Some symmetric boundary value problems and non-symmetric solutions

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Abstract. We consider the equation $-\Delta u = wf'(u)$ on a symmetric bounded domain in $\mathbb{R}^n$ with Dirichlet boundary conditions. Here w is a positive function or measure that is invariant under the (Euclidean) symmetries of the domain. We focus on solutions u that are positive and/or have a low Morse index. Our results are concerned with the existence of non-symmetric solutions and the non-existence of symmetric solutions. In particular, we construct a solution u for the disk in $\mathbb{R}^2$ that has index 2 and whose modulus $|u|$ has only one reflection symmetry. We also provide a corrected proof of [12, Theorem 1].

1. Introduction and main results

Let Ω a bounded open Lipschitz domain in $\mathbb{R}^n$. A classical result by Gidas, Ni, and Nirenberg [1] implies that if Ω is symmetric with respect to some codimension 1 hyperplane and convex in the direction orthogonal to this plane, then any positive solution u of the equation

$$-\Delta u = wf'(u)\,, \qquad u\,\Big|_{\partial\Omega} = 0\,, \tag{1.1}$$

is necessarily symmetric as well, provided that $w:\Omega\to\mathbb{R}$ is symmetric and satisfies some monotonicity condition. Here f' is the derivative of a function $f\in \mathrm{C}^2(\mathbb{R})$. Subsequent extensions include, among other things, classes of solutions that are not necessarily positive [2,7,8,9,10,11]. In particular, a results in [10] implies that, if Ω is a ball or annulus, w is radially symmetric, and f'' is convex, then any solution u of (1.1) with Morse index n or less has an axial symmetry.

In these cases, a solution u of (1.1) with low Morse index inherits at least one symmetry of the equation. One may wonder whether the same property forces u to have additional symmetries, if not all symmetries in the case $u\ge 0$. In this paper we present some results that give a negative answer to this question in several cases. This includes radially symmetric domains as well as domains that have only discrete symmetries, such as regular polytopes. For the square in $\mathbb{R}^2$, the existence of a non-symmetric index-2 solution was proved in [12].

To simplify the discussion, assume for now that $f(u)=\frac{1}{p}|u|^p$ with $p>2$, and p subcritical if $n\ge 3$, and that w is a nonnegative bounded measurable function on Ω. Then a solution $u\in \mathrm{H}^1_0(\Omega)$ of the equation (1.1) is a critical point if the following functional J,

$$J(u)=\frac{1}{2}\langle u,u\rangle - F(u)\,, \qquad \langle u,v\rangle=\int_\Omega(\nabla u)\cdot(\nabla v)\,, \quad F(u)=\int_\Omega wf(u)\,. \tag{1.2}$$

The Morse index of u is defined to be the dimension of the largest subspace of $\mathrm{H}^1_0(\Omega)$ where the second derivative $D^2J(u)$ of J is negative definite. Since f is superquadratic,

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this index is always at least 1, except at the trivial solution $u = 0$. Minimization of J on the Nehari manifold $\mathfrak{N} = \{u \in \mathrm{H}_0^1(\Omega) : DJ(u)u = 0\,, u \neq 0\}$ shows that index-1 solutions always exist and that they do not vanish anywhere on Ω.

Let $0 < \theta < 1$ be fixed but arbitrary. We start with the case where Ω is either a ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ or an annulus $A_R = \{x \in \mathbb{R}^n : \theta R < |x| < R\}$ with $R > 0$. Here $|x|$ denotes the Euclidean length of x. A function $u : \Omega \to \mathbb{R}$ is said to be radially symmetric if it is constant on spheres $|x| = r$.

Theorem 1.1. *Let $\Omega = B_R$ or $\Omega = A_R$. Let $f(u) = \frac{1}{p}|u|^p$ with $p > 2$. Then there exists a nonnegative radially symmetric function $w \in \mathrm{C}_0^\infty(\Omega)$ such that every positive radially symmetric solution of (1.1) has index $n + 1$ or larger.*

This theorem shows in particular that, under the given assumptions, no solution of index 1 can be radially symmetric. The absence of sign-changing radially symmetric solutions of (1.1) with index $\leq n$ was proved in [2], for any C^2 function f with $f'(0) \geq 0$. Other results in [2] are concerned with geometric properties of the nodal regions of sign-changing solutions.

After proving the existence of a non-symmetric index-2 solution for the square in [12], one of our goals has been to prove an analogous theorem for the disk. In this case, we know by [10] that any index-2 solution has one reflection symmetry. As we will describe in Section 2, it is possible to find a smooth function $w > 0$ on the disk such that, numerically, the corresponding equation (1.1) admits an index-2 solution u whose modulus $|u|$ has only one reflection symmetry. So far we have not yet been able to prove that there exists a true index-2 solution nearby.

The following result concerns a simplified version of the above-mentioned disk problem. Let Ω be the unit disk in $\mathbb{R}^2$, centered at the origin. We consider the equation (1.1) in a distributional sense, where the weight w is not a function but a measure, concentrated on two circles,

$$w(x) = \frac{4}{3}\delta\bigl(|x| - \tfrac{3}{4}\bigr) + 3\delta\bigl(|x| - \tfrac{1}{8}\bigr)\,. \tag{1.3}$$

Theorem 1.2. *The equation (1.1) with weight (1.3) admits a continuous index-2 solution $u \in \mathrm{H}_0^1(\Omega)$ that is symmetric with respect to one reflection symmetry of Ω but neither symmetric nor antisymmetric with respect to any other reflection symmetry of Ω.*

The function u described in this theorem is depicted in Figure 1. We note that any solution of (1.1) is harmonic outside the support of w. For the weight w defined in (1.3), this implies that a solution u is determined uniquely by its restriction U to the union of two circles $S_{1/8} \cup S_{3/4}$. The function U is obtained by solving a suitable fixed point problem $\mathcal{N}(U) = U$ on a space of real analytic functions on $S_{1/8} \cup S_{3/4}$. The Morse index in $\mathrm{H}_0^1(\Omega)$ of the corresponding solution u is related to the spectrum of the derivative $D\mathcal{N}(u)$ of $\mathcal{N}$ at u. Our analysis of the map $\mathcal{N}$ involves estimates that have been carried out by a computer.

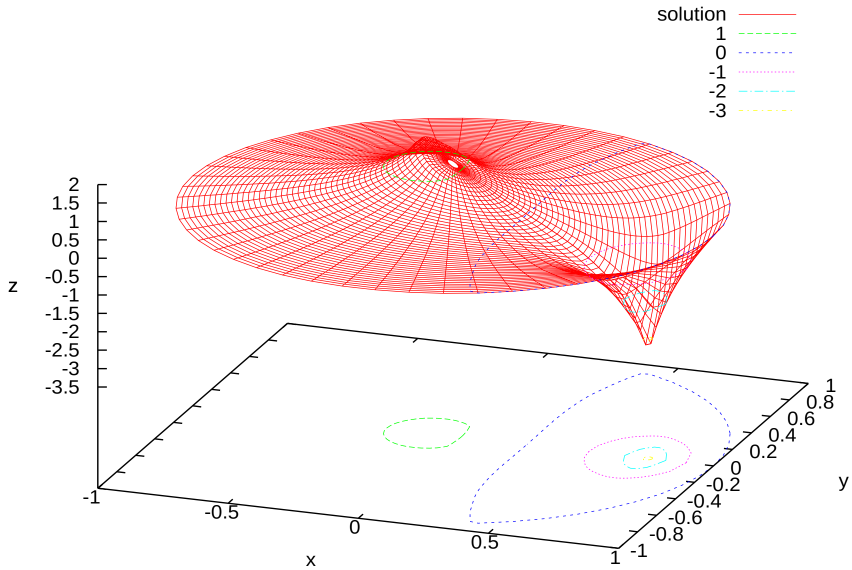

Figure 1. The solution u from Theorem 1.2.

Our remaining results are concerned with discrete symmetries. To be more precise, let $\mathcal{S}$ be a nontrivial finite group of Euclidean symmetries $\sigma : \mathbb{R}^n \to \mathbb{R}^n$. We assume that the (bounded open Lipschitz) domain Ω is invariant under every symmetry $\sigma \in \mathcal{S}$. A function u on Ω is said to be invariant under $\mathcal{S}$ if $u \circ s = u$ for all $s \in \mathcal{S}$.

Theorem 1.3. *Let $f(u) = \frac{1}{p}|u|^p$ with $p > 2$. Then there exists a nonnegative function $w \in \mathrm{C}_0^\infty(\Omega)$ that is invariant under $\mathcal{S}$, such that no minimizer of J on the Nehari manifold $\mathfrak{N} = \{u \in \mathrm{H}_0^1(\Omega) : DJ(u)u = 0\,, u \neq 0\}$ is invariant under $\mathcal{S}$.*

It is well known that a minimizer of J on $\mathfrak{N}$ has Morse index 1 and does not vanish anywhere on Ω. Our proof in Section 3 of Theorem 1.3 illustrates nicely how symmetries can prevent a function $u \in \mathfrak{N}$ from being a minimizer of J. The following simple case served as a starting point: Let Ω be a union of two mutually disjoint balls of radius 1. Let σ be a reflection that exchanges the two balls. A positive solution u of (1.1) that is invariant under σ is a sum of two solutions that have disjoint supports. Each of them has index ≥ 1, so u has index ≥ 2. The idea is to mimic such a situation inside an arbitrary symmetric domain Ω.

We note that the special case $n = 2$ and $p = 4$ of Theorem 1.3 is already covered in [12, Theorem 1.1]. However, the proof given in [12] contains an error. This was one of the main motivations for re-visiting discrete symmetries in this paper.

One of the shortcomings of Theorem 1.3 is that it does not exclude the existence of an index-1 solution that is invariant under a nontrivial subgroup of $\mathcal{S}$. This is overcome in part in the following theorem. We say that $\sigma \in \mathcal{S}$ is an involution if $\sigma \circ \sigma = \mathrm{I}$.

