tau) = \frac{m!}{(m-k)!} \left(\frac{\sigma}{\tau}\right)^m \sigma^{-k}. \quad (9.1)$$

Proposition 9.1. [17] *Let $r = (r_u, r_v)$, with $0 < r_u < \rho_u$ and $0 < r_v \leq \rho_v$. If $\phi \in \mathcal{A}_\rho$ and $k \geq 0$ then $\partial_u^k \phi \in \mathcal{A}_r$ and $\|\partial_u^k \phi\|_r \leq W_k(r_u, \rho_u) \|\phi\|_\rho$.*

An analogous bound holds of course for derivatives with respect v .

9.2. Composition

The composed function $f(\mathcal{V}_1, \mathcal{V}_2)$ in (6.2) is estimated by using the Banach algebra property of the spaces $\mathcal{B}_{V_g}(r)$. More generally, let \mathcal{B} be any commutative Banach algebra over \mathbb{C} , with unit $\mathbf{1}$. Let $X, Y \in \mathcal{B}$ and $T = X + Y$. Pick $\varrho = (t_0, s_0, \rho_u, \rho_v)$. Given f as in (4.7), define $f(X, Y) = \phi(U, V) + T\psi(U, V)$, where $V = X - Y - s_0\mathbf{1}$ and $U = T^2 - t_0^2\mathbf{1} + bV$.

Proposition 9.2. [17] *Assume that $\|T\| \leq \rho_t$ and $\|U\| \leq \rho_u$ and $\|V\| \leq \rho_v$. If f belongs to \mathcal{B}_ϱ then $f(X, Y)$ belongs to \mathcal{B} and $\|f(X, Y)\| \leq \|f\|_\varrho$. Furthermore, the map $(X, Y) \mapsto f(X, Y)$ is analytic, on any open domain in $\mathcal{B} \times \mathcal{B}$ where the assumptions above are satisfied.*

In the special case $\mathcal{B} = \mathbb{C}$ we get the estimate $|f(x, y)| \leq \|f\|_\varrho$, for all $(x, y) \in \mathcal{D}_\varrho$.

The proof of this proposition is a simple exercise in power series. In particular, from the representation (4.6) one gets

$$\|\phi(U, V)\| \leq \sum_{m,n} |\phi_{m,n}| \|U^m\| \|V^n\| \leq \sum_{m,n} |\phi_{m,n}| \|U\|^m \|V\|^n, \quad (9.2)$$

and the last sum is bounded by $\|\phi\|_\rho$ with $\rho = (\rho_u, \rho_v)$, if $\|U\| \leq \rho_u$ and $\|V\| \leq \rho_v$.

The second inequality in (9.2) can be far from optimal if \mathcal{B} is a function space and the norm of U (or V) is significantly larger than its sup-norm. In our program we deal with this problem as follows. For $k > 1$ define

$$\|U\|_k = \|U^k\|^{1/k}, \quad C_U = \max_{0 \leq r < k} \|U\|_k^{-r} \|U^r\|, \quad (9.3)$$

if U is nonzero. Notice that $\lim_k \|U\|_k$ is the spectral radius of the operator “multiplication by U ”, which in turn is the sup-norm of U . Let us now fix $k > 1$. Writing a nonnegative integer m as $m = qk + r$ with $0 \leq r < k$, we have

$$\|U^m\| \leq \|U^k\|^q \|U^r\| = \|U\|_k^{kq} \|U^r\| \leq C_U \|U\|_k^m. \quad (9.4)$$

Similarly we have $\|V^n\| \leq C_V \|V\|_k^n$. These are the bounds that we use (for large m or n) to estimate the first sum in (9.2). Notice also that $\phi(U, V) \in \mathcal{B}$ whenever $\phi \in \mathcal{A}_\rho$ and $\|U\|_k \leq \rho_u$ and $\|V\|_k \leq \rho_v$.

