A renormalization group approach to quasiperiodic motion with Brjuno frequencies

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Abstract. We introduce a renormalization group scheme that applies to vector fields on $\mathbb{T}^d \times \mathbb{R}^m$ with frequency vectors that satisfy a Brjuno condition. Earlier approaches were restricted to Diophantine frequencies, due to a limited control of multidimensional continued fractions. We get around this restriction by avoiding the use of a continued fractions expansion. Our results concerning invariant tori generalize those of reference [17] from Diophantine to Brjuno type frequency vectors. In particular, each Brjuno vector $\omega \in \mathbb{R}^d$ determines an analytic manifold $\mathcal{W}$ of infinitely renormalizable vector fields, and each vector field on $\mathcal{W}$ is shown to have an elliptic invariant d-torus with frequencies $\omega_1, \omega_2, \ldots, \omega_d$.

1. Introduction and main results

The renormalization of Hamiltonian flows and related maps has a relatively long history; see e.g. [21,7,16] and references therein. The approach considered here was motivated originally by the problem of describing the breakup of invariant tori for Hamiltonian systems with two degrees of freedom [1–11]. Some of these questions have been answered in [10,11]. The same methods also apply to the study of near-integrable Hamiltonians or near-linear flows. In this regime, the renormalization group approach can be viewed as an alternative to purely perturbative methods, based on KAM theory or Lindstedt series. Over the past few years, its scope has been extended from a small set of "self-similar" frequency vectors [8] to arbitrary Diophantine frequencies [13–18], and from Hamiltonian flows to a large class of vector fields [17]. As far as the construction of smooth invariant tori is concerned, this work covers the classical KAM results, but not the later extensions to Brjuno type frequency vectors [22–31]. This is due to the fact that the current approach requires good bounds on a continued fractions expansion, such as the ones obtained in [15] for Diophantine frequency vectors. Unfortunately, there seem to be significant obstacles to obtaining such bounds for Brjuno vectors in dimensions $d > 2$.

This has motivated us to develop a renormalization scheme that does not rely on continued fractions. As it turns out, it applies quite naturally to rotation vectors $\omega \in \mathbb{R}^d$ that satisfy the following Brjuno condition [22]:

$$\sum_{n=1}^{\infty} 2^{-n} \ln(1/\Omega_n) < \infty \,, \qquad \Omega_n = \min_{0<|\nu|\le 2^n} |\omega \cdot \nu| \,, \tag{1.1}$$

where ν denotes lattice points in $\mathbb{Z}^d$. Our new renormalization group transformations share some important features with those used in [8–20]. Thus, before discussing the differences, we shall first describe the transformations used in [8–17], starting with some general remarks about renormalization.

Renormalization can be viewed as a tool for classifying systems by the value of a given observable that describes asymptotic properties of the system. A renormalization

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transformation is a map on the space of systems being considered, contracting directions within the same equivalence class, and expanding directions along which the observable changes, preferably in a way that induces a natural action on the set of observed values. In the case at hand, the systems are vector fields on $\mathcal{M} = \mathbb{T}^d \times \mathbb{R}^m$, and the observed quantities are the ratios among the d frequencies of rotation. Among the natural actions on frequency vectors $\omega \in \mathbb{R}^d$ are the steps in a continued fractions algorithm. They typically involve integer matrices with a distinguished expanding direction, such that rational approximants to ω approach (up to rescaling) the orbit of ω under iteration of the algorithm.

Let now T be a fixed matrix in $\mathrm{SL}(d, \mathbb{Z})$ whose transpose $T^\star$ contracts the orthogonal complement of ω. Let T' be the $m \times m$ identity matrix. (An alternative choice for T' will be mentioned later). To start with a simple case, consider vector fields of the form $X(x, y) = (w, v(y))$, with $w \neq 0$ and $v(0) = 0$. Then a possible renormalization transformation associated with T is given by $\mathcal{R}(X) = \eta^{-1} \mathcal{T}_\mu^* X$, where

$$\mathcal{T}_\mu(x, y) = (Tx, \mu T' y) . \tag{1.2}$$

Here, and in what follows, we use the notation $\mathcal{U}^* X = (D\mathcal{U})^{-1}(X \circ \mathcal{U})$ for the pullback of a vector field X under a diffeomorphism $\mathcal{U} : \mathcal{M} \to \mathcal{M}$. The parameters η and μ in the definition of $\mathcal{R}$ rescale time and the variable y, respectively. There is no natural choice for η, but since all members of the family $\eta \mapsto \eta^{-1} \mathcal{T}_\mu^* X$ are equivalent, in the sense that they yield the same frequency ratios, it is useful to choose $\eta = \eta(X)$ in such a way that $\mathcal{R}(X)$ becomes a specific (normalized) representative of the family. This ensures contraction within this family of equivalent systems. The same considerations apply in principle to the scaling μ, but for the vector fields considered here, it suffices to choose for μ a small positive constant that makes $\mu T'$ a contraction. Our precise choices for T, η, and μ, will be described in Sections 3.

When considering more general vector fields, we also have to achieve contraction within families $\mathcal{U} \mapsto \mathcal{U}^* X$, obtained from changes of coordinates $\mathcal{U} : \mathcal{M} \to \mathcal{M}$ close to the identity. This suggests defining

$$\mathcal{R}(X) = \eta^{-1} \mathcal{T}_\mu^* \mathcal{U}_X^* X , \tag{1.3}$$

where $\mathcal{U}_X$ is some change of coordinates designed to bring the renormalized vector field into some appropriate normal form. More details about the choice of normal forms will be given later. In particular, $\mathcal{U}_X$ will be the identity map whenever $X(x, y) = (w, v(y))$.

If $\omega \in \mathbb{R}^d$ admits a periodic continued fractions expansion, then it suffices to work with a single RG transformation [8–12]. More general frequency vectors require a sequence of RG transformations $\mathcal{R}_n$, one for each of the matrices T_n in the continued fractions expansion of ω. In the single frequency case ($d = 2$), such an analysis was carried out for Diophantine [13,14] and Brjuno type frequencies [19,20]. What makes this case special is that there is a canonical continued fractions expansion, and the corresponding matrices T_n are known explicitly. More recent results [15–18] extend the scope of renormalization to Diophantine vectors $\omega \in \mathbb{R}^d$, for arbitrary $d \geq 2$, using a multidimensional continued fractions algorithm developed in [32,33,15]. Here, the matrices T_n are no longer known explicitly. They are the increments $T_n = P_{n-1}^{-1} P_n$ of integer approximants $P_k \in \mathrm{SL}(d, \mathbb{Z})$ to matrices $WE(t_k) \in \mathrm{SL}(d,\mathbb{R})$, where $\{t_k\}$ is some appropriately chosen increasing sequence of positive real numbers, $E(t) = \mathrm{diag}(e^{-t},\ldots,e^{-t},e^{(d-1)t})$, and W is a fixed matrix in $\mathrm{SL}(d,\mathbb{R})$ that maps the expanding eigenvector of $E(t)$ to ω. The approximation is well controlled for Diophantine frequency vectors [15], but attempts to extend this to Brjuno vectors have not been successful so far.

The idea pursued here is to avoid the integer approximation and renormalize directly with real matrices. This leads us to consider tori of the form $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$, where $\mathcal{Z}$ is a simple lattice in $\mathbb{R}^d$. A nonsingular real matrix T defines a map $x+\mathcal{Z} \mapsto Tx + T\mathcal{Z}$ from the torus $\mathbb{R}^d/\mathcal{Z}$ to the torus $\mathbb{R}^d/(T\mathcal{Z})$. In order to simplify notation, this map will again be denoted by T. Furthermore, whenever the lattice $\mathcal{Z}$ is fixed or irrelevant, we will simply write x in place of $x+\mathcal{Z}$.