Assume that f is even, and that there exists a positive real number $\gamma < 1$ such that

$$0 < f'(t) \leq (1-\gamma) f''(t) t \,, \qquad t > 0 \,. \tag{1.4}$$

Notice that this condition is satisfied for $f(t) = \frac{1}{p}|t|^p$ if $p > 2$.

Theorem 1.4. *Under the above-mentioned assumptions on f, there exists a nonnegative function $w \in \mathrm{C}_0^\infty(\Omega)$ that is invariant under all symmetries in $\mathcal{S}$, such that the following holds. Let $\mathcal{S}_k$ be a subgroup of $\mathcal{S}$ of order 2^k, generated by k mutually commuting involutions. If u is a positive solution of (1.1) that is invariant under all symmetries in $\mathcal{S}_k$, then u has index $k+1$ or larger.*

This theorem and Theorem 1.3 are proved in Section 3. A proof of Theorem 1.1 is given in Section 4. In Section 5 we prove Theorem 1.2, based on three technical lemmas. Our proof of these lemmas is computer-assisted and is described in Section 6.

2. Some numerical results

Here we describe some numerical results concerning index-2 solutions of the equation

$$-\Delta u = w u^3 \,, \qquad u \,\big|_{\partial\Omega} = 0 \,, \tag{2.1}$$

for the disk $\Omega = \bigl\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\bigr\}$, with $w : \Omega \to [0,\infty)$ radially symmetric. It is known [10] that any such solution is invariant under a reflection symmetry of Ω. Thus, we restrict our analysis to solutions that are invariant under $R_y : (x,y) \mapsto (x,-y)$. Our main goal is to find a radially symmetric weight function $w : \Omega \to [0,\infty)$ such that (2.1) admits an index-2 solution u whose modulus $|u|$ is not invariant under any reflection symmetry of the disk Ω other than R_y.

It is convenient to reformulate (2.1) as the fixed point problem $\mathcal{F}(u) = u$, where

$$\mathcal{F}(u) = (-\Delta)^{-1} w u^3 \,, \qquad u \in \mathrm{H}_0^1(\Omega) \,. \tag{2.2}$$

The Morse index of a fixed point u coincides with the number of eigenvalues in $(1,\infty)$ of the derivative $D\mathcal{F}(u)$, as the following identity shows:

$$D^2 J(u)(v_1,v_2) = -\int_\Omega \bigl[\Delta v_1 + 3 w u^2 v_1\bigr] v_2 = \bigl\langle v_1, [\mathrm{I} - D\mathcal{F}(u)] v_2 \bigr\rangle \,. \tag{2.3}$$

We note that $D\mathcal{F}(u)$ has a trivial eigenvalue 1 due to the rotation invariance of $\mathcal{F}$.

For the constant weight $w = 1$, it is easy to find a fixed point $u = u_0$ of index 2, but u_0 is antisymmetric under $R_x : (x,y) \mapsto (-x,y)$. So $|u_0|$ is symmetric with respect to both R_x and R_y. This fixed point u_0 is depicted in Figure 2 on the left.

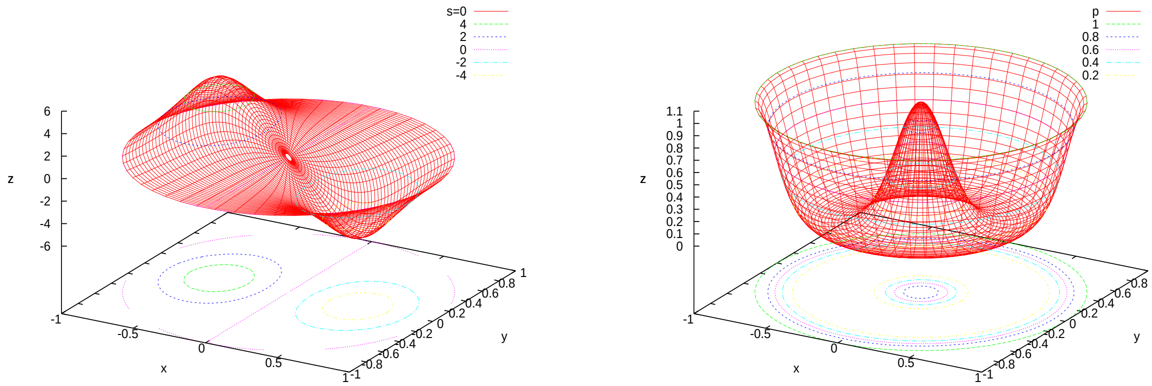

Figure 2. The solution u_0 for $w = 1$ (left) and the function p (right).

Numerically, $D\mathcal{F}(u_0)$ has no nontrivial eigenvector with eigenvalue 1. Thus, $\mathcal{F}$ should have a fixed point $u \approx u_0$ for any radially symmetric weight $w \approx 1$. But $\mathcal{F}$ preserves R_x-antisymmetry, so the perturbed solution u still has the undesired antisymmetry with respect to R_x.

We now increase the perturbation along a one-parameter family of weights $w_s = (1 - s) + sp$. After some experimentation we found the function $p : \Omega \to [0, \infty)$ shown in Figure 2 right, which has the following property. Denote by $\mathcal{F}_s$ the map (2.2) with weight $w = w_s$. As the value of s is increased from $s = 0$ to $s \approx 0.9$, the map $\mathcal{F}_s$ is observed numerically to have a fixed point $u = u_s$ that depends smoothly on s. This curve $s \mapsto u_s$ will be referred to as the primary branch. At a value $s = s_0 \approx 0.7$, the derivative of $D\mathcal{F}_s(u_s)$ has an eigenvector with eigenvalue 1 which is not antisymmetric with respect to R_x. In the direction of this eigenvector, a second branch of fixed points bifurcates off the primary branch. For $s > s_0$ the solutions u_s on the primary branch have Morse index 3, while those on the second branch have Morse index 2 and are not antisymmetric with respect to R_x. The solutions of (2.1) on the two branches for the value $s = 0.84$ are depicted in Figure 3.

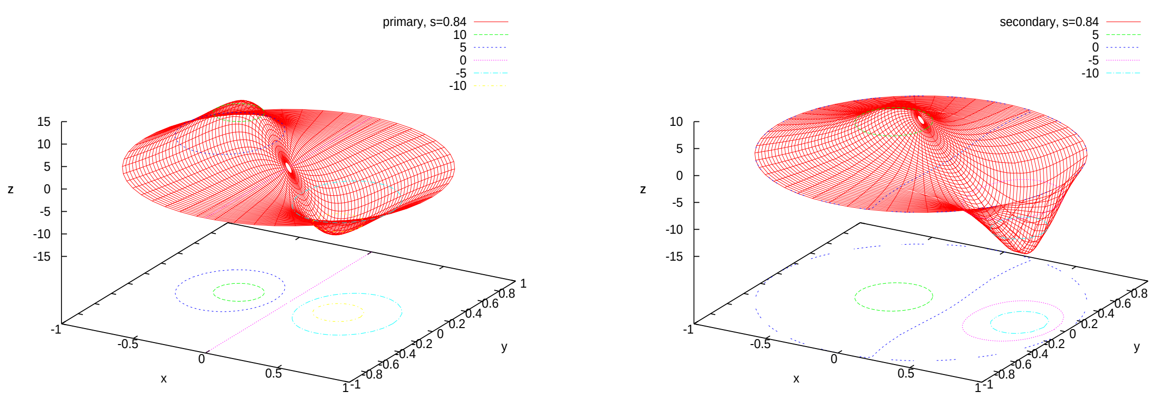

Figure 3. The solutions on the primary (left) and secondary (right) branch for $s = 0.84$.

In our numerical implementation of the map $\mathcal{F}$ we use polar coordinates (r,ϑ) and represent a function $u:\Omega\to\mathbb{R}$ that is invariant under R_y as a Fourier series

$$u(r,\vartheta)=\sum_{k=0}^{\infty}u_k(r)\cos(k\vartheta)\,. \tag{2.4}$$

Such a representation is well suited for both basic operations that are involved in the computation of $\mathcal{F}(u)$, namely the product $(u,v)\mapsto uv$ and the inverse of $-\Delta$. In particular, $u=(-\Delta)^{-1}v$ is given by the integrals

$$u_k(r)=r^k\int_1^r s^{-2k-1}\left(\int_0^s t^{1+k}v_k(t)\,dt\right)ds\,,\qquad k=0,1,2,\ldots\,. \tag{2.5}$$

The main problem is to find a representation of the functions u_k and v_k that is accurate and efficient for the computation of both (2.5) and products. Ideally, such a representation also allows for good error estimates. We have implemented several representations, including an expansion of u_k or $r^{-k}u_k$ into orthogonal polynomials (Chebyshev and others). Unfortunately, none of them yielded estimates that allowed us to prove a result analogous to Theorem 1.2 for the weight function w_s described above.

Interestingly, the most accurate numerical results were obtained with the following "germ" representation. For integers n and j define $r_j=\frac{2j-2}{2n-1}$ and $t_j=\frac{2j-1}{2n-1}$. Let now $n\ge 2$ be fixed. We consider a partition of $[0,1]$ into n subintervals $I_1=[0,t_1]$ and $I_j=(t_{j-1},t_j]$ for $j=2,3,\ldots,n$. On each subinterval I_j we represent u_k by a Taylor series

$$u_k(r)=U_{k,j}(r-r_j)\,,\qquad r\in I_j\,,\qquad U_{k,j}(z)=\sum_{m=0}^{\infty}U_{k,j,m}\,z^m\,, \tag{2.6}$$

where $U_{k,1,m}=0$ whenever $m-k$ is odd.