9.3. Implicit equations

Implicit equation such as the midpoint equation $\mathcal{K}'(\mathcal{V}_0) = 0$ are solved by using a quasi-Newton method of the following type.

Let \mathcal{X} and \mathcal{Y} be Banach spaces. Let F be a C^1 function defined on some open domain in \mathcal{X} , taking values in \mathcal{Y} . The goal is to solve the equation $F(x) = 0$. To accomplish this, we choose an approximate solution x_0 in the domain of F , and a bounded linear operator $M_0 : \mathcal{Y} \rightarrow \mathcal{X}$ that approximates $DF(x_0)^{-1}$. In addition, we choose a positive integer n .

Proposition 9.3. *Assume that M_0 is one-to-one, that F is of class C^1 in an open neighborhood of a ball $B = \{x \in \mathcal{A} : \|x - x_0\| \leq r\}$, and that*

$$\|M_0 F(x_0)\| \leq \varepsilon \leq \frac{n}{n+1} r, \quad \|I - M_0 DF(x)\| \leq \frac{1}{n+1},$$

for all $x \in B$. Then there exists a unique $x \in B$ such that $F(x) = 0$.

The proof is a straightforward: apply the contraction mapping theorem to the map $x \mapsto x - M_0 F(x)$.

9.4. Linear operators

The spaces \mathcal{A}_ρ and \mathcal{B}_ρ are also convenient for estimating linear operators. In particular, the operator norm of a continuous linear map $L : \mathcal{A}_\rho \rightarrow \mathcal{A}_\rho$ is given by

$$\|L\| = \sup_{m,n} \|Le_{m,n}\|_\rho, \quad e_{m,n}(u, v) = c_{m,n} u^m v^n, \quad (9.5)$$

where $c_{m,n}$ is defined in such a way that $e_{m,n}$ has norm one.

Linear equations $Lx = y$ are considered in the context of Proposition 9.3. The linear operator $L : \mathcal{X} \rightarrow \mathcal{Y}$ is assumed to be bounded, and $M_0 : \mathcal{Y} \rightarrow \mathcal{X}$ is an approximate inverse of L . Applying Proposition 9.3 to the function F defined by $F(x) = Lx - y$, we obtain the

Corollary 9.4. *Assume that M_0 is one-to-one, and that*

$$\|M_0(Lx_0 - y)\| \leq \varepsilon \leq \frac{n}{n+1}r, \quad \|I - M_0L\| \leq \frac{1}{n+1}. \quad (9.6)$$

Then there exists a unique $x \in \mathcal{X}$ such that $\|x - x_0\| \leq r$ and $Lx = y$.

This corollary is also used (with $y = I$ and $x_0 = M_0$) to estimate the inverse of an operator L . Similarly, we estimate the reciprocal of a function f by inverting the multiplication operator $L : h \mapsto hf$. In this case, or whenever M_0 commutes with L , the one-to-one condition on M_0 can be dropped.

9.5. Contracting iterates

The following proposition, with a suitable choice of parameters, is used to prove the claim in Lemma 8.1 concerning the contraction property of the map \mathcal{F}^6 defined in (8.5).

Let \mathcal{B} be any Banach space, and let $m \geq 1$.

Proposition 9.5. *Let $p_0 \in \mathcal{B}$ and $0 < r < r'$. Define $B = \{p \in \mathcal{B}_p : \|p - p_0\| \leq r\}$ and $B' = \{p \in \mathcal{B}_p : \|p - p_0\| < r'\}$. Let $\mathcal{F} : B' \rightarrow \mathcal{B}$ be of class C^1 . Assume that for $n = 1, 2, \dots, m$,*

$$\|\mathcal{F}^n(p_0) - p_0\| \leq \varepsilon_n, \quad \|D\mathcal{F}^n(p)\| \leq K_n, \quad \varepsilon_n + K_n r < r', \quad (9.7)$$

for all $p \in B$ in the domain of \mathcal{F}^n . Then \mathcal{F}^n maps B into B' , for all $n \leq m$. Furthermore, if $\varepsilon_m + K_m r \leq r$ then \mathcal{F}^m is a contraction on B .