With this in mind, we choose the matrix T in the definition (1.2) of the scaling $\mathcal{T}_\mu$ to be of the form

$$T(x) = \eta^{-1}x_{\parallel} + \beta x_{\perp}\,, \qquad 0<\eta\,,\beta<1\,, \tag{1.4}$$

where $x = x_{\parallel} + x_{\perp}$ is the decomposition of $x\in\mathbb{R}^d$ into a component $x_{\parallel}$ parallel to ω, and a component $x_{\perp}$ perpendicular to ω. Notice that, if $d=2$ and $\omega_1/\omega_2 = 1/(k+1/(k+\ldots))$, then the choice $\eta=\beta=(\omega_1/\omega_2)^2$ makes T in fact an integer matrix. A matrix of the type (1.4) will be referred to as a *scaling matrix*. The corresponding RG transformation $\mathcal{R}$ is taken to be again of the form (1.3), with η^{-1} the expanding eigenvalue of T. Our choice of μ and $\mathcal{U}_X$ will be described later. Clearly, $K=(\omega,0)$ is a fixed point for $\mathcal{R}$. We note that, by choosing the time scaling η^{-1} in (1.3) to be the same as the spatial scaling η^{-1} in (1.4), which is independent of the vector field X, we allow $\mathcal{R}$ to have a non-contracting direction. However, since time scaling commutes with our renormalization transformations, this direction can easily be taken care of later (in applications of our main result).

Functions on the torus $\mathbb{R}^d/\mathcal{Z}$ can be identified with functions on $\mathbb{R}^d$ that are invariant under $\mathcal{Z}$-translations, or equivalently, with quasiperiodic functions on $\mathbb{R}^d$ whose frequency module lies in the dual lattice (the set of points $v\in\mathbb{R}^d$ satisfying $\exp(iv\cdot z)=1$, for all $z\in\mathcal{Z}$). For convenience, we will now perform a linear change of coordinates in $\mathbb{R}^d$, such that $\omega=(1,0,\ldots,0)$. The lattice obtained from $2\pi\mathbb{Z}^d$ under this transformation will be denoted by $\mathcal{Z}_0$, and its dual lattice by $\mathcal{V}_0$. The frequencies ν in (1.1) now range over $\mathcal{V}_0$.

Our analysis applies to vector fields that are close to $K=(\omega,0)$. We assume analyticity on a complex neighborhood D_ϱ of $D_0 = \mathbb{T}^d\times\{0\}$, characterized by the conditions $|\mathrm{Im}\,x_i|<\varrho$ and $|y_j|<\varrho$. Denote by Φ_X the flow for a vector field X. An invariant torus for X, with frequency vector ω, is a continuous embedding Γ of D_0 into the domain of X, such that $\Gamma\circ\Phi_K^t = \Phi_X^t\circ\Gamma$. Denote by A^u the space of all vector fields $Y(x,y)=(w,My+v)$, with (w,v) a vector in $\mathbb{C}^d\times\mathbb{C}^m$ and M a complex $m\times m$ matrix. In Section 2, we will introduce Banach spaces $\mathcal{A}_\varrho(\mathcal{V})$ of analytic vector fields on D_ϱ, having frequency module in $\mathcal{V}$, and a projection operator $\mathbb{P}$ from $\mathcal{A}_\varrho(\mathcal{V})$ onto A^u. The subspace of functions in $\mathcal{A}_\varrho(\mathcal{V})$, that do not depend on the coordinate $y\in\mathbb{C}^m$, will be denoted by $\mathcal{A}^0_\varrho(\mathcal{V})$. A function will be called “real” if it takes real values for real arguments. Our main result is the following.

Theorem 1.1. *Assume that ω satisfies the Brjuno condition (1.1). Then there exists a sequence of scaling matrices T_n, and a corresponding sequence of RG transformations $\mathcal{R}_n$ of the form (1.3), such that the following holds. Define $\mathcal{V}_n = T_n\mathcal{V}_{n-1}$ for $n=1,2,\ldots$.*

Then $\mathcal{R}_n$ is an analytic map, from some open neighborhood $\mathcal{D}_{n-1}$ of K in $\mathcal{A}_\varrho(\mathcal{V}_{n-1})$, to $\mathcal{A}_\varrho(\mathcal{V}_n)$. The set $\mathcal{W}$ of infinitely renormalizable vector fields X_0 in $\mathcal{D}_0$, characterized by the property that $X_n = \mathcal{R}_n(X_{n-1})$ belongs to $\mathcal{D}_n$ for $n = 1, 2, \ldots$, is the graph of an analytic function $W : (\mathbb{I} - \mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. If $\rho > \varrho + \delta$, with $\delta > 0$, then every vector field $X \in \mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$ has an elliptic invariant torus $\Gamma_X \in \mathcal{A}_\delta^0(\mathcal{V}_0)$ with frequency vector ω. The map $X \mapsto \Gamma_X$ is real analytic on $\mathcal{W} \cap \mathcal{A}_\rho(\mathcal{V}_0)$.

The bounds obtained in the proof of this theorem are uniform within classes of Bruno vectors $\mathcal{B}(\Omega')$ described at the end of Section 3. Concerning our choice of matrices T_n, we note that $\eta_n \to 0$, and that $\beta_n \to 1$, with $\prod_n \beta_n$ positive.

In addition, we obtain results analogous to those in [17], concerning the restriction of W to special types of vector fields (Hamiltonian, reversible, divergence free, symmetric) and the reduction of the number of parameters via nondegeneracy conditions. Since the proofs are completely analogous as well, we refer to [17] for details.

The main new aspect in this paper is the choice of the scaling matrices T_n, and the control of the corresponding sequence of RG transformations $\mathcal{R}_n$. The choice of the coordinate change $\mathcal{U}_X$ is determined by the same considerations as in earlier work [8–20]. Its role is to compensate for the loss of analyticity that results from the scaling $X \mapsto \mathcal{T}_\mu^* X$. In this step, we use a normal form theorem proved in [17]. Thus, controlling a single RG step is quite simple; see Section 2. In Section 3, we define the matrices T_n and give estimates on the transformations $\mathcal{R}_n$. Then we apply a stable manifold theorem for sequences of maps [17] to obtain the manifold $\mathcal{W}$ described in Theorem 1.1. The construction of invariant tori for vector fields $X \in \mathcal{W}$ is described in Section 4.

2. A single renormalization step

As mentioned in the introduction, we work with coordinates where the frequency vector is $\omega = (1, 0, \ldots, 0)$. The torus considered in this section is $\mathbb{T}^d = \mathbb{R}^d/\mathcal{Z}$, where $\mathcal{Z}$ is some simple lattice in $\mathbb{R}^d$. The dual lattice will be denoted by $\mathcal{V}$.