At this point we should mention that the weight functions w_s described above have been chosen real analytic. Thus, if n is chosen sufficiently large, we can expect the functions $U_{k,j}$ associated with a solution u of (2.1) to be analytic in a disk $|z|<\rho$ with $\rho>\frac{1}{2n-1}$. After choosing a suitable Banach algebra $\mathcal{B}$ of real analytic functions on such a disk, we identify each Fourier coefficient u_k of u with an n-tuple $U_k=(U_{k,1},U_{k,2},\ldots,U_{k,n})$ of functions $U_{k,j}\in\mathcal{B}$.

Now we rewrite the equation (2.6) in terms of the functions $U_{k,j}$. This yields an extension of the map $\mathcal{F}$ to a space of functions $u:\Omega\to\mathbb{R}$ whose Fourier coefficients u_k are piecewise real analytic functions (2.6) with $U_k\in\mathcal{B}^n$ unconstrained. The equation $\mathcal{F}(u)=u$ is now solved by iterating a quasi-Newton map $\mathcal{M}$ associated with $\mathcal{F}$, of the type described in Section 5.

This method seems well adapted to problems on the disk which can be shown to have only real analytic solutions. In this case, the functions $r\mapsto U_{k,j}(r-r_j)$ associated with a solution u are the true germs of its Fourier coefficients u_k. If n can be chosen relatively small ($n=3$ in our case), then these germs can be computed efficiently and with high accuracy.

3. Discrete symmetries

In this section we prove Theorem 1.3 and Theorem 1.4. The idea in both proofs is to choose a weight function w that forces a symmetric solution $u > 0$ of the equation (1.1) to have several well-separated local maxima. As we will see, this is incompatible with u having a low Morse index (in the setup of Theorem 1.4) or minimizing J on the Nehari manifold $\mathfrak{N}$ (in the setup of Theorem 1.3).

We first consider the case $f(u) = \frac{1}{p}|u|^p$ which is more transparent. Assume that $p > 2$. In this case, a function in $\mathrm{H}^1_0(\Omega)$ is a minimizer of J on $\mathfrak{N}$ if and only if a constant multiple $u \neq 0$ of this function minimizes the ratio R,

$$R(u) = \frac{\langle u, u\rangle^{p/2}}{F(u)}\,, \qquad F(u) = \frac{1}{p}\int_\Omega w|u|^p\,. \tag{3.1}$$

Intuitively, since the denominator $F(u)$ is a sum of powers, while the numerator $\langle u, u\rangle^{p/2}$ includes a power of a sum, a function $u > 0$ that is too "spread out" cannot be a minimizer of R. To be more precise, assume that u is a sum of functions $u_1, u_2, \ldots, u_m$ that are close to having mutually disjoint supports. (Later we will also have $u_j > 0$.) Then it is natural to consider the quantities q and φ, defined by

$$q = \sum_{j=1}^m \frac{\langle u_j, u_j\rangle^{p/2}}{\langle u, u\rangle^{p/2}}\,, \qquad \varphi = \sum_{j=1}^m \frac{F(u_j)}{F(u)}\,, \qquad u = \sum_{j=1}^m u_j\,. \tag{3.2}$$

The following proposition shows that if $q < \varphi$ then u cannot be a minimizer of R.

Proposition 3.1. *If $q < \varphi$ then $R(u) > R(u_j)$ for some j.*

Proof. Assume that $R(u_j) \geq r$ for all j. Then

$$F(u) = \frac{1}{\varphi}\sum_{j=1}^m F(u_j) \leq \frac{1}{r\varphi}\sum_{j=1}^m \langle u_j, u_j\rangle^{p/2} = \frac{q}{r\varphi}\langle u, u\rangle^{p/2}\,, \tag{3.3}$$

and thus $R(u) \geq r\varphi/q$. This proves the claim. **QED**

In our proof of Theorem 1.3, we will choose w to be a symmetric sum of m bump functions w_j with mutually disjoint supports. Then a symmetric solution u of (1.1) can be written as a sum $u = \sum_j u_j$ with $u_j = (-\Delta)^{-1} w_j f'(u)$. By symmetry, $\langle u_j, u_j\rangle$ is independent of j. Assuming $u > 0$, we will see that $\langle u_i, u_j\rangle > 0$ for all i and j. This immediately implies that $q \leq m^{1-p/2} < 1$. So our goal is to choose w in such a way that the (positive) functions $u_1, u_2, \ldots, u_m$ are close to having mutually disjoint supports, in the sense that φ is close to 1, Then $q < \varphi$ and Proposition 3.1 applies.

In some sense we are considering a perturbation about a (singular) limit $\varphi = 1$. This is similar in spirit to the approach taken in [4], where a solution u of (1.1) is constructed on a domain $\Omega \approx \bigcup_j B_j$ that is close to a union of m mutually disjoint balls $B_1, B_2, \ldots, B_m$.

In this case $u \approx \sum_j u_j$, with u_j supported on B_j. For a precise statement of this result we refer to [4].

In what follows we use the notation $\mathcal{H} = \mathrm{H}_0^1(\Omega)$ and assume that $u \in \mathcal{H}$.

Proof of Theorem 1.3. Let $\mathcal{S}$ be a finite group of Euclidean symmetries that leave Ω invariant. For $s \in \mathcal{S}$ and $u : \Omega \to \mathbb{R}$ define $s^* u = u \circ s$. Let $s_1 = \mathrm{I}$ and $s_2, \ldots, s_m$ be the elements of $\mathcal{S}$, where m is the order of $\mathcal{S}$. Let x_1 be a point in Ω that is not invariant under any s_j with $j \geq 2$. Define $x_j = s_j^{-1}(x_1)$ for $2 \leq j \leq m$. Then $\{x_1, x_2, \ldots, x_m\}$ is the orbit of x_1 under the group $\mathcal{S}$.

Choose $r > 0$ such that $\mathrm{dist}(x_j, \partial\Omega) > r$ for all j, and such that $|x_i - x_j| > 3r$ whenever $i \neq j$. Given a positive real number $\varepsilon < r$ to be determined later, consider the disks $D_j = \{x \in \mathbb{R}^n : |x - x_j| < \varepsilon\}$. Let ϕ be a monotone C^∞ function on $[0, \infty)$ taking the value 1 on $[0, 1/2]$ and 0 on $[1, \infty)$. Define

$$w = \sum_{j=1}^m w_j \,, \qquad w_j(x) = \phi\bigl(\varepsilon^{-1}|x - x_j|\bigr) \,, \qquad x \in \Omega \,. \tag{3.4}$$

We will identify w_j with the multiplication operator $u \mapsto w_j u$. Let now $u \in \mathcal{H}$ be a positive solution of (1.1) for the weight function w defined above. Define

$$u_j = (-\Delta)^{-1} w_j f'(u) \,, \qquad e_j = u - u_j \,, \qquad 1 \leq j \leq m \,. \tag{3.5}$$

Denote by G the Dirichlet Green's function for $-\Delta$ on Ω with zero boundary conditions. It is well known that $G(x, y) > 0$ for any two distinct points $x, y \in \Omega$. This implies in particular that $u_j \geq 0$ for all j. Furthermore,

$$\langle u_i, u_j \rangle = \int_{D_i \times D_j} f'\bigl(u(x)\bigr) w_i(x) G(x, y) w_j(y) f'\bigl(u(y)\bigr)\, dx dy \; > 0 \,. \tag{3.6}$$

Assume now that u is invariant under $\mathcal{S}$. Using that Δ commutes with s_j^* we have

$$e_1 = (-\Delta)^{-1} \sum_{j=2}^m s_j^* w_1 u^{p-1} = \sum_{j=2}^m s_j^* (-\Delta)^{-1} w_1 u^{p-1} \,. \tag{3.7}$$

In terms of the Green's function G,

$$\begin{aligned} u_1(x) &= \int_{D_1} G(x, y) w_1(y) u(y)^{p-1}\, dy \,, \\ e_1(x) &= \int_{D_1} \mathcal{E}_1(x, y) w_1(y) u(y)^{p-1}\, dy \,, \qquad \mathcal{E}_1(x, y) = \sum_{j=2}^m G\bigl(s_j(x), y\bigr) \,. \end{aligned} \tag{3.8}$$

A possible representation for G is

$$G(x, y) = \gamma_n \bigl[g_n(|x - y|) - h_n(x, y)\bigr] \,, \qquad g_n(s) = \begin{cases} -\ln(s) & \text{if } n = 2, \\ s^{2-n} & \text{if } n \geq 3, \end{cases} \tag{3.9}$$

where γ_n is some positive constant, and where h_n is a function on $\bar{\Omega}\times\Omega\cup\Omega\times\bar{\Omega}$ such that $x\mapsto h_n(x,z)$ and $y\mapsto h_n(z,y)$ are harmonic in Ω, with boundary values $g_n(x,z)$ for $x\in\partial\Omega$ and $g_n(z,y)$ for $y\in\partial\Omega$, respectively, for every $z\in\Omega$. Clearly h_n is bounded on $D_1\times D_1$. Thus, given any $\delta>0$, if $\varepsilon>0$ is chosen sufficiently small then

$$\mathcal{E}_1(x,y)\le\delta G(x,y)\,,\qquad x,y\in D_1\,. \tag{3.10}$$

By (3.8) this inequality implies that $e_1\le\delta u_1$ on D_1. And by symmetry we have $e_j\le\delta u_j$ on D_j for all j. Equivalently, $u\le(1+\delta)u_j$ on D_j. This in turn implies that

$$\int_{D_j} w_j\frac{u^p}{p}\le(1+\delta)^p\int_{D_j} w_j\frac{u_j^p}{p}\le(1+\delta)^pF(u_j)\,. \tag{3.11}$$

Summing over j we obtain

$$F(u)\le(1+\delta)^p\sum_{j=1}^{m}F(u_j)\,. \tag{3.12}$$

Consider now the sums q and φ defined in (3.2). Since $\langle u_i,u_j\rangle\ge 0$ and $\langle u_j,u_j\rangle=\langle u_1,u_1\rangle$ for all i and j, we have $q<m^{1-p/2}$. Choosing $\delta>0$ such that $(1+\delta)^p<m^{p/2-1}$, we also have $\varphi^{-1}<m^{p/2-1}$ by (3.12). Consequently $\varphi^{-1}q<1$, which by Proposition 3.1 implies that u is not a minimizer of R. **QED**

Proof of Theorem 1.4. We use the same notation and assumptions as in the proof above, up to (3.5). The equation (3.6) applies here as well.