The proof of this proposition is straightforward and thus will be omitted.

We note that $r' \gg r$ in our application of this proposition, since our map \mathcal{F} is far from being a contraction (for the given norm).

9.6. Analyticity

The analyticity properties mentioned in Lemma 6.1 follow “by construction”. In order to explain why, let us recall some facts about analytic maps.

Let \mathcal{X} and \mathcal{Y} be Banach spaces over \mathbb{C} , and let $B \subset \mathcal{X}$ be open. A map $F : B \rightarrow \mathcal{Y}$ is said to be analytic if it is Fréchet differentiable. Thus, sums, products, and compositions of analytic maps are analytic. Equivalently, F is analytic if it is locally bounded, and if $\psi \circ F \circ \phi$ is analytic for arbitrary continuous linear maps $\phi : \mathbb{C} \rightarrow \mathcal{X}$ and $\psi : \mathcal{Y} \rightarrow \mathbb{C}$. This shows in particular that uniform limits of analytic functions are analytic. These and other useful facts can be found e.g. in [1].

So the reason why e.g. the solution \mathcal{V}_0 of the midpoint equation $\mathcal{K}'(\mathcal{V}_0) = 0$ depends analytically on the functions f and g is that $\mathcal{K}'(\mathcal{V}_0)$ depends analytically (in fact linearly)

on f and g , and that the solution is obtained via an iteration $\mathcal{V} \mapsto \mathcal{V} - M_0\mathcal{K}'(\mathcal{V})$ that converges uniformly. The same applies to all of our implicit equations: We always use the contraction mapping theorem, which yields a solution via uniform limits.

9.7. Remaining tasks

In Sections 4–8 we have reduced the proofs of Theorems 1.1 and 1.2 to the task of proving Theorem 4.2 and several technical lemmas. Among these lemmas, only Lemmas 6.1, 8.2, and 8.1 are sufficiently involved to require arguments that go beyond straightforward “computations”. What we have done in the preceding subsections was to reduce the proof of these three lemmas to computations as well.

The objects in these computations are elements in one of our Banach spaces, such as \mathbb{R} , \mathcal{A}_ρ or \mathcal{B}_ρ . By “computing” an element x in some space \mathcal{X} we mean finding a set $X \subset \mathcal{X}$ that contains x . The types of enclosures X that we use, and the procedures for finding them, will be described in the next section.

One task that is potentially infinite is the computation of an operator norm. However, the operators L whose norms we have to estimate are compact. In fact, they owe their compactness to the scaling $f \mapsto f \circ \lambda^3$. So in the notation of (9.5), the norm $\|Le_{m,n}\|_\rho$ tends to zero as $m+n$ increases. In this case, it suffices to compute $Le_{m,n}$ for $m+n \leq N$, plus the image LE of a set E that encloses all functions $e_{m,n}$ with $m+n > N$. To get an accurate bound on $\|L\|$, if desired, it suffices to choose N in such a way that $\|LE\|_\rho \leq \|Le_{m,n}\|_\rho$ for some pair (m,n) with $m+n \leq N$.

Concerning Theorem 4.2, we recall that a similar result has already been proved in reference [17], so we refer to [17,23] for details.

10. Organization of the programs

In this section we describe how the remaining proofs are organized. The precised definitions, and all other details of the proof, can be found in the source code of our programs [23], written in the programming language Ada [19]. Similar techniques have been used in other computer-assisted proofs, including [6,12,17,18], to list just a few on area-preserving maps.

10.1. Basic strategy

The remaining task is compute the quantities that appear in any of the yet-unproved lemmas, and to verify that they have the desired properties. It is important to notice that the original claims all have been reduced to inequalities. At this point, the main objects besides numbers are functions in the spaces \mathcal{A}_ρ or \mathcal{B}_ρ , and no operation involved is more complex than the composition of such functions.