Unless specified otherwise, our norm on $\mathbb{C}^n$ is $\|v\| = \sup_j |v_j|$. Another norm that will be used is $|v| = \sum_j |v_j|$. For linear operators between normed linear spaces, we will always use the operator norm, unless stated otherwise. Denote by D_ρ the set of all vectors (x, y) in $\mathbb{C}^d \times \mathbb{C}^m$ characterized by $\|\operatorname{Im} x\| < \rho$ and $\|y\| < \rho$. Define $\mathcal{A}_\rho(\mathcal{V})$ to be the space of all analytic vector field X on D_ρ, with frequency module in $\mathcal{V}$, and with a finite norm

$$\|X\|_\rho = \sum_{v,\alpha} \|X_{v,\alpha}\| e^{\rho|v|} \rho^{|\alpha|}, \qquad X(x, y) = \sum_{v,\alpha} X_{v,\alpha} e^{iv\cdot x} y^\alpha, \tag{2.1}$$

where $v \cdot x = \sum_j v_j x_j$ and $y^\alpha = \prod_j y_j^{\alpha_j}$. The sums in this equation range over all $v \in \mathcal{V}$ and $\alpha \in \mathbb{N}^m$. In Section 4, we will also use functions with domain $D_0 = \mathbb{T}^d \times \{0\}$. Denote by $\mathcal{A}_0(\mathcal{V})$ the Banach space of continuous functions $F : D_0 \to \mathbb{C}^{d+m}$, with frequency module in $\mathcal{V}$, for which the norm $\|F\|_0 = \sum_v \|F_v\|$ is finite, where $\{F_v\}$ are the Fourier coefficients of F. Since the lattice $\mathcal{V}$ is fixed in this section, we will simply write $\mathcal{A}_\rho$ in place of $\mathcal{A}_\rho(\mathcal{V})$.

Proposition 2.1. *Let $X \in \mathcal{A}_\rho$ and $Z \in \mathcal{A}_{\rho'}$, with $0 \le \rho' \le \rho$. Then*

(a) $\|X(x,y)\| \le \|X\|_\rho$ *for all* $(x,y)\in D_\rho$.
(b) $(DX)Z\in\mathcal{A}_{\rho'}$ *and* $\|(DX)Z\|_{\rho'}\le(\rho-\rho')^{-1}\|X\|_\rho\|Z\|_{\rho'}$, *if* $\rho'<\rho$.
(c) $X\circ(\mathrm{I}+Z)\in\mathcal{A}_{\rho'}$ *and* $\|X\circ(\mathrm{I}+Z)\|_{\rho'}\le\|X\|_\rho$, *if* $\rho'+\|Z\|_{\rho'}\le\rho$.

The proof of these estimates is straightforward and will be omitted. In what follows, we always assume that $\rho>0$, unless specified otherwise.

We assume that the components of ω are rationally independent with respect to $\mathcal{V}$, in the sense that the first component $v_\parallel$ of any nonzero vector $v\in\mathcal{V}$ is nonzero. Then, given any $L\ge 1$, we can find $\ell>0$ such that

$$|v_\perp|>L \quad\text{or}\quad |v_\parallel|\ge\ell\,,\qquad \forall v\in\mathcal{V}\setminus\{0\}\,. \tag{2.2}$$

In other words, all points in $\mathcal{V}$, except for the origin, lie outside the rectangle $|v_\perp|\le L$ and $|v_\parallel|<\ell$. Notice that the scaling (1.4) shrinks the length L of the excluded rectangle, and expands its width ℓ. In what follows, the parameters L,ℓ,η,β are assumed to be given, subject to the conditions (1.4) and (2.2).

Definition 2.2. *Denote by S the generator of the one-parameter group of scalings $\mu\mapsto\mathcal{S}_\mu^*$, defined by $\mathcal{S}_\mu(x,y)=(x,\mu y)$. Given any subset J of $I=\mathcal{V}\times\{-1,0,1,2,\ldots\}$, define $P(J)$ to be the joint spectral projection in $\mathcal{A}_\rho(\mathcal{V})$ for the operators $(-i\nabla_x,S)$, associated with the eigenvalues (v,k) in J. Let $\tau=(1+\beta)/2$. Given $\gamma\ge1$ to be chosen later, let I^+ be the set of pairs $(v,k)\in I$ satisfying $|Tv|\le\tau|v|$ or $|Tv|\le\tau(k-\gamma)$, and let I^- be the complement of I^+ in I. Define $\mathbb{I}^\pm=P(I^\pm)$. The resonant and nonresonant parts of a vector field $X\in\mathcal{A}_\rho$ are defined as $\mathbb{I}^+X$ and $\mathbb{I}^-X$, respectively. In addition, we define $\mathbb{E}_k=P\bigl(\{(0,k)\}\bigr)$, for each $k\ge-1$. The torus averaging operator is then given by $\mathbb{E}=\sum_k\mathbb{E}_k$.*

As we will see later, the scaling $X\mapsto\mathcal{T}_\mu^*X$ is well behaved when restricted to resonant vector fields. Thus, before applying this scaling, we try to perform a change of variables $X\mapsto\mathcal{U}_X^*X$ that eliminates the nonresonant part of X. Theorem 5.2 in [17] shows that this is possible, provided that the problem can be solved to first order in the size of $X-K$. The equation for the map $\mathcal{U}_X$, and for the vector field $Z=\mathbb{I}^-Z$ generating its first order approximation Φ_Z^1 , is

$$\mathbb{I}^-(X+[Z,X])=0\,,\qquad \mathbb{I}^-\mathcal{U}_X^*X=0\,, \tag{2.3}$$

where $[Z,X]=(DX)Z-(DZ)X$. The following proposition is used to solve the first part of this equation. Given any positive real number r, denote by $\mathcal{A}_r'$ the set of vector fields $X\in\mathcal{A}_r$ whose first partial derivatives belong to $\mathcal{A}_r$. Assume that

$$2\sigma L<\ell\,,\qquad \sigma=\tfrac12(1-\beta)\eta\,. \tag{2.4}$$

Proposition 2.3. *If $r>0$ and $Z\in\mathcal{A}_r'$ is nonresonant, then*

$$\|[Z,K]\|_r\ge\sigma\|Z\|_r\,,\qquad \|[Z,K]\|_r\ge\frac{\sigma r}{\sigma r+r+\gamma+2}\|DZ\|_r\,. \tag{2.5}$$

Proof. Assume that (v,k) belongs to I^-. In particular, we have $|Tv|>\tau|v|$, or equivalently, $\eta^{-1}|v_\parallel|+\beta|v_\perp|>\tau|v_\parallel|+\tau|v_\perp|$. This immediately implies that $|v_\parallel|>\sigma|v_\perp|$. Combining this with the condition $|Tv|>\tau(k-\gamma)$, we also have

$$\sigma^{-1}|v_\parallel|=\tau^{-1}\bigl(\eta^{-1}|v_\parallel|+\beta\sigma^{-1}|v_\parallel|\bigr)>\tau^{-1}\bigl(\eta^{-1}|v_\parallel|+\beta|v_\perp|\bigr)=\tau^{-1}|Tv|>k-\gamma\,.$$

The inequality $|v_\parallel| > \sigma|v_\perp|$, together with (2.2) and (2.4), also implies that $|v_\parallel| > \sigma$. These bounds show that if $Z \in \mathbb{I}^-\mathcal{A}'_r$ and $Y = [Z, K]$, then $\|Z\|_r \le \sigma^{-1}\|Y\|_r$ and

$$\sum_{j=2}^{d} \left\| \frac{\partial}{\partial x_j} Z \right\|_r \le \frac{1}{\sigma}\|Y\|_r\,, \qquad \sum_{j=1}^{m} \left\| \frac{\partial}{\partial y_j} Z \right\|_r \le \frac{\gamma+2}{\sigma r}\|Y\|_r\,. \tag{2.6}$$

As a result we obtain (2.5). **QED**

The existence of a solution for equation (2.3) follows from the normal form theorem in [17, Section 5]. More specifically, this theorem proves a generalization of Lemma 2.4 below, assuming just the bounds (2.5), and part (*c*) of Proposition 2.1. A similar theorem can also be found in [16], and purely Hamiltonian versions in [8-10,12,14]. The proofs involve Nash-Moser type methods, which are needed due to the domain loss during composition.