Let $u\in\mathcal{H}$ be a positive solution of (1.1). Using that $DJ(u)=0$ we have

$$\begin{aligned}D^2J(u)(u,u)&=-D^2F(u)(u,u)+DF(u)u\\&=-\int_\Omega w\bigl[f''(u)u-f'(u)\bigr]u\le-\gamma\int_\Omega wf''(u)u^2\,,\end{aligned} \tag{3.13}$$

by the assumption (1.4). In particular, $D^2J(u)(u,u)<0$. Thus u has index ≥ 1. This proves the assertion in the case $k=0$.

Consider now $k=1$. Assume that $\mathcal{S}$ contains a nontrivial involution S. Then m is even. Let $\mathcal{I}$ and $\mathcal{J}$ be two disjoint $\frac{m}{2}$-element subsets of $\{1,2,\ldots,m\}$ that are exchanged by the map s defined by $S(x_j)=x_{s(j)}$. Define

$$\hat{u}=\sum_{i\in\mathcal{I}}u_i\,,\qquad\check{u}=\sum_{j\in\mathcal{J}}u_j\,,\qquad\check{w}=\sum_{j\in\mathcal{J}}w_j\,. \tag{3.14}$$

Then $\hat{u}+\check{u}=u$. Assume now that u is invariant under S^*. Let $v=\hat{u}-\check{u}$. Pick $i\in\mathcal{I}$. Using (3.13), together with the fact that $\langle\hat{u},\check{u}\rangle\ge 0$ by (3.6), we obtain

$$\begin{aligned}D^2J(u)(v,v)&=D^2J(u)(u,u)-4D^2J(u)(\hat{u},\check{u})\\&\le-\gamma D^2F(u)(u,u)-4\langle\hat{u},\check{u}\rangle+4D^2F(u)(\hat{u},\check{u})\\&\le-\int_D wf''(u)\bigl[\gamma u^2-4\hat{u}\check{u}\bigr]=-m\int_{D_i}w_if''(u)\bigl[\gamma u^2-4\hat{u}\check{u}\bigr]\\&\le-m\int_{D_i}w_if''(u)u\bigl[\gamma u-4\check{u}\bigr]\,.\end{aligned} \tag{3.15}$$

Here we have used the symmetry of u, and the fact that $0 \le \hat{u} \le u$.

Our goal is to show that $\gamma u - 4\check{u} > 0$ on D_i, provided that $\varepsilon > 0$ has been chosen sufficiently small. As a starting point we note that

$$\gamma u - 4\check{u} = (-\Delta)^{-1}[\gamma w - 4\check{w}]f'(u) \ge (-\Delta)^{-1}[\gamma w_i - 4\check{w}]f'(u)\,, \tag{3.16}$$

since $w \ge w_i$ and $(-\Delta)^{-1}$ preserves positivity. For each j there exist $\sigma_j \in \mathcal{S}$ such that $w_j = \sigma_j^* w_i$. This allows us to write

$$\begin{aligned}(-\Delta)^{-1}\check{w}f'(u) &= (-\Delta)^{-1}\sum_{j\in\mathcal{J}} w_j f'(u) = \sum_{j\in\mathcal{J}}(-\Delta)^{-1}\sigma_j^* w_i f'(u)\\ &= \sum_{j\in\mathcal{J}}\sigma_j^*(-\Delta)^{-1}w_i f'(u)\,.\end{aligned} \tag{3.17}$$

Here we have used that the Laplacean commutes with σ_j^*. Combining the last two equations yields

$$\gamma u(x) - 4\check{u}(x) \ge \int_{D_i}\Bigl[\gamma G(x,y) - 4\sum_{j\in\mathcal{J}} G\bigl(\sigma_j(x),y\bigr)\Bigr] w_i(y) f'\bigl(u(y)\bigr)\,dy\,. \tag{3.18}$$

Consider now $x,y \in D_i$ and $j \in \mathcal{J}$. Then $|\sigma_j(x) - y| > r$. Thus, there exists a constant $C > 0$, depending only on Ω and r, such that $G(\sigma_j(x),y) \le C$. This shows that the sum in (3.18) is bounded from above by $\frac{m}{2}C$. By using the representation (3.9) of the Green's function G, together with the fact that h_n is bounded on $D_i \times D_i$, we see that by choosing $\varepsilon > 0$ sufficiently small, $\gamma G(x,y) > 2mC + 1$ for all $x,y \in D_i$. This makes the term $[\cdots]$ in equation (3.18) larger than 1, and by (3.15) this yields

$$D^2J(u)(v,v) \le -m\int_{D_i} w_i f''(u)u \;< 0\,. \tag{3.19}$$

Recall also that $D^2J(u)(u,u) < 0$ by (3.13). Below we will show that $D^2J(u)(u,v) = 0$. Thus, the restriction of $D^2J(u)$ to the 2-dimensional subspace spanned by u and v is a negative quadratic form. This implies that u has index 2 or larger.

From (2.3) one easily sees that

$$D^2J(u)(\sigma^* v_1, v_2) = D^2J(u)(v_1, \sigma^* v_2)\,, \tag{3.20}$$

for every $v_1, v_2 \in \mathcal{H}$ and every involution $\sigma \in \mathcal{S}$. Thus, if v_1 and v_2 are eigenfunctions of σ^* for different eigenvalues, then $D^2J(u)(v_1,v_2) = 0$. In particular, since $S^*u = u$ and $S^*v = -v$, we have $D^2J(u)(u,v) = 0$.

Consider now the case $k \ge 2$. Let $S_1, S_2, \ldots, S_k$ be involutions from $\mathcal{S}$ that generate $\mathcal{S}_k$. For $\alpha = 1,2,\ldots,k$ we can construct as above a function $v = v_\alpha$ such that $D^2J(u)(v_\alpha,v_\alpha) < 0$ and $S_\alpha^* v_\alpha = -v_\alpha$. It is useful to choose the index sets $\mathcal{I}_\alpha$ and $\mathcal{J}_\alpha$ in advance, in such a way that $S_\beta^* v_\alpha = v_\alpha$ when $\beta \ne \alpha$. It is not hard to see that this is possible. Then, setting $v_0 = u$, we have $D^2J(u)(v_\alpha,v_\beta) = 0$ whenever $0 \le \alpha < \beta \le k$. This shows that the restriction of $D^2J(u)$ to the $k+1$-dimensional subspace spanned by $\{v_0, v_1, \ldots, v_k\}$ is a negative quadratic form. Thus u has index $k+1$ or larger. **QED**

4. Proof of Theorem 1.1

Since the Laplacean and f and are homogeneous, a solution of (1.1) for $\Omega = B_R$ yields a solution for $\Omega = B_1$ via scaling, and vice versa. Similarly for annuli with fixed ratio θ. Thus we may choose any value of $R > 0$.

We will use the following estimates [3,5] for the Green's function G of $-\Delta$ on Ω with zero boundary conditions. Consider first $R = 1$. Then

$$G(x,y) \le \frac{1}{4\pi} \ln\left(1 + C_0 \frac{d_x d_y}{|x-y|^2}\right), \qquad (n=2), \tag{4.1}$$

and

$$G(x,y) \le C_0 |x-y|^{2-n}\left(1 \wedge \frac{d_x d_y}{|x-y|^2}\right), \qquad (n \ge 3), \tag{4.2}$$

where C_0 is some fixed constant that depends only on n, and on θ if Ω is an annulus. Here we have used the notation $d_z = \operatorname{dist}(z, \partial\Omega)$ and $a \wedge b = \min\{a, b\}$. The Green's function for a ball or annulus with outer radius R is given by $G_R(x,y) = R^{n-2} G(Rx, Ry)$. Thus G_R satisfies the same bound (4.1) or (4.2), with the same constant C_0. In order to simplify notation, we will drop the subscript R.

Our aim is to choose a weight function w that is supported very close to the outer boundary of $\partial\Omega$, relative to R. It is convenient to do this by choosing R large and w supported near the circle $|x| = R - 1$. To be more precise, we choose

$$w(x) = \phi\big(|x| - R + 1\big), \tag{4.3}$$

where $\phi \in \mathrm{C}^\infty(\mathbb{R})$ is nonnegative, has support in $\left[-\frac{1}{16}, \frac{1}{16}\right]$, and satisfies $\int \phi = 1$. Then w is supported in the annulus $D = \{x \in \mathbb{R}^n : a \le |x| \le b\}$, where $a = R - \frac{17}{16}$ and $b = R - \frac{15}{16}$.