The first step is to implement procedures that allows us to work at the level of these main objects. In particular, if \mathbf{F} and \mathbf{G} represent enclosures for two functions $F \in \mathcal{B}_\rho$ and $G \in \mathcal{B}_\rho$, respectively, then we want $\mathbf{F}*\mathbf{G}$ to yield a rigorous enclosure for the product FG . Once such basic operations are implemented, we can use the propositions from Section 9 to do the same for more complex operations, such as quotients \mathbf{F}/\mathbf{G} or composed maps $\mathbf{Comp}(\mathbf{Phi}, \mathbf{U}, \mathbf{V})$ or partial derivatives $\mathbf{DerX}(\mathbf{F})$ etc. Then it becomes possible to verify a

claim like $\|M_0\mathcal{K}'(\mathcal{V})\| < \varepsilon$ by executing a command like `Norm(M0*KPrime(V))<Eps` and checking that the result is `True`.

Our choice of enclosures will be described in detail below. This sets the framework for implementing bounds like `Comp` or `Norm`. The actual implementation is best (most efficiently and accurately) described by the source code of our programs [23]. As a rough guide to this code we will mention the program units (Ada packages) where the entities being discussed are defined.

10.2. Enclosures

To every space \mathcal{X} considered we associate a finite collection $\mathfrak{E}(\mathcal{X})$ of subsets of \mathcal{X} that are representable on the computer. For a product space $\mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_n$ we choose $\mathfrak{E}(\mathcal{X})$ to be the collection of all sets $X_1 \times \dots \times X_n$ with $X_j \in \mathfrak{E}(\mathcal{X}_j)$ for each j , unless specified otherwise. An “enclosure” for an element $x \in \mathcal{X}$ is a set $X \in \mathfrak{E}(\mathcal{X})$ that contains x . A “bound” on a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a map $F : \mathfrak{E}(\mathcal{X}) \rightarrow \mathfrak{E}(\mathcal{Y}) \cup \{\text{undefined}\}$, with the property that $f(x) \in F(X)$ whenever $x \in X \in \mathfrak{E}(\mathcal{X})$, unless $F(X) = \text{undefined}$. In practice, if $F(X) = \text{undefined}$ then the program halts with an error message.

Each collection $\mathfrak{E}(\mathcal{X})$ corresponds to a data type in our programs. A simple type of enclosure is associated with a data type `Ball2`, which consists of all triples $\mathbf{S}=(\mathbf{S.C}, \mathbf{S.R}, \mathbf{S.B})$, where $\mathbf{S.C}$ is a representable real number (type `Rep`), and where $\mathbf{S.R}$ and $\mathbf{S.B}$ are nonnegative representable real numbers (type `Radius`). If \mathcal{X} is any Banach algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ with unit $\mathbf{1}$, we include in $\mathfrak{E}(\mathcal{X})$ all sets $\mathbf{S}^b = (\mathbf{S.C} + \mathbf{S.R} \mathbb{U}_{\mathbb{F}})\mathbf{1} + \mathbf{S.B} \mathbb{U}_{\mathcal{X}}$ where \mathbf{S} can be any `Ball2`. Here $\mathbb{U}_{\mathbb{F}}$ and $\mathbb{U}_{\mathcal{X}}$ denote the closed unit balls in \mathbb{F} and \mathcal{X} , respectively.

For the type `Radius` we use a standard Ada type `Long_Long_Float` with 64 mantissa bits (with the Gnat compiler [21]). The type `Rep` is either `Long_Long_Float` as well, or an MPFR floating point type [22] with up to 320 mantissa bits, depending on the program. Both types support controlled rounding [20] for the basic arithmetic operations. It is straightforward to implement bounds on the corresponding operations in a Banach algebra \mathcal{X} , using enclosures \mathbf{S}^b with \mathbf{S} of type `Ball2`. For details we refer to the packages `Flts.Std.Balls2` and `MPFR.Floats.Balls2` in [23]. In the cases $\mathcal{X} = \mathbb{R}$ and $\mathcal{X} = \mathbb{C}$, these are our only sets in $\mathfrak{E}(\mathcal{X})$. For the spaces $\mathcal{X} = \mathcal{A}_\rho$ and $\mathcal{X} = \mathcal{B}_\varrho$ we now define additional sets and bounds.