Let $\varrho > 0$ be fixed once and for all. What we will call a "universal constant" may depend on the choice of ϱ, but not on any other parameter.

Lemma 2.4. *There exist universal constants C_1 and C_2, such that the following holds. Let $\rho' > 0$ and $\rho \ge \rho' + \sigma\varrho$. If X is any vector field in $\mathcal{A}'_\rho$, satisfying*

$$\|X - K\|'_\rho \le C_1(\sigma/\gamma)\,, \qquad \|\mathbb{I}^-X\|_\rho \le C_1(\sigma/\gamma)^2\,, \tag{2.7}$$

then there exists a vector field $Z \in \mathbb{I}^-\mathcal{A}_\rho$ and a change of coordinates $\mathcal{U}_X : D_{\rho'} \to D_\rho$, solving equation (2.3). The vector field $\mathcal{U}_X^* X$ belongs to $\mathcal{A}_{\rho'}$, and

$$\begin{aligned} \|Z\|_\rho\,, \|\mathcal{U}_X - \mathrm{I}\|_{\rho'} &\le C_2(\gamma/\sigma)\|\mathbb{I}^-X\|_\rho\,,\\ \|\mathcal{U}_X^*X - X\|_{\rho'} &\le C_2(\rho-\rho')^{-1}(\gamma/\sigma)\|\mathbb{I}^-X\|_\rho\,,\\ \|\mathcal{U}_X^*X - X - [Z,X]\|_{\rho'} &\le C_2(\rho-\rho')^{-3}(\gamma/\sigma)^3\|\mathbb{I}^-X\|_\rho^2\,. \end{aligned} \tag{2.8}$$

The map $X \mapsto \mathcal{U}_X$ is continuous in the region defined by (2.7), and analytic in its interior.

Next, we assume that the scaling parameters η, β and μ satisfy

$$\eta < 1/2\,, \qquad e^{-\varrho\frac{(1-\beta)}{6}L} \le (4\mu)^{\gamma+1}\,, \qquad 4\mu \le e^{-\varrho}\,. \tag{2.9}$$

Lemma 2.5. *If $\varrho(2+\beta)/3 \le \rho' \le \varrho$, then $\mathcal{T}_\mu^*$ defines a bounded linear operator from $\mathbb{I}^+\mathcal{A}_{\rho'}(\mathcal{V})$ to $\mathcal{A}_\varrho(T\mathcal{V})$, with the property that*

$$\begin{aligned} \|\mathcal{T}_\mu^*\mathbb{E}_k X\|_\varrho &\le 8\eta^{-1}(4\mu)^k\|\mathbb{E}_k X\|_{\rho'}\,,\\ \|\mathcal{T}_\mu^*\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_\varrho &\le 2\eta^{-1}\bigl(4e^\varrho\mu\bigr)^\gamma\|\mathbb{I}^+(\mathbb{I}-\mathbb{E})X\|_{\rho'}\,. \end{aligned} \tag{2.10}$$

Proof. By our choice of norm (2.1), it suffices to verify the given bounds for vector fields $X = P(J)Y$, with $J \subset I^+$ containing a single point, say $J = \{(v,k)\}$. Let $b = \varrho/(\rho'\beta)$. Then it follows essentially from the definitions that

$$\|\mathcal{T}_\mu^*P(J)Y\|_\varrho \le 2\eta^{-1}e^A\|P(J)Y\|_{\rho'}\,, \qquad A = \varrho|Tv| - \rho'|v| + k\ln(b\mu)\,. \tag{2.11}$$

Setting $v = 0$, and using that $1 < b < 4$, yields the first bound in (2.10).

In order to prove the second bound, assume that (v, k) belongs to I^+, and that $v \neq 0$. Consider first the case $|Tv| \le \tau|v|$. It leads to $|v_{\parallel}| < 2\sigma|v_{\perp}|$, if we use that $\eta\tau < 1/2$. This inequality excludes frequencies v that satisfy $|v_{\perp}| \le L$ and $|v_{\parallel}| \ge \ell$, due to the condition (2.4). Thus, we must have $|v_{\perp}| > L$ by condition (2.2). Consequently,

$$A \le -\varrho\Bigl(\frac{\rho'}{\varrho} - \tau\Bigr)|v| + k\ln(b\mu) \le -\varrho\frac{1-\beta}{6}L + k\ln(b\mu)\,, \tag{2.12}$$

and the second bound in (2.10) follows by using (2.9).

Next, consider the case $|Tv| \le \tau(k - \gamma)$. Notice that $k > \gamma$ here, since v is nonzero. By using the bound $A \le \varrho(k-\gamma) + k\ln(b\mu)$, together with (2.11), we obtain

$$\|\mathcal{T}_\mu^* P(J)Y\|_\varrho \le 2\eta^{-1}\bigl(be^\varrho\mu\bigr)^k \|P(J)Y\|_{\rho'}\,.$$

This again implies the second bound in (2.10). **QED**

Combining the preceding two lemmas, we obtain the following theorem. Notice that, by property (2.8), the restriction of $\mathcal{R}$ to the subspace $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V})$ defines a linear operator from $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V})$ to $\mathbb{P}\mathcal{A}_\varrho(T\mathcal{V})$. This operator will be denoted by $\mathcal{L}$.

Theorem 2.6. *There exist universal constants $C, R > 0$, such that the following holds, under the given assumptions on $L, \ell, \eta, \beta, \gamma$ and μ. Let B be the open ball in $\mathcal{A}_\varrho(\mathcal{V})$ of radius $R(\sigma/\gamma)^2$, centered at K. Then $\mathcal{R}$ is a bounded analytic map from B to $\mathcal{A}_\varrho(T\mathcal{V})$, satisfying $\|\mathcal{L}^{-1}\| \le 1$ and*

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &\le \eta^{-2}(1-\beta)^{-1}(\gamma/\sigma)(C\mu)^\gamma\|(\mathbb{I}-\mathbb{E})X\|_\varrho\,,\\ \|(\mathbb{I}-\mathbb{P})\mathcal{R}(X)\|_\varrho &\le C\eta^{-2}(1-\beta)^{-1}(\gamma/\sigma)\mu\|(\mathbb{I}-\mathbb{P})X\|_\varrho\,,\\ \|\mathbb{E}\mathcal{R}(X) - \mathcal{R}(\mathbb{E}X)\|_\varrho &\le C\eta^{-2}(1-\beta)^{-3}(\gamma/\sigma)^3\mu^{-1}\|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,. \end{aligned} \tag{2.13}$$

Proof. Let $\rho = \varrho - \varrho(1-\beta)/12$ and $\rho' = \rho - \varrho(1-\beta)/4$. Then there exists a universal constant $R > 0$, such that the conditions (2.7) in Lemma 2.4 hold, whenever X belongs to the domain B, defined by $\|X - K\|_\varrho < R(\sigma/\gamma)^2$. Here, we have used that $\sigma < (1-\beta)/4$.