Let u be a positive solution of (1.1) that only depends on $r = |x|$. Let $1 \le j \le n$. Consider the half-annuli $D_\pm = \{x \in D : \pm x_j \ge 0\}$, and define

$$w_\pm(x) = \chi(x \in D_\pm) w(x), \qquad u_\pm = \Delta^{-1} w_\pm u^{p-1}, \qquad v_j = u_+ - u_-, \tag{4.4}$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. Notice that $u = u_+ + u_-$. Clearly

$$D^2 J(u)(u,u) = -(p-2) \int_\Omega w u^p = -2(p-2) \int_{D_+} w u^p \tag{4.5}$$

is negative. Our goal is to show that $D^2 J(u)(v_j, v_j)$ is negative as well. As in (3.15) we have

$$D^2 J(u)(v_j, v_j) \le -2(p-1) \int_{D_+} w_+ u^{p-1} \big[\gamma u - 4u_-\big]. \tag{4.6}$$

Here $\gamma = \frac{p-2}{p-1}$. We expect $u_-(y)$ to be small when y_j is large, so that the term $[\ldots]$ in the above integral is positive on most of D_+. To make this more precise, we can use the bounds (4.1) and (4.2), which imply that

$$G(y,z) \le C_1 |y-z|^{-n}, \qquad y, z \in D, \qquad |y-z| \ge C_2. \tag{4.7}$$

Here, and in what follows, $C_1, C_2, \ldots$ denote positive constants that are independent of R and j. In Lemma 4.1 below we will show that there exist positive constants C_3 and C_4 such that

$$C_3 \le u(z) \le C_4\,, \qquad z \in D\,, \tag{4.8}$$

provided that R has been chosen sufficiently large (which we shall henceforth assume). Thus, if $y \in D_+$ with $y_j \ge C_2$, then

$$u_-(y) = \int_{D_-} G(y,z) w_-(z) u(z)^{p-1}\,dz \le C_5 \int_{D_-} |y-z|^{-n}\,dz\,. \tag{4.9}$$

Here we have used the upper bound on u from (4.8). This shows that for every $\varepsilon > 0$ there exists $C_6 > 0$ such that $|u_-(y)| < \varepsilon$ whenever $y \in D_+$ with $y_j \ge C_6$. Thus, using the lower bound on u from (4.8) we see that there exists $C_7 > 0$ such that

$$\gamma u(y) - 4u_-(y) \ge \tfrac{1}{2}\gamma u(y)\,, \tag{4.10}$$

for all y in the domain $D_2 = \{y \in D : y_j \ge C_7\}$. Let $D_1 = \{y \in D : 0 \le y_j \le C_7\}$. Then by (4.6) we have

$$D^2 J(u)(v_j, v_j) \le 8(p-1) \int_{D_1} w_+ u^{p+1} - (p-1)\gamma \int_{D_2} w_+ u^{p+1}\,. \tag{4.11}$$

Now consider the behavior of the two integrals in this equation, as $R \to \infty$. Using (4.8) the integral of $w_+ u^{p+1}$ over D_1 can be bounded from above by $C_8 R^{n-2}$, and the integral of $w_+ u^{p+1}$ over D_2 can be bounded from below by $C_9 R^{n-1}$. Thus, if R is chosen sufficiently large, then $D^2 J(u)(v_j, v_j) < 0$.

Setting $v_0 = u$, we also have $D^2 J(u)(v_i, v_j) = 0$ whenever $0 \le i < j \le n$. This follows from an argument analogous to the one used in in the proof of Theorem 1.4. Thus, the restriction of $D^2 J(u)$ to the $n+1$-dimensional subspace spanned by $\{v_0, v_1, \ldots, v_n\}$ is a negative quadratic form. This implies that u has index $n+1$ or larger.

What remains to be proved is the following lemma. Consider still $\Omega = B_R$ or $\Omega = A_R$, and $f(u) = \frac{1}{p}|u|^p$ with $p > 2$.

Lemma 4.1. *Let w be the weight function defined in (4.3). Then there exists $C > 1$ such that the following holds if $R > 1$ is chosen sufficiently large. Let u be a positive solution of (1.1) that only depends on the radial variable $r = |x|$. Then $C^{-1} \le u(x) \le C$ for all x in the support of w.*

Proof. Let $a = R - \frac{17}{16}$ and $b = R - \frac{15}{16}$. To simplify notation we regard both w and u functions of $r = |x|$. Then the equation (1.1) can be written as

$$\partial_r\bigl(r^{n-1}\partial_r u\bigr) = -r^{n-1} w u^{p-1}\,. \tag{4.12}$$

Let u be a positive solution of this equation, with $u(R) = 0$. Then (4.12) shows that

$$u'(r) \geq u'(R)(r/R)^{-n+1}\,, \qquad r \leq R\,, \tag{4.13}$$

and equality holds for $r \geq b$. This immediately yields the bound

$$u(r) \leq 2\big|u'(R)\big|(R-r)\,, \qquad R-2 \leq r \leq R\,, \tag{4.14}$$

for sufficiently large R (depending only on n). We also assume that u is constant on $[0, a]$ if $\Omega = B_R$, and that $u(\theta R) = 0$ if $\Omega = A_R$.

Notice that $r^{n-1}\partial_r u$ is decreasing by (4.12). In the case $\Omega = B_R$ this implies that $u' \leq 0$, so the inequality (4.13) is an upper bound on $|u'|$. Consider now the case $\Omega = A_R$. Then $u'(r) \geq u'(a) \geq 0$ for $r \leq a$. Thus $u(a) \geq u'(a)(a - \theta R)$. Combined with (4.14) this yields $u'(a) \leq \frac{1}{2}u(a) \leq 2|u'(R)|$ for sufficiently large R. So both for the ball and annulus we have

$$|u'(r)| \leq 2|u'(R)|\,, \qquad r \geq a\,, \tag{4.15}$$

for sufficiently large R. Using that $u(b) \geq |u'(R)|(R-b) \geq \frac{1}{2}|u'(R)|$, and that $b - a = \frac{1}{8}$, this implies the first inequality in

$$\frac{1}{4}\big|u'(R)\big| \leq u(r) \leq 4\big|u'(R)\big|\,, \qquad r \in [a, b]\,. \tag{4.16}$$

The second inequality follows (4.14).

Now we estimate $|u'(R)|$. Using (4.12) and the fact that $\int w\,dr = 1$, we have

$$a^{n-1}u'(a) - b^{n-1}u'(b) = \int_a^b w u^{p-1} r^{n-1}\,dr = u(s)^{p-1}s^{n-1}\,, \tag{4.17}$$

for some $s \in [a, b]$. Since $u'(b) < 0 \leq u'(a)$, this implies

$$|u'(b)| \leq u(s)^{p-1} \leq u'(a) + (b/a)^{n-1}|u'(b)|\,. \tag{4.18}$$

Combining this bound with (4.15) and (4.16) yields the two inequalities

$$|u'(R)| \leq \big(4|u'(R)|\big)^{p-1}\,, \qquad \big(\tfrac{1}{4}|u'(R)|\big)^{p-1} \leq 6|u'(R)|\,. \tag{4.19}$$

Dividing by $|u'(R)|$ yields constant lower and upper bounds on $|u'(R)|^{p-2}$. These in turn yield lower and upper bound on $u(r)$ for $r \in [a, b]$ via (4.16). **QED**

5. Results implying Theorem 1.2

In this section we state three lemmas which imply Theorem 1.2, as will be shown. Our proof of these lemmas is computer-assisted and will be described in Section 6.

Let Ω be the unit disk in $\mathbb{R}^2$, centered at the origin. Let $\mathcal{H} = \mathrm{H}_0^1(\Omega)$. The boundary value problem considered here is the same as the problem described at the beginning of Section 2, except that w is not a function but a measure, supported on two circles $C_j = \{x \in \mathbb{R}^2 : |x| = \rho_j\}$ with positive radii $\rho_j < 1$. More specifically, assume that

$$w(x) = W_1\,\delta_1(|x|) + W_2\,\delta_1(|x|)\,, \qquad \delta_j(r) = \rho_j^{-1}\delta(r-\rho_j)\,, \quad W_j > 0\,. \tag{5.1}$$

Clearly every solution u of the equation $-\Delta u = wu^3$ is harmonic outside the support of w. Thus, we will restrict our analysis of this equation to functions $u \in \mathcal{H}$ that admit a representation

$$(-\Delta u)(r,\vartheta) = \delta_1(r)Y_1(\vartheta) + \delta_2(r)Y_2(\vartheta)\,, \tag{5.2}$$

where Y_1 and Y_2 are 2π-periodic functions on $\mathbb{R}$. Assume for now that Y_1 and Y_2 are continuous. Let G be the Green's function for $-\Delta$ on Ω, with zero boundary conditions. By rotation invariance, $G(r,\vartheta\,,\rho_j,\varphi)$ depends on the angles ϑ and φ only via their difference. Applying $(-\Delta)^{-1}$ to both sides of (5.2) yields

$$u(r,\vartheta) = \sum_{j=1}^{2}\int_0^{2\pi}\Gamma_{r,\rho_j}(\vartheta-\varphi)Y_j(\varphi)\,d\varphi\,, \qquad \Gamma_{r,\rho_j}(t) = G(r,t\,,\rho_j,0)\,. \tag{5.3}$$

Consider the traces $U_j(\vartheta) = u(\rho_j,\vartheta)$. If u is a solution of the equation $-\Delta u = wu^3$, then by (5.2) we must have $Y = WU^3$, meaning that $Y_j = W_jU_j^3$ for both $j=1$ and $j=2$. Combining this with (5.3), we see that $-\Delta u = wu^3$ if and only if U is a fixed point of $\mathcal{N}$,

$$\mathcal{N}(U)_i = \sum_{j=1}^{2} W_j\Gamma_{\rho_i,\rho_j} * U_j^3\,, \qquad i = 1,2\,. \tag{5.4}$$

Here "$*$" denotes the standard convolution operator. Defining $\Gamma_{i,j}h = \Gamma_{\rho_i,\rho_j} * h$, we can write (5.4) more succinctly as

$$\mathcal{N}(U) = \Gamma\bigl(WU^3\bigr)\,, \qquad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}\,, \qquad \Gamma = \begin{bmatrix} \Gamma_{1,1} & \Gamma_{1,2} \\ \Gamma_{2,1} & \Gamma_{2,2} \end{bmatrix}\,.$$

An explicit computation shows that

$$\Gamma_{r,\rho} * \cos(k\,\boldsymbol{.}) = \psi_k(r,\rho)\cos(k\,\boldsymbol{.})\,, \tag{5.5}$$

with

$$\psi_k(r,\rho) = \frac{1}{2k}\Bigl(\bigl[(r/\rho)\wedge(\rho/r)\bigr]^k - (r\rho)^k\Bigr)\,, \quad \psi_0(r,\rho) = \ln\bigl(r^{-1}\wedge\rho^{-1}\bigr)\,, \tag{5.6}$$

for $k \ge 1$. Here we have used the notation $a \wedge b = \min\{a,b\}$. To be more specific, consider the function v defined by $v(r,\vartheta) = \psi_k(r,\rho)\cos(k\vartheta)$. Clearly v is harmonic for

$r \neq \rho$, continuous at $r = \rho$, and vanishes for $r = 1$. Furthermore, $\partial_r \psi_k(r,\rho)$ has a jump discontinuity at $r = \rho$ with a jump of size $-\rho^{-1}$. Thus we have $-\Delta v(r,\vartheta) = \rho^{-1}\delta(r-\rho)\cos(k\vartheta)$. This implies (5.5).