Our collection $\mathfrak{E}(\mathcal{A}_\rho)$ also contains sets \mathbf{T}^b associated with pairs $\mathbf{T}=(\mathbf{T.R}, \mathbf{T.C})$ from a data type `Taylor2`. The component $\mathbf{T.R}$ is $\rho = (\rho_u, \rho_v)$, which we always take to be a pair of type `Radius`. To be more specific about the component $\mathbf{T.C}$, let us first describe the representation used in [17]. There, $\mathbf{T.C}$ is a two-dimensional array with entries $\mathbf{T.C}(1..D, 1..D)$ of type `Ball2`. The corresponding set in $\mathfrak{E}(\mathcal{A}_\rho)$ is given by $\mathbf{T}^b = \sum \mathbf{T.C}(M, N)^b \mathbf{u}^M \mathbf{v}^N$, with the sum being restricted to $M + N \leq D$. Here $\mathbf{u}(u, v) = u$ and $\mathbf{v}(u, v) = v$. Notice that each term $\mathbf{T.C}(M, N)^b$ in the above sum is itself a set in $\mathfrak{E}(\mathcal{A}_\rho)$ and can contain non-polynomial functions. Our implementation in [23] differs from what we have just described only in that the elements $\mathbf{T.C}(M, N)$ with $M + N \leq D$ are stored in a one-dimensional array $\mathbf{T.C}(0..L)$ with $L = D(D + 3)/2$. Using our `Ball2`-based bounds, it is straightforward to implement `Taylor2`-based bounds on the basic operations in \mathcal{A}_ρ . Our `Taylor2`-based bounds are defined in the package `Taylor2`.

Next consider a space \mathcal{B}_ϱ with $\varrho = (t_0, s_0, b, \rho)$. For $\mathfrak{E}(\mathcal{B}_\varrho)$ we use a data type `Fun`,

whose members are quadruplets $F=(F.A,F.E,F.P,F.Q)$, where $F.A$ specifies the parameters (t_0, s_0, b) and $F.E$ is of type `Boolean`; while $F.P$ and $F.Q$ are of type `Taylor2`, with $F.P.R$ and $F.Q.R$ equal to ρ . If $F.E=True$ then F^b consists of all functions (4.7) with $\phi \in F.P^b$ and $\psi \in F.Q^b$. The type `Fun` with $F.E=False$ exists only for reasons of efficiency. For a discussion of this type we refer to [17]. `Fun`-based bounds on operations involving functions in the spaces \mathcal{B}_ρ are defined in the package `Funs`.

Bounds on solutions of implicit equations use Proposition 9.3 via one of the generic packages `Newton` or `Zeros`. For basic linear algebra we use generic packages `Matrices` and `Vectors` etc. The type `FunVec` is a `Vector` with components of type `Fun`. It is used e.g. to store powers (type `FunPowers`) of functions $U, V \in \mathcal{B}_\rho$ that are used repeatedly in compositions $\phi(U, V) = \sum \phi_{m,n} U^m V^n$. Enclosures for the operators \mathcal{L}'_{-1} and \mathcal{L}''_{-1} use the types `Flop` and `Flop2`, respectively, where `Flop` is a `FunVec(1..2)`, and where `Flop2` is a `Matrix(1..2, 1..2)` with components of type `Flop`. Bounds based on these types are also defined in `Funs`.

We should add that the packages `Taylor2` and `Funs` are based on a generic type `Scalar`, and the above description applies only when these packages are instantiated with `Scalar => Ball2`.