By Lemma 2.5, we have

$$\begin{aligned} \|(\mathbb{I}-\mathbb{E})\mathcal{R}(X)\|_\varrho &= \eta^{-1}\|\mathcal{T}_\mu^*(\mathbb{I}-\mathbb{E})\mathcal{U}_X^* X\|_\varrho \\ &\le 2\eta^{-2}\bigl(4e^\rho\mu\bigr)^\gamma\bigl[\|(\mathbb{I}-\mathbb{E})X\|_{\rho'} + \|\mathcal{U}_X^*X - X\|_{\rho'}\bigr]\,. \end{aligned} \tag{2.14}$$

Using the bound in (2.8) on the norm of $\mathcal{U}_X^*X - X$, together with the fact that $\mathbb{I}^-\mathbb{E} = 0$, we obtain the first inequality in (2.13). Similarly, Lemma 2.5 implies that

$$\|\mathbb{E}_k\mathcal{R}(X)\|_\varrho \le C_1\eta^{-2}\mu\bigl[\|\mathbb{E}_kX\|_{\rho'} + \|\mathbb{E}_k(\mathcal{U}_X^*X - X)\|_{\rho'}\bigr]\,, \tag{2.15}$$

for all $k \ge 1$. Here, and in what follows, $C_1, C_2, \ldots$ denote positive universal constants. Summing over $k \ge 1$ to get a bound on $\|(\mathbb{E}-\mathbb{P})\mathcal{R}(X)\|_\varrho$, and then adding (2.14), yields a bound analogous to (2.15), but with $\mathbb{E}_k$ replaced by $\mathbb{I}-\mathbb{P}$. Using again the bound in (2.8) on $\mathcal{U}_X^* X - X$, and the fact that $\mathbb{I}^-\mathbb{P}=0$, we obtain the second inequality in (2.13).

By Lemma 2.5, we also have a bound

$$\begin{aligned} \|\mathbb{E}\mathcal{R}(X)-\mathcal{R}(\mathbb{E}X)\|_\varrho &= \eta^{-1}\|\mathcal{T}_\mu^*\mathbb{E}(\mathcal{U}_X^*X-X)\|_\varrho \\ &\le 2\eta^{-2}\mu^{-1}\|\mathbb{E}(\mathcal{U}_X^*X-X)\|_{\rho'}\,. \end{aligned} \tag{2.16}$$

Using Lemma 2.4, the norm on the right hand side of (2.16) can be estimated as follows:

$$\|\mathbb{E}(\mathcal{U}_X^*X-X)\|_{\rho'} \le C_2(1-\beta)^{-3}(\gamma/\sigma)^3\|(\mathbb{I}-\mathbb{E})X\|_\rho^2 + \|\mathbb{E}[Z,X]\|_{\rho'}\,, \tag{2.17}$$

where $Z=\mathbb{I}^-Z$ is the vector field described in Lemma 2.4. Since $\mathbb{E}Z=0$, we have $\mathbb{E}[Z,\mathbb{E}X]=0$. As a result,

$$\begin{aligned} \|\mathbb{E}[Z,X]\|_{\rho'} = \|\mathbb{E}[Z,(\mathbb{I}-\mathbb{E})X]\|_{\rho'} &\le C_3(1-\beta)^{-1}\|Z\|_\rho\|(\mathbb{I}-\mathbb{E})X\|_\rho \\ &\le C_4(1-\beta)^{-1}(\gamma/\sigma)\|(\mathbb{I}-\mathbb{E})X\|_\rho^2\,. \end{aligned} \tag{2.18}$$

Here, we have used Proposition 2.1 and the bound on $\|Z\|_\rho$ from Lemma 2.4. Combining the last three equations yields the third inequality in (2.13).

In order to bound the inverse of $\mathcal{L}$, let Y be a vector field in $\mathbb{P}\mathcal{A}_\rho$. Then Y can be written as $Y(x,y)=(w,My+v)$, and the last inequality in (2.13) now follows from the fact that

$$\bigl(\mathcal{L}^{-1}Y\bigr)(x,y)=\eta(Tw,My+\mu v)\,. \tag{2.19}$$

Here, we have used that $T'=\mathrm{I}$, except (optionally) for the renormalization of purely Hamiltonian vector fields, where M and v are zero. **QED**

3. Iterated RG transformations

Let now $\mathcal{V}_0$ be a simple lattice in $\mathbb{R}^d$, such that the Brjuno condition (1.1) holds if the frequencies ν are chosen from $\mathcal{V}_0$. Using the same set of frequencies, define

$$a_n=\sum_{k=n}^{\infty}2^{n-k}\Bigl[2^{-k-\kappa}\ln(1/\Omega'_{k+\kappa})+(k+\kappa')^{-2}\Bigr]\,,\qquad \Omega'_n=\min_{0<|\nu_\perp|<2^n}|\nu_\parallel|\,, \tag{3.1}$$

for all positive integers n. Here $\kappa,\kappa'>2$ are two integer constants to be determined later. Then the Brjuno condition (1.1) is equivalent to the condition that the resulting sequence $\{a_n\}$ is summable. We remark that the weighted sum has been included in the definition (3.1) in order to limit the local growth of the sequence $\{a_n\}$. And the term $(k+\kappa')^{-2}$ has been included to avoid sequences $\{a_n\}$ that decrease too rapidly. This allows for a more uniform treatment of all Brjuno vectors.

Define $\lambda_0=1$ and

$$\lambda_n=2^{-n-\kappa}e^{-2^{n+\kappa}a_n}\,,\quad \eta_n=\frac{\lambda_n}{\lambda_{n-1}}\,,\quad A_n=\sum_{k=n}^{\infty}a_k\,,\quad \beta_n=\frac{A_{n+1}}{A_n}\,, \tag{3.2}$$

for all positive integers n. Consider the corresponding scaling transformations

$$P_n(x) = \lambda_n^{-1} x_\parallel + A_1^{-1} A_{n+1} x_\perp\,, \qquad T_n(x) = \eta_n^{-1} x_\parallel + \beta_n x_\perp\,. \tag{3.3}$$

Notice that $P_n = T_1 T_2 \cdots T_n$ by equation (3.2). These quantities will now be used to define the n-th step RG transformation $\mathcal{R} = \mathcal{R}_n$. To this end, we need to verify the assumptions made in Section 2. Clearly, $\beta = \beta_n$ is positive and less than one, since $n \mapsto A_n$ is a decreasing sequence. And the condition on $\eta = \eta_n$ in equation (2.9) follows from the fact that $a_n > a_{n-1}/2$ for $n > 1$, and that $\lambda_1 < 1/2$.

The geometric data $\mathcal{V}$, L and ℓ used in step n are

$$\mathcal{V}_{n-1} = P_{n-1}\mathcal{V}_0\,, \qquad L_{n-1} = A_1^{-1} A_n 2^{n+\kappa}\,, \qquad \ell_{n-1} = 2^{n+\kappa}\eta_n\,. \tag{3.4}$$

These definitions immediately imply (2.4). The following proposition shows that the condition (2.2) holds for all $v \in \mathcal{V}$.

Proposition 3.1. *If $v \in \mathcal{V}_{n-1}$ is nonzero, then either $|v_\parallel| \ge \ell_{n-1}$ or $|v_\perp| > L_{n-1}$.*

Proof. Assume that $v \in \mathcal{V}_{n-1}$ satisfies $0 < |v_\perp| \le L_{n-1}$. Then the corresponding lattice point $\nu = P_{n-1}^{-1} v$ in $\mathcal{V}_0$ satisfies $|\nu_\perp| \le A_1 A_n^{-1} L_{n-1} = 2^{n+\kappa}$, and thus $|\nu_\parallel| \ge \Omega'_{n+\kappa}$ by (3.1). Since we have $\lambda_n < 2^{-n-\kappa}\Omega'_{n+\kappa}$, this yields

$$|v_\parallel| = \lambda_{n-1}^{-1} |\nu_\parallel| \ge \eta_n \lambda_n^{-1} \Omega'_{n+\kappa} > \eta_n 2^{n+\kappa} = \ell_{n-1}\,, \tag{3.5}$$

as claimed. **QED**

The second condition in (2.9) is satisfied simply by choosing $\mu = \mu_n$, where

$$\mu_k = \exp\Bigl\{-\frac{\varrho}{6} \cdot \frac{1-\beta_k}{\gamma+1} L_{k-1}\Bigr\} = \exp\Bigl\{-\frac{\varrho}{6(\gamma+1)A_1} \cdot 2^{k+\kappa} a_k\Bigr\}\,, \qquad k \ge 1\,. \tag{3.6}$$

Finally, the third inequality in (2.9) is taken care of by choosing κ' and κ sufficiently large, as the following proposition shows.