Based on the result in [11] mentioned earlier, we expect that solutions of $-\Delta u = wu^3$ are symmetric with respect to one reflection. Thus, we restrict our analysis to functions U_j that are even. To be more precise, given $\varrho > 0$, denote by $\mathcal{S}_\varrho$ the strip in $\mathbb{C}$ defined by the condition $\operatorname{Im}(z) < \varrho$. Denote by $\mathcal{A}(\varrho)$ the Banach space of all real analytic 2π-periodic functions on $\mathcal{S}_\varrho$ that extend continuously to the boundary of $\mathcal{S}_\varrho$ and have a finite norm

$$\|h\| = \sum_{k=-\infty}^{\infty} |h_k| \cosh(\varrho k)\,, \qquad h(z) = \sum_{k=0}^{\infty} h_k \cos(kz) + \sum_{k=1}^{\infty} h_{-k}\sin(kz)\,. \tag{5.7}$$

Most of our analysis uses a fixed value of ϱ that will be specified below. Thus, in order to simplify notation, we will also write $\mathcal{A}$ in place of $\mathcal{A}(\varrho)$. The even subspace of $\mathcal{A}$ will be denoted $\mathcal{A}_e$.

Remark 1. As defined above, $\mathcal{A}$ is a Banach space over $\mathbb{R}$. When discussing eigenvectors of linear operators on $\mathcal{A}$, we will also need the corresponding space over $\mathbb{C}$. Since it should be clear from the context which number field is being used, we will denote both spaces by $\mathcal{A}$. Since we are only interested in real solution, the default field is $\mathbb{R}$.

Notice that $\mathcal{A}$ and $\mathcal{A}_e$ are Banach algebras. In particular, $h \mapsto h^3$ is an analytic map on $\mathcal{A}_e$. And from (5.6) we see that the convolution operators $\Gamma_{i,j}$ are bounded (and in fact compact) on $\mathcal{A}_e$. Thus, the equation (5.4) defines an analytic map $\mathcal{N} : \mathcal{A}_e^2 \to \mathcal{A}_e^2$. Here $\mathcal{A}_e^2$ denotes the Banach space of all vectors $U = [U_1\ U_2]^\top$ with $U_1, U_2 \in \mathcal{A}_e$ and $\|U\| = \|U_1\| + \|U_2\|$. Such pairs of functions can (and will) be identified with functions on $C = C_1 \cup C_2$.

Consider the trace $T : \mathrm{C}_0^\infty(\omega) \to \mathbb{R}$ defined by $Tu = [U_1\ U_2]^\top$ with $U_j(\vartheta) = u(\rho_j,\vartheta)$. It is well known that T extends to a bounded linear operator from $\mathcal{H}$ to $\mathrm{L}^p(C)$, for every finite $p \geq 1$. So in what follows, T stands for any one (or each) of these extensions.

Denote by $\mathcal{H}_e$ be the subspace of $\mathcal{H}$ consisting of all function $u \in \mathcal{H}$ that are even under the reflection $\vartheta \mapsto -\vartheta$. Let $\mathcal{Z}$ be the (closed) null space of $T : \mathcal{H}_e \to \mathrm{L}^p(C)$. Clearly this null space is independent of the choice of $p \geq 1$. Denote by $\mathcal{H}_e^0$ the orthogonal complement of $\mathcal{Z}$ in $\mathcal{H}_e$. Since $\langle v, u\rangle = \int_\Omega v(-\Delta)u$ on a dense subspace of $\mathcal{H}_e$, we see that $\mathcal{H}_e^0$ consists precisely of those functions $u \in \mathcal{H}_e$ for which Δu vanishes (in the sense of distributions) on $\Omega \setminus C$. Clearly, every even solution $u \in \mathcal{H}_e$ of the equation $-\Delta u = wu^3$ belongs to $\mathcal{H}_e^0$.

Proposition 5.1. *Denote by $\bar\Gamma$ the map $Y \mapsto u$ defined by (5.3). Then $T^* = \bar\Gamma\Gamma^{-1}$ maps $\mathcal{A}_e^2$ into a dense subspace of $\mathcal{H}_e^0$. Furthermore, if $U \in \mathcal{A}_e$ and $v \in \mathcal{H}_e$ then*

$$\langle v, T^*U\rangle = \langle Tv, U\rangle_0\,, \qquad \langle V, U\rangle_0 \stackrel{\text{def}}{=} \int_0^{2\pi} V^\top \Gamma^{-1} U\,. \tag{5.8}$$

Proof. First, notice that Γ is a convolution operator whose Fourier multipliers are the 2×2 matrices Ψ_k with entries $\psi_k(\rho_i,\rho_j)$. Since $(-\Delta)^{-1}$ is a positive operator, the eigenvalues of the matrices Ψ_k are all positive. So $\langle\,.\,,.\rangle_0$ defines an inner product on $\mathcal{A}_e^2$, since Ψ_k^{-1} grows only linearly in k, while the Fourier coefficients h_k of a function $h\in\mathcal{A}_e$ decrease exponentially with k. Clearly $U\mapsto\langle U,U\rangle_0$ is continuous on $\mathcal{A}_e^2$.

Let $\mathcal{P}\subset\mathcal{A}_e$ be the space of all 2π-periodic Fourier polynomials. Let $U\in\mathcal{P}^2$. Then $Y=\Gamma^{-1}U$ belongs to $\mathcal{P}^2$ as well. Clearly $u=\bar{\Gamma}Y$ belongs to $\mathcal{H}_e^0$ and satisfies $Tu=U$. Using (5.2) we have

$$\langle v,u\rangle=\int_\Omega v(-\Delta u)=\int_0^{2\pi}V^\top Y=\langle V,U\rangle_0\,, \tag{5.9}$$

for every $v\in\mathrm{C}_0^\infty(\Omega)$, where $V=Tv$. Given that $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathcal{H}$, we can take a limit in (5.9) to obtain $\langle v,u\rangle=\langle V,U\rangle_0$ for any $v\in\mathcal{H}_e$. Here we have used the continuity of $T:\mathcal{H}\to\mathrm{L}^1(C)$, which implies $\int_0^{2\pi}V_n^\top Y\to\int_0^{2\pi}V^\top Y$ whenever $v_n\to v$ in $\mathcal{H}_e$. Thus $\langle v,u\rangle=\langle V,U\rangle_0$ holds for any $v\in\mathcal{H}_e$. In particular, $\langle u,u\rangle=\langle U,U\rangle_0$. Taking limits again, using that $\mathcal{P}^2$ is dense in $\mathcal{A}_e^2$, we find that $\bar{\Gamma}\Gamma^{-1}$ extends to a continuous linear operator $T^*:\mathcal{A}_e\to\mathcal{H}_e^0$, and that T^* satisfies (5.8).

Let now v be a function in $\mathcal{H}_e$ that is perpendicular to every function $u\in T^*\mathcal{A}_e^2$. Then $\langle Tv,U\rangle_0=0$ for every $U\in\mathcal{A}_e^2$, which clearly implies that $v\in\mathcal{Z}$. This shows that $T^*\mathcal{A}_e^2$ is dense in $\mathcal{H}_e^0$. **QED**

In what follows, the parameters ρ_j and W_j that appear in (5.1) are assumed to take the values

$$\rho_1=\tfrac{3}{4}\,,\quad W_1=1\,,\quad \rho_2=\tfrac{1}{8}\,,\quad W_2=\tfrac{3}{8}\,. \tag{5.10}$$

In order to solve the fixed point equation $\mathcal{N}(U)=U$, we first determine numerically an approximate solution $P=(P_1,P_2)$. Then we consider a quasi-Newton map

$$\mathcal{M}(H)=H+\mathcal{N}(P+AH)-(P+AH)\,,\qquad H\in\mathcal{A}_e^2\,, \tag{5.11}$$

where A is an approximation to $[\mathrm{I}-D\mathcal{N}(P)]^{-1}$. Given $\delta>0$ and $H\in\mathcal{A}_e^2$, denote by $B_\delta(H)$ the closed ball of radius δ in $\mathcal{A}_e^2$, centered at H. Let $\varrho=\log\bigl(17/16\bigr)$. Our proofs of the following three lemmas are computer-assisted and will be described in Section 6.

Lemma 5.2. *There exist a pair of Fourier polynomials $P=(P_1,P_2)$, a linear isomorphism $A:\mathcal{A}_e^2\to\mathcal{A}_e^2$, and positive constants K,δ,ε satisfying $\varepsilon+K\delta<\delta$, such that the map $\mathcal{M}$ given by (5.11) is well-defined on $B_\delta(0)$ and satisfies*

$$\|\mathcal{M}(0)\|<\varepsilon\,,\qquad \|D\mathcal{M}(H)\|<K\,,\qquad H\in B_\delta(0)\,. \tag{5.12}$$

This lemma, together with the contraction mapping principle, implies that the map $\mathcal{M}$ has a unique fixed point $H_*\in B_\delta(0)$. So $U_*=P+H_*$ is a fixed point of $\mathcal{N}$. The corresponding function $u_*=T^*U_*$ belongs to $\mathcal{H}_e^0$ and solves the equation $-\Delta u_*=wu_*^3$.

The following lemma shows that $|u_*|$ cannot be invariant under any reflection symmetry of Ω other than $\vartheta\mapsto-\vartheta$. Notice that $U_*\in B_r(P)$ for $r=\|A\|\delta$.