10.3. Main procedures and programs

The procedures that are directly related to renormalization are defined in the package `RG` and its children: `RG.Fix` (used to solve the fixed point problem for \mathcal{N}) and `RG.Maps` (used to analyze map-domains) and `RG.Spec` (used for the analysis of \mathcal{M}'). Our main programs do little else than invoking a few procedures from these packages.

`RG` is again based on a generic type `Scalar`. It uses two instantiations of `Funs` side by side: the main version `Funs(Scalar => Scalar)` named `SFuns`, and a “numeric” version `Funs(Scalar => Reals)` named `NFuns`. The latter is used whenever an estimate needs an approximate solution, which is the case for all of our implicit equations. But approximate solutions that are needed repeatedly can be (and have been) saved to file and are read from file if available.

Among the data files in [23] are approximate fixed point for the maps \mathcal{N} and \mathcal{F} , approximate midpoints functions, approximate inverses for the operators $DK'(\mathcal{V}_0)$ and $DK''(\mathcal{V}_1, \mathcal{V}_2)$, and some approximate eigenvectors for $DM_*(p_*)$. The labeling of eigenvalues and eigenvectors that is used in our programs is the one given in Tables 1, 2. The eigenvalue δ^3 and eigenvector \bar{p} are labeled `03`.

The proof of Theorem 4.2 uses a quasi-Newton map for the operator \mathcal{N} near the approximate fixed point. As in [17], the main work is carried out by the procedures `RG.Fix.Renorm` and `RG.Fix.DContractNorm`. The fixed point g_* is shown to lie in a ball of radius less than 10^{-89} , centered at a polynomial approximation of degree 200. We note that the enclosure for g_* is taken slightly larger than necessary. As a by-product, our enclosure for λ_* is wider than necessary, and this ensures e.g. the existence of the $\varepsilon > 0$ in Lemma 7.1.

Lemma 5.2 is proved by running the program `CheckMaps`, which calls procedures from the package `RG.Maps`. The same program is also used to verify the inequalities given in Theorem 4.2. The claims in Lemma 5.3 concerning the map \mathbb{A} are verified by the program

`AllEigen3`. The proof of Lemma 6.1 uses the procedures `MidPointVf` and `MidPointVg` from `RG.Spec`. They are called from the program `WriteRG3Data`, which also carries out the domain checks for Lemma 8.4.

Lemma 8.1 is proved in three steps. Using an approximate fixed point p_0 for \mathcal{F} and fixed constants K_n defined by `RGParam.XC3`, the procedure `RG.Spec.Contr3Fix` determines suitable values of the remaining constants in Proposition 9.5 and verifies all inequalities in this proposition, except for the bounds $\|D\mathcal{F}^n(p)\| \leq K_n$. An enclosure for $D\mathcal{F}(p)$ for arbitrary $p \in B'$ is determined in `RG.Spec.XContr3Mat`, and the necessary bounds on $\|D\mathcal{F}^n(p)\|$ for $n = 1, 2, \dots, 6$ are computed by the program `XMatNorm`. The same program is used also to prove Lemma 8.2.

We note that controlling the derivative of \mathcal{M}' is much more delicate than controlling the map itself, since $D\mathcal{M}'(p_*)p$ is not stationary with respect to variations of the midpoint functions \mathcal{V}_j . Getting good bounds on these functions was rather challenging. Besides high degrees, large mantissas, and other technicalities [23], it required a careful choice of domains.

What our programs effectively do is to reduce the original problem to a finite sequence of trivial numerical computations (with specified rounding). These computations have been carried out successfully by a standard desktop computer. We used the Gnat compiler [21] to generate the executable code and to link with the MPFR library [22]. On a current desktop computer with four 3.4GHz processors, the total running time of our programs is roughly 240 hours. The files in [23] include the source code of our programs, data files, log files, and a `README` file with instructions on how to compile and run the programs. Bounds on λ , μ , α , β , and δ that are more accurate than the ones given in this paper can be found in the log file `alleigen3.log`.

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