Proposition 3.2. *For all $k \ge 1$, $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$. Furthermore, given $\gamma \ge 1$ and $C, N > 0$, if κ' and then κ are chosen sufficiently large, then for all $k \ge 1$,*

$$\mu_k \le C e^{-N 2^{k+\kappa} a_k}\,, \qquad \mu_k \le C \eta_k^N\,, \qquad \mu_k \le C(1-\beta_k)^N\,. \tag{3.7}$$

Proof. The inequality $\mu_{k+1} < \mu_k < \mu_{k+1}^{1/4}$ follows from the fact that $a_{k+1}/2 < a_k < 2a_{k+1}$. Let now $c = \varrho/(6(\gamma+1))$. By choosing κ and κ' sufficiently large, we have $c/A_1 \ge 2N$. Keeping κ' fixed, and increasing κ further, if necessary, we obtain the first two bounds in (3.7) by using that $2^{k+\kappa} a_k \ge 2^{k+\kappa}(k+\kappa')^{-1} \ge c' 2^\kappa k$, for some constant $c' > 0$. The same inequality, together with $1 - \beta_k = a_k/A_k > (k+\kappa')^{-2} 2N/c$, implies the third bound in (3.7). **QED**

Having verified all of the assumptions made in Section 2, we can now apply Theorem 2.6 to the n-th step RG transformation $\mathcal{R}_n$, defined by the parameters introduced above. Denote by $\mathcal{L}_n$ the corresponding linear operator from $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V}_{n-1})$ to $\mathbb{P}\mathcal{A}_\varrho(\mathcal{V}_n)$.

Define $\mathcal{A}_{\varrho,k} = \mathcal{A}_\varrho(\mathcal{V}_k)$, for all non-negative integers k. To simplify notation, the norm in $\mathcal{A}_{\varrho,k}$ and the projections $\mathbb{E}$ and $\mathbb{P}$ on this space will not be given indices. From Theorem 2.6 we immediately obtain

Theorem 3.3. *Let $\gamma \geq 1$. There exist constants $r, C > 0$, such that the following holds, for every positive integer n. Let B_{n-1} be the open ball in $\mathcal{A}_{\varrho,n-1}$ of radius $r\sigma_n^2$, centered at K, where $\sigma_n = \frac{1}{2}(1-\beta_n)\eta_n$. Then $\mathcal{R}_n$ is a bounded analytic map from B_{n-1} to $\mathcal{A}_{\varrho,n}$, satisfying $\|\mathcal{L}_n^{-1}\| \leq 1$ and*

$$
\begin{aligned}
\|(\mathbb{I}-\mathbb{E})\mathcal{R}_n(X)\|_\varrho &\leq C\sigma_n^{-3}\mu_n^\gamma \|(\mathbb{I}-\mathbb{E})X\|_\varrho\,,\\
\|(\mathbb{I}-\mathbb{P})\mathcal{R}_n(X)\|_\varrho &\leq C\sigma_n^{-3}\mu_n \|(\mathbb{I}-\mathbb{P})X\|_\varrho\,,\\
\|\mathbb{E}\mathcal{R}_n(X)-\mathcal{R}_n(\mathbb{E}X)\|_\varrho &\leq C\sigma_n^{-6}\mu_n^{-1} \|(\mathbb{I}-\mathbb{E})X\|_\varrho^2\,.
\end{aligned}
\tag{3.8}
$$

In what follows, a domain $\mathcal{D}_{n-1}$ for $\mathcal{R}_n$ is a subset of the ball B_{n-1} described in Theorem 3.3, that is open in $\mathcal{A}_{\varrho,n-1}$ and contains the vector field K. Given a domain $\mathcal{D}_{n-1}$ for each $\mathcal{R}_n$, the domain $\widetilde{\mathcal{D}}_n$ of the combined RG transformation $\widetilde{\mathcal{R}}_{n+1} = \mathcal{R}_{n+1} \circ \mathcal{R}_n \circ \ldots \circ \mathcal{R}_1$ is defined recursively as the set of all vector fields in the domain of $\widetilde{\mathcal{R}}_n$ that are mapped under $\widetilde{\mathcal{R}}_n$ into the domain $\mathcal{D}_n$ of $\mathcal{R}_{n+1}$. By Theorem 3.3, these domains are open and non-empty, and the transformations $\widetilde{\mathcal{R}}_n$ are analytic.

To prove the following result, we apply the stable manifold theorem for sequences of mappings, given in [17, Section 6].

Theorem 3.4. *Let $\gamma \geq 4$. If κ' and then κ are chosen sufficiently large, then there exist a sequence of domains $\mathcal{D}_0, \mathcal{D}_1, \ldots$ for the RG transformations $\mathcal{R}_1, \mathcal{R}_2, \ldots$, such that the set $\mathcal{W} = \cap_n \widetilde{\mathcal{D}}_n$ is the graph of an analytic function $W : (\mathbb{I}-\mathbb{P})\mathcal{D}_0 \to \mathbb{P}\mathcal{D}_0$, satisfying $W(0) = K$ and $DW(0) = 0$. For every $X \in \mathcal{W}$, if $n \geq 1$ and $\psi_n = \mu_1\mu_2\cdots\mu_n$, then*

$$
\begin{aligned}
\big\|\widetilde{\mathcal{R}}_n(X)-K_n\big\|_\varrho &\leq \psi_n^{1/2}\|(\mathbb{I}-\mathbb{P})X\|_\varrho\,,\\
\big\|\mathbb{P}[\widetilde{\mathcal{R}}_n(X)-K_n]\big\|_\varrho &\leq \psi_n\|(\mathbb{I}-\mathbb{P})X\|_\varrho^2\,,\\
\big\|(\mathbb{I}-\mathbb{E})\widetilde{\mathcal{R}}_n(X)\big\|_\varrho &\leq \psi_n^{\gamma-1/2}\|(\mathbb{I}-\mathbb{E})X\|_\varrho\,.
\end{aligned}
\tag{3.9}
$$

Proof. We start by rescaling the transformations $\mathcal{R}_n$. Let $r_n = r_{n-1}\sigma_{n+1}^2$ for every positive integer n, with $r_0 > 0$ smaller than half the constant r from Theorem 3.3.