Lemma 5.3. *Let $r = \|A\|\delta$. Then the components U_1 and U_2 of every $U \in B_r(P)$ are strictly increasing on the interval $[0, \pi]$, and $U_1(\pi/2) \neq 0$.*

What remains to be proved is that u_* has Morse index 2. Given the relation (2.3) between $D^2J(u)$ and $D\mathcal{F}(u)$, it suffices to prove e.g. that the (compact) linear operator $D\mathcal{F}(u_*)$ has exactly two eigenvalues in the interval $[1, \infty)$ and in its interior. Our first goal now is to prove an analogous result for the derivative $D\mathcal{N}(U_*)$ of $\mathcal{N}$ at the fixed point U_*. Notice that all eigenvalues of $D\mathcal{N}(U_*)$ are real and positive, since

$$\left\langle V, D\mathcal{N}(U)V'\right\rangle_0 = 3\int_0^{2\pi} V^\top\bigl(WU^2V'\bigr) = 3\int_\Omega vwu^2v' = \left\langle v, D\mathcal{F}(u)v'\right\rangle, \tag{5.13}$$

where $u = T^*U$, $v = T^*V$, and $v' = T^*V'$.

In order to estimate the largest 3 eigenvalues of $D\mathcal{N}(U_*)$, we approximate $D\mathcal{N}(U_*)$ numerically by a simple operator $\mathcal{L}_0$.

Lemma 5.4. *With P, r as in Lemma 5.2 and Lemma 5.3, there exists a continuous finite-rank operator $\mathcal{L}_0$ on $\mathcal{A}_e$ with eigenvalues $\mu_1 > \mu_2 > 1 > \mu_3 > \ldots \geq 0$, such that*

$$\left\|\bigl[D\mathcal{N}(U) - \mathcal{L}_0\bigr]\bigl(\mathcal{L}_0 - \mathbb{I}\bigr)^{-1}\right\| < 1\,, \qquad \forall U \in B_r(P)\,. \tag{5.14}$$

Furthermore, $\mathcal{L}_0$ is symmetric with respect to the inner product (5.8).

Based on these three lemmas, we can now give a

Proof of Theorem 1.2. As described earlier, Lemma 5.2 implies the existence of a fixed point $U_* \in B_r(P)$ of $\mathcal{N}$, and the corresponding function $u_* \in \mathcal{H}_e^0$ is a fixed point of $\mathcal{F}$. Furthermore, Lemma 5.3 rules out the existence of any reflection symmetry of u_* other than $u_*(r, \vartheta) = u_*(r, -\vartheta)$. What remains to be proved is that u_* has Morse index 2.

First we note that the map $F : u \mapsto \frac{1}{4}\int_\Omega wu^4$ and its derivatives (as multilinear forms) are well-defined on $\mathcal{H}$ and continuous, since $T : \mathcal{H} \to \mathrm{L}^4(C)$ is bounded. The same holds for $J : u \mapsto \langle u, u\rangle - F(u)$. Similarly for the map $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ defined by (2.2), as can be seen from the identity $\langle \mathcal{F}(u), v\rangle = DF(u)v$. Furthermore, $D\mathcal{F}(u)$ is compact for any $u \in \mathcal{H}$ since the trace $T : \mathcal{H} \to \mathrm{L}^4(C)$ is in fact compact. Notice also that $D\mathcal{F}(u)$ is symmetric.

Consider now the orthogonal splitting $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_o$, where $\mathcal{H}_o$ is the subspace of $\mathcal{H}$ consisting of all functions $u \in \mathcal{H}$ that are odd under the reflection $\vartheta \mapsto -\vartheta$. Clearly, both $\mathcal{H}_e$ and $\mathcal{H}_o$ are invariant subspaces for $D\mathcal{F}(u_*)$. By (2.3) and Proposition 5.5 below, $D^2J(u_*)(u + v, v) \geq 0$ for all $u \in \mathcal{H}_e$ and all $v \in \mathcal{H}_o$. Thus, given that we are trying to identify the largest subspace of $\mathcal{H}$ where $D^2J(u_*)$ is negative definite, it suffices to consider subspaces of $\mathcal{H}_e$.

Next consider the splitting $\mathcal{H}_e = \mathcal{Z} \oplus \mathcal{H}_e^0$, where $\mathcal{Z}$ is the (closed) null space of T. If $v \in \mathcal{Z}$ then $\left\langle u, D\mathcal{F}(u_*)v\right\rangle = 3\int_\Omega wu_*^2uv = 0$ for every $u \in \mathcal{H}_e$. Thus, we can restrict our analysis further to $\mathcal{H}_e^0$.

Since $T^*\mathcal{A}_e^2$ is dense in $\mathcal{H}_e^0$ by Proposition 5.1, we start by discussing the spectrum of $D\mathcal{N}(U_*)$. Let $U \in B_r(P)$ be fixed but arbitrary. Consider the operators $\mathcal{L}_s = sD\mathcal{N}(U) +$

$(1-s)\mathcal{L}_0$, for $0 \le s \le 1$, with $\mathcal{L}_0$ as described in Lemma 5.4. Each of these operators is compact, symmetric with respect to the inner product (5.8), and positive in the sense that $\langle H, \mathcal{L}_s H\rangle_0 \ge 0$ for all $H \in \mathcal{A}_e^2$. Furthermore, $\mathcal{L}_s - \mathbb{I}$ has a bounded inverse,

$$(\mathcal{L}_s - \mathbb{I})^{-1} = (\mathcal{L}_0 - \mathbb{I})^{-1}(\mathbb{I} + s\mathcal{V})^{-1}\,, \qquad \mathcal{V} = \bigl[D\mathcal{N}(U) - \mathcal{L}_0\bigr](\mathcal{L}_0 - \mathbb{I})^{-1}\,, \tag{5.15}$$

since $\|\mathcal{V}\| < 1$ by (5.14). In other words, $\mathcal{L}_s$ has no eigenvalue 1. Since the positive eigenvalues of $\mathcal{L}_s$ vary continuously with s, this implies that the operators $\mathcal{L}_0$ and $\mathcal{L}_1$ have the same number of eigenvalues (counting multiplicities) in the interval $[1,\infty)$ and its interior. By Lemma 5.4, this number is 2.

By (2.3) and (5.13) we have

$$D^2J(u_*)(v,v) = \bigl\langle Tv, [\mathbb{I} - D\mathcal{N}(U_*)]Tv\bigr\rangle_0\,, \tag{5.16}$$

for every function $v \in T^*\mathcal{A}_e^2$. Let $\mathcal{P}$ be the subspace of $\mathcal{A}_e$ spanned by the two eigenvectors of $D\mathcal{N}(U_*)$ for the two eigenvalues that are larger than 1. From (5.16) we see that $D^2J(u_*)$ is negative definite on the two-dimensional subspace $T^*\mathcal{P}$ of $\mathcal{H}_e^0$. Since $D\mathcal{N}(U_*)$ is symmetric with respect to the inner product $\langle\,.\,,.\,\rangle_0$, we have $\mathcal{A}_e = \mathcal{P} \oplus \mathcal{Q}$ with $\mathcal{Q}$ a subspace of $\mathcal{A}_e$ that is perpendicular to $\mathcal{P}$. Furthermore, $\bigl\langle V, [\mathbb{I} - D\mathcal{N}(U_*)]V\bigr\rangle_0 \ge 0$ for every $V \in \mathcal{Q}$.

Let now u be any vector in $\mathcal{H}_e^0$ that is perpendicular to $T^*\mathcal{P}$. Since $T^*\mathcal{A}_e^2$ is dense in $\mathcal{H}_e^0$ by Proposition 5.1, there exists a sequence of vectors $v_n \in T^*\mathcal{Q}$ that converges to u. By (5.16) we have $D^2J(u_*)(v_n,v_n) \ge 0$ for all n, and thus $D^2J(u_*)(u,u) \ge 0$. This shows that the plane $T^*\mathcal{P}$ is the largest subspace of $\mathcal{H}_e^0$ where $D^2J(u_*)$ is negative definite. Hence u_* has index 2, as claimed. **QED**

Denote by $\mathcal{H}_o$ the subspace of $\mathcal{H}$ consisting of all functions $u \in \mathcal{H}$ that are odd under the reflection $\vartheta \mapsto -\vartheta$.

Proposition 5.5. *The restriction of $D\mathcal{F}(u_*)$ to $\mathcal{H}_o$ has no eigenvalue larger than* 1.

Proof. Since functions in $\mathcal{H}_o$ vanish on the x_1-axis, we can (and will) identify $\mathcal{H}_o$ with $\mathrm{H}_0^1(B)$, where B is the half-disk $B = \{x \in \Omega : x_1 > 0\}$. Denote by $\mathcal{L}$ be the restriction of $D\mathcal{F}(u_*)$ to $\mathcal{H}_o$. Clearly $\mathcal{L}$ is compact, symmetric, and positive. Let λ_1 be the largest eigenvalue of $\mathcal{L}$, and let v_1 be an eigenvector of $\mathcal{L}$ with eigenvalue λ_1. Then v_1 maximizes the Rayleigh quotient

$$R(v) = \frac{\langle v, \mathcal{L}v\rangle}{\langle v, v\rangle} = \frac{3\int_B w u_*^2 v^2}{\int_B |\nabla v|^2}\,, \qquad v \in \mathcal{H}_o\,, \tag{5.17}$$

and $R(v_1) = \lambda_1$. Using that $|\nabla|v||^2 = |\nabla v|^2$ almost everywhere [6, Theorem 6.17] if $v \in \mathcal{H}$, we see that $|v_1|$ is also an eigenvector of of $\mathcal{L}$ with eigenvalue λ_1.