Consider the transformations $R_1, R_2, \ldots$, given by the equation

$$
R_n(Z) = r_n^{-1}\big[\mathcal{R}_n(K + r_{n-1}Z) - K\big]\,, \qquad n = 1, 2, \ldots\,. \tag{3.10}
$$

The restriction $R_n\mathbb{P}$ defines a linear map from $\mathbb{P}\mathcal{A}_{\varrho,n-1}$ to $\mathbb{P}\mathcal{A}_{\varrho,n}$, which will be denoted by L_n. By Theorem 3.3, R_n is analytic and bounded on the ball $\|Z\|_\varrho < 2$, and satisfies

$$
\begin{aligned}
\|(\mathbb{I}-\mathbb{E})R_n(Z)\|_\varrho &\leq \varepsilon_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho\,,\\
\|(\mathbb{I}-\mathbb{P})R_n(Z)\|_\varrho &\leq \vartheta_n\|(\mathbb{I}-\mathbb{P})Z\|_\varrho\,,\\
\|\mathbb{P}R_n(Z)-R_n'(\mathbb{P}Z)\|_\varrho &\leq \varphi_n\|(\mathbb{I}-\mathbb{E})Z\|_\varrho^2\,,
\end{aligned}
\tag{3.11}
$$

where

$$\varepsilon_n = C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n^{\gamma}\,, \quad \vartheta_n = C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n\,, \quad \varphi_n = C\sigma_n^{-6}\sigma_{n+1}^{-2}\mu_n^{-1}\,. \tag{3.12}$$

Here, $C \geq 1$ is a constant that may depend on γ, but not on any other RG parameters. In addition, we have $\|L_n^{-1}\| < 1/4$. We will restrict R_n to the domain $D_{n-1} \subset \mathcal{A}_{\varrho,n-1}$, defined by

$$\|\mathbb{P}Z\|_{\varrho} < 1\,, \qquad \|(\mathbb{I}-\mathbb{P})Z\|_{\varrho} < 1\,, \qquad \|(\mathbb{I}-\mathbb{E})Z\|_{\varrho} < \delta_{n-1}\,, \tag{3.13}$$

where $\delta_{n-1} = (6\varphi_n)^{-1}$. By Proposition 3.2, if κ' and κ are chosen sufficiently large, then $C\sigma_n^{-3}\sigma_{n+1}^{-2}\mu_n^{1/2} \leq 1/6$ and $C\mu_n^{\gamma-3} \leq \sigma_{n+1}^{6}\sigma_{n+2}^{2}$, for all positive integers n. These inequalities imply

$$\varepsilon_n \leq \mu_n^{\gamma-1/2}/6\,, \qquad \vartheta_n \leq \mu_n^{1/2}/4\,, \qquad \varepsilon_n\delta_{n-1} \leq \delta_n\,, \tag{3.14}$$

for all $n \geq 1$. The hypotheses of Theorem 6.1 in [17] are now verified, with $\varepsilon = 1/6$ and $\vartheta = 1/4$, and the conclusions of this theorem imply the statements in Theorem 3.4. **QED**

We note that the "min" in equation (3.1) could be replaced by "a lower bound", as long as $n \mapsto \Omega'_n$ is a non-increasing sequence of positive real numbers, converging to zero, and the corresponding numbers a_n are summable. Our estimates are then uniform in the class $\mathcal{B}(\Omega')$ of vectors $\omega \in \mathbb{R}^d$ that admit the same sequence $n \mapsto \Omega'_n$ of lower bounds.

4. Invariant tori

Our construction of invariant tori follows closely the ideas used in [9,11,14,17].

Consider the RG transformation $\mathcal{R}$ defined in Section 2, and a vector field X in the domain of $\mathcal{R}$. If F is any map from D_0 into the domain of $\Lambda_X = \mathcal{U}_X \circ \mathcal{T}_\mu$, define

$$\mathcal{M}_X(F) = \Lambda_X \circ F \circ \mathcal{T}_\mu^{-1}\,. \tag{4.1}$$

Formally, if $\widetilde{\Gamma}$ is an invariant torus for $\mathcal{R}(X)$, then $\Gamma = \mathcal{M}_X(\widetilde{\Gamma})$ is an invariant torus for X. This can be seen easily from the identity $\Lambda_X \circ \Phi^{\eta t}_{\mathcal{R}(X)} = \Phi^t_X \circ \Lambda_X$. In order to make such identities more precise, we estimate the difference between the flow for X and the flow for the constant vector field $K = (\omega, 0)$.

Proposition 4.1. *Let τ be a positive real number and X a vector field in $\mathcal{A}_\varrho$, such that $\tau\|X-K\|_\varrho < r < \varrho$. Then for all times t in the interval $[-\tau,\tau]$,*

$$\|\Phi^t_X - \Phi^t_K\|_{\varrho-r} \leq \|t(X-K)\|_\varrho\,. \tag{4.2}$$

The proof of this proposition follows standard arguments, using the contraction mapping principle applied to the integral equation

$$Y(t_2) = Y(t_1) + \int_{t_1}^{t_2} [(X-K)\circ\Phi^t_K]\circ[\mathrm{I}+Y(t)]\,dt\,, \tag{4.3}$$

for the difference $Y(t) = \Phi_X^t - \Phi_K^t$. Notice that Φ_K^t is an isometry; the domain loss in Proposition 4.1 comes from the composition with $\mathrm{I} + Y(t)$, using Proposition 2.1.c.

Consider now a fixed but arbitrary vector field X on the stable manifold $\mathcal{W}$ described in Theorem 3.4. Let $X_0 = X$, and $X_n = \mathcal{R}_n(X_{n-1})$ for $n \geq 1$. In order to simplify notation, we will write $\mathcal{U}_k$ and $\mathcal{M}_{k+1}$ in place of $\mathcal{U}_{X_k}$ and $\mathcal{M}_{X_k}$, respectively. Our goal is to construct an appropriate sequence of functions $\Gamma_k \in \mathcal{A}_0(\mathcal{V}_k)$, satisfying

$$\Gamma_{n-1} = \mathcal{M}_n(\Gamma_n) = \Lambda_n \circ \Gamma_n \circ \mathcal{T}_{\mu_n}^{-1}\,, \qquad \Lambda_n = \mathcal{U}_{n-1} \circ \mathcal{T}_{\mu_n}\,, \tag{4.4}$$

for all $n > 0$. Then we will show that Γ_0 is an invariant torus for X_0.

For every $n \geq 0$, define $\mathcal{B}_n$ to be the vector space $\mathcal{A}_0(\mathcal{V}_n)$, equipped with the norm

$$\|f\|_n' = r_n^{-1}\|f\|_0 = r_n^{-1} \sum_{v \in \mathcal{V}_n} \|f_v\|\,, \qquad r_n = \psi_n^{1/3}\,, \tag{4.5}$$

where $\psi_0 = 1$. Denote by B_n the unit ball in $\mathcal{B}_n$, centered at the identity function I.

Proposition 4.2. *Let $\gamma \geq 5$. If κ' and then κ are chosen sufficiently large, then there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \geq 1$, the map $\mathcal{M}_n$ is well defined and analytic, as a function from B_n to $\mathcal{B}_{n-1}$. Furthermore, $\mathcal{M}_n$ takes values in $B_{n-1}/2$, and $\|D\mathcal{M}_n(F)\| \leq \mu_n^{1/4}$, for all $F \in B_n$.*

Proof. Clearly, $\mathcal{M}_n$ is well defined in some open neighborhood of I in $\mathcal{B}_n$, and

$$\mathcal{M}_n(F) = \mathrm{I} + g + (\mathcal{U}_{n-1} - \mathrm{I}) \circ (\mathrm{I} + g)\,, \qquad g = \mathcal{T}_{\mu_n} \circ f \circ \mathcal{T}_{\mu_n}^{-1}\,, \tag{4.6}$$

where $f = F - \mathrm{I}$. In order to estimate $\mathcal{U}_{n-1} - \mathrm{I}$, we can apply Lemma 2.4, with ρ' equal to $\varrho - \varrho(1 - \beta_n)/3$, as in the proof of Theorem 2.6. We will use Proposition 3.2 and assume that κ' and then κ have been chosen sufficiently large, without always mentioning it. By Lemma 2.4 and Theorem 3.4, there exist a constant $C > 0$, such that

$$\begin{aligned} \|\mathcal{U}_{n-1} - \mathrm{I}\|_{\rho'} &\leq C\sigma_n^{-1}\|\mathbb{I}^- X_{n-1}\|_\varrho \leq C\sigma_n^{-1}\psi_{n-1}^{\gamma-1/2}\|(\mathbb{I} - \mathbb{E})X\|_\varrho \\ &\leq \psi_{n-1}^{\gamma-1}\|(\mathbb{I} - \mathbb{E})X\|_\varrho \leq \psi_n^{3/4}\,, \end{aligned} \tag{4.7}$$

for all $n > 1$, and for all $X \in \mathcal{W} \cap B$, provided that the neighborhood B of K has been chosen sufficiently small (depending on κ' and κ). The first inequality in (4.7) and the final bound also hold for $n = 1$.