Now we already know one eigenvector of $\mathcal{L}$: Since $(r,\vartheta) \mapsto u_*(r,\vartheta-\varphi)$ is a fixed point of $\mathcal{F}$ for any angle φ, the function $u' = \partial_\vartheta u_*$ is an eigenvector of $\mathcal{L}$ with eigenvalue 1. Given that $-\Delta u' = 3wu^2u'$, we have

$$\langle v, u'\rangle = \sum_{j=1}^{2} 3W_j \int_0^\pi v(\rho_j,\vartheta)U_j(\vartheta)^2 U_j'(\vartheta)\,, \qquad v \in \mathcal{H}_o\,,$$

where $U = Tu_*$, and where U_j' denotes the derivative of U_j. Now we use that $U_j' > 0$ on the interval $(0, \pi)$ by Lemma 5.3. Thus $\langle |v_1|, u' \rangle > 0$. This implies that $\lambda_1 = 1$; otherwise $|v_1|$ would have to be orthogonal to u'. **QED**

6. Estimates done by computer

What remains to be proved are Lemmas 5.2, 5.3, and 5.4. The claims in these lemmas are (or can be written) in the form of strict inequalities. Thus, our approach is to discretize the objects involved and to estimate the discretization errors. Since Γ is a limit of finite rank operators, this can be done to sufficient precision in a finite number of steps. Still, the task is too involved to be carried out by hand, so we enlist the help of a computer. For the types of operations needed here, the techniques are quite standard by now. Thus we will restrict our description mainly to the problem-specific parts. The complete details of our proofs can be found in [16].

To every space X considered we associate a finite collection $\mathfrak{R}(X)$ of subsets of X that are "representable" on the computer. For the computer, a bound on an element $s \in S$ is an enclosure $S \ni s$ that belongs to $\mathfrak{R}(X)$. A "bound" on a map $f : X \to Y$ is a map $F : \mathfrak{R}(X) \to \mathfrak{R}(Y) \cup \{\texttt{undefined}\}$, with the property that $f(s) \in F(S)$ whenever $s \in S \in \mathfrak{R}(X)$, unless $F(S) = \texttt{undefined}$. In practice, if $F(S) = \texttt{undefined}$ then the program halts with an error message.

Each collection $\mathfrak{R}(X)$ corresponds to a data type in our programs. For $\mathfrak{R}(\mathbb{R})$ we use a type `Ball`, which consists of all pairs `S=(S.C,S.R)`, where `S.C` is a representable number (`Rep`) and `S.R` a nonnegative representable number (`Radius`). The representable set defined by such a `Ball S` is the interval $\texttt{S}^\flat = \{s \in \mathbb{R} : |s - \texttt{S.C}| \le \texttt{S.R}\}$.

For the representable numbers, we choose a numeric data type named `Rep`, for which elementary operations are available with controlled rounding [14]. This makes it possible to implement a bound `Balls.Sum` on the function $(s,t) \mapsto s+t$ on $\mathbb{R}\times\mathbb{R}$, as well as bounds on other elementary functions on $\mathbb{R}$ or $\mathbb{R}^n$, including operations like the matrix product. Unless specified otherwise, $\mathfrak{R}(X \times Y)$ is taken to be the collection of all sets $S \times T$ with $S \in \mathfrak{R}(X)$ and $T \in \mathfrak{R}(Y)$.

Consider now the function space $\mathcal{A} = \mathcal{A}(\varrho)$ defined in Section 5, with e^ϱ a representable number. Denote by $\mathcal{A}_e$ and $\mathcal{A}_o$ the even and odd subspaces of $\mathcal{A}$, respectively. Let $\mathcal{E}_k$ be the subspace of $\mathcal{A}$ consisting of all functions $h \in \mathcal{A}$ whose Fourier coefficients h_k are zero for $|k| < m$. Let D be a fixed positive integer. Our representable subsets of $\mathcal{A}$ are associated with a data type `Fourier1`, which is a triple `F=(F.R,F.C,F.E)`, where `F.R` is a `Radius` with value e^ϱ, `F.C` is an `array(-D..D)` with components `F.C(K)` of type `Ball`, and `F.E` is an `array(-2*D..2*D)` with components `F.E(M)` of type `Radius`. The corresponding set $F^\flat \in \mathfrak{R}(\mathcal{A})$ is the set of all function f that admit a representation

$$f = \sum_{k=0}^{D} C_k \cos(k.) + \sum_{k=1}^{D} C_{-k} \sin(k.) + \sum_{m=-2D}^{2D} E_m \,, \tag{6.1}$$

with $E_m \in \mathcal{E}_m \cap \mathcal{A}_e$ for $m \ge 0$ and $E_m \in \mathcal{E}_m \cap \mathcal{A}_o$ for $m < 0$, such that $C_k \in \texttt{F.C}(k)^\flat$ and $\|E_m\| \le \texttt{F.E}(m)$. Here $-D \le k \le D$ and $-2D \le m \le 2D$. Using `Balls.Sum`, it is straightforward to implement a bound `Fouriers1.Sum` on the function "+": $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. For details we refer to the package `Fouriers1` in [16]. This package also defines bounds on maps like $(f, g) \mapsto fg$ and $f \mapsto \|f\|$ etc.

$\mathfrak{R}(\mathcal{A}_e)$ is defined in an obvious way as a subset of $\mathfrak{R}(\mathcal{A})$. For $\mathfrak{R}(\mathcal{A}_e^2)$ we use pairs of even `Fourier1`. The corresponding bounds are defined in the package `Fouriers1.Green`. This package also implements bounds on various functions on $\mathcal{A}_e^2$, including the map $\mathcal{M}$ defined in (5.11) and its derivative $D\mathcal{M}$. In order to estimate the operator norm $\|\mathcal{L}\|$ of a continuous linear operator $\mathcal{L} : \mathcal{A}_e^2 \to \mathcal{A}_e^2$, we use that

$$\|\mathcal{L}\| = \sup_{\substack{k \geq 0 \\ j=1,2}} \|\mathcal{L}p_{k,j}\|, \qquad p_{k,j} = \|P_{k,j}\|^{-1} P_{k,j}, \tag{6.2}$$

where $P_{k,1}(\vartheta) = (\cos(k\vartheta), 0)$ and $P_{k,2}(\vartheta) = (0, \cos(k\vartheta))$. For the operators needed in our analysis, it is easy to determine $m \geq 0$ such that $\|\mathcal{L}p_{k,j}\|$ is "sufficiently small" for all $k \geq m$ and $j = 1, 2$. Then (6.2) reduces to a finite computation. This is how we prove e.g. the bound $\|D\mathcal{M}(H)\| < K$ claimed in Lemma 5.2.

To prove Lemma 5.3 we compute (for $j = 1, 2$) the first and second derivative of U_j, as elements in the spaces $\mathcal{A}_o(\varrho')$ and $\mathcal{A}_e(\varrho'')$, respectively, with $0 < \varrho'' < \varrho' < \varrho$. Then we verify that $U_j''(\theta) > 0$ for all $\theta \in [0, 1/16]$, that $U_j'(\theta) > 0$ for all $\theta \in [1/16, 25/8]$, and that $U_j''(\theta) < 0$ for all $\theta \in [25/8, \pi]$. Since $U_j'(0) = U_j'(\pi) = 0$, it follows that U_j is strictly monotone on $[0, \pi]$, as claimed.

We should add that we are not using the canonical bound on the evaluation map $(\vartheta, f) \mapsto f(\vartheta)$ for functions $f \in \mathcal{A}$. For the functions considered here, such bound would require subdividing an interval like $I = [1/16, 25/8]$ into extremely small subintervals. Instead, we cover I with reasonably small intervals $[x - r, x + r]$. On each such subinterval we first compute a Taylor expansion (a quadratic polynomial with error estimates) for the function $z \mapsto f(x + rz)$. This function is then evaluated on $[-1, 1]$ in one step. For details on this procedure we refer to the packages `Quadrs` and `Fouriers1`.

The operator $\mathcal{L}_0$ described in Lemma 5.4 has rank $n = 140$ and is constructed as follows. Denote by $\mathbb{P}$ the orthogonal projection in $\mathcal{A}_e^2$ onto the n-dimensional subspace spanned by the vectors $P_{k,j}$ for $0 \leq k \leq 69$ and $j = 1, 2$. The inner product used here is the one defined in (5.8). Consider $L = \mathbb{P}D\mathcal{N}(P)\mathbb{P}$, regarded as a linear operator on $\mathbb{P}\mathcal{A}_e^2$. As a first step, we determine n approximate eigenvalue-eigenvector pairs (μ_i, v_i) for this operator. As expected, $\mu_1 > \mu_2 > 1 > \mu_3 > \ldots > \mu_n > 0$, and the vectors v_i are almost mutually orthogonal. Now we apply a rigorous Gram-Schmidt procedure to convert $[v_1, v_2, \ldots, v_n]$ into an orthonormal basis $B = [b_1, b_2, \ldots, b_n]$. Identifying $\mathbb{P}\mathcal{A}_e^2$ with $\mathbb{R}^n$, and B with the $n \times n$ matrix whose columns are the vectors b_i, we have $B^{-1} = B^\top \Gamma$, where $B^\top$ denotes the transposed matrix. Now we extend the matrices B and $D = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$ to operators on $\mathcal{A}_e^2$ by setting $BU = U$ and $DU = 0$, for all U in the orthogonal complement of $\mathbb{P}\mathcal{A}_e^2$. Then $\mathcal{L}_0 = BDB^{-1}$ is self-adjoint with eigenvalues $\mu_1 > \mu_2 > 1 > \mu_3 > \ldots \geq 0$. Furthermore, the operator $(\mathcal{L}_0 - \mathbb{I})^{-1} = B(D - \mathbb{I})^{-1}B^{-1}$ appearing in (5.14) is easy to compute. The operator norm in (5.14) is now estimated as described earlier.

For a precise and complete description of all definitions and estimates, we refer to the source code and input data of our computer programs [16]. The source code is written in Ada2005 [13]. Our programs were compiled and run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [15].

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