The composition with $\mathrm{I} + g$ in equation (4.6) is controlled by Proposition 2.1, using that $\|g\|_0 \leq \eta_n^{-1} r_n \|f\|_n'$ is less than $\varrho/2$. Here, and in what follows, we assume that $F \in B_n$. By using that $r_n/r_{n-1} = \mu_n^{1/3}$, we obtain $\|g\|_{n-1}' \leq \eta_n^{-1}\mu_n^{1/3} \leq \mu_n^{2/7}$. When combined with (4.7), this shows that $\mathcal{M}_{n-1}$ maps B_n into $B_{n-1}/2$. Using now $\rho' = \varrho/2$, we obtain a bound analogous to (4.7) for the derivative of $\mathcal{U}_{n-1}$. This, together with the fact that the inclusion map from B_n into B_{n-1} is bounded in norm by $\mu_n^{1/3}$, shows that $\|D\mathcal{M}_n(F)\| \leq \mu_n^{1/4}$, for all $n \geq 1$, and for all $F \in B_n$. **QED**

Denote by Φ_n and Φ_∞ the flows for the vector fields X_n and K, respectively. In order to prove that a solution to (4.4) yields an invariant torus Γ_0 for X, we will use the identity

$$\Phi_{n-1}^t \circ \mathcal{M}_n(F) \circ \Phi_\infty^{-t} = \mathcal{M}_n\bigl(\Phi_n^{\eta_n t} \circ F \circ \Phi_\infty^{-\eta_n t}\bigr)\,, \tag{4.8}$$

which follows from the relation described after (4.1), between the flow for a vector field and the flow for the corresponding renormalized vector field. This requires an estimate of the following type.

Proposition 4.3. *Under the same assumptions as in Proposition 4.2, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that for every $X \in \mathcal{W} \cap B$, and for every $n \ge 1$, the function $\Phi_n^s \circ F \circ \Phi_\infty^{-s}$ belongs to B_n, whenever $F \in B_n/2$ and $|s| \le \psi_n^{-1/6}$.*

Proof. We will use the identity

$$\Phi_n^s \circ F \circ \Phi_\infty^{-s} = \mathrm{I} + f \circ \Phi_\infty^{-s} + \bigl[\Phi_n^s \circ \Phi_\infty^{-s} - \mathrm{I}\bigr] \circ \bigl(\mathrm{I} + f \circ \Phi_\infty^{-s}\bigr)\,. \tag{4.9}$$

By Proposition 4.1 and Theorem 3.4, we have the bound

$$\bigl\|\Phi_n^s \circ \Phi_\infty^{-s} - \mathrm{I}\bigr\|_{\varrho/2} \le \|s(X_n - K)\|_\rho \le C\psi_n^{1/3}\|(\mathbb{I} - \mathbb{P})X\|_\varrho\,, \tag{4.10}.$$

provided e.g. that the right hand side of this inequality is less than $\varrho/2$. This is certainly the case, for any n, if $\|X - K\|_\varrho$ is sufficiently small. The composition by $\mathrm{I} + f \circ \Phi_\infty^{-s}$ in equation (4.9) is controlled in the same way as the composition by $\mathrm{I} + g$ in the proof of Proposition 4.2, using also that $\|f \circ \Phi_\infty^{-s}\|_0 = \|f\|_0$. As a result, the third term on the right hand side of (4.9) belongs to $\mathcal{B}_n$ and is bounded in norm by $C\|X - K\|_\varrho$, which is less than $1/2$ for any $n \ge 1$, if X is sufficiently close to K. **QED**

Now we are ready to construct invariant tori. A function f defined on $\mathcal{W}$ is said to be analytic if $f \circ W$ is analytic on the domain of W.

Theorem 4.4. *Under the same assumptions as in Proposition 4.2, there exists an open neighborhood B of K in $\mathcal{A}_\varrho$, such that the following holds. Given any $X \in \mathcal{W} \cap B$, and any sequence of functions $F_k \in B_k$, define*

$$\Gamma_{n,k} = \bigl(\mathcal{M}_{n+1} \circ \ldots \circ \mathcal{M}_k\bigr)(F_k)\,, \qquad 0 \le n < k\,. \tag{4.11}$$

Then the limits $\Gamma_n = \lim_{k\to\infty} \Gamma_{n,k}$ exist in $\mathcal{B}_n$, are independent of the choice of $F_0, F_1, \ldots$, and satisfy the identities (4.4). Furthermore, Γ_0 is an elliptic invariant torus for X, and the map $X \mapsto \Gamma_0$ is analytic and bounded on $\mathcal{W} \cap B$.

Proof. By Proposition 4.2 and Proposition 3.2, the map $\mathcal{M}_n : B_n \to B_{n-1}/2$ contracts distances by a factor of at least $1/2$. Thus, if $1 \le n < k < k'$, then the difference $\Gamma_{n,k'} - \Gamma_{n,k}$ is bounded in norm by 2^{n-k+1}. This shows that the sequence $k \mapsto \Gamma_{n,k}$ converges in $\mathcal{B}_n$ to a limit Γ_n, which is independent of the choice of the functions F_k. By choosing $F_k = \Gamma_k$ for all k, we obtain the identities (4.4). The analyticity of $X \mapsto \Gamma_0$ follows via chain rule from the analyticity of the maps used in our construction, and from uniform convergence.

In order to prove that Γ_0 is an invariant torus for X, we will use the identity (4.8). To be more precise, given a real number $-1 < t < 1$, define $t_n = \lambda_n t$ for all $n \geq 0$. By using that $\lambda_n \leq {\psi_n}^{-1/6}$, independently of n, if κ' and κ have been chosen sufficiently large (which we assume), Proposition 4.3 allows us to iterate (4.8), to get the identity

$$\Phi_0^t \circ \Gamma_{0,k} \circ \Phi_\infty^{-t} = \bigl(\mathcal{M}_1 \circ \ldots \circ \mathcal{M}_k\bigr)\bigl(\Phi_k^{t_k} \circ \Phi_\infty^{-t_k}\bigr)\,, \tag{4.12}$$

for all $k > 0$. As proved above, the right (and thus left) hand side of this equation converges in $\mathcal{A}_0$ to Γ_0. In addition, $\Gamma_{0,k} \to \Gamma_0$ in $\mathcal{A}_0$, and the convergence is pointwise as well, by part (a) of Proposition 2.1. Thus, since the flow Φ_0^t is continuous, we have $\Phi_0^t \circ \Gamma_0 \circ \Phi_\infty^{-t} = \Gamma_0$. This identity now extends to arbitrary $t \in \mathbb{R}$, due to the group property of the flow, and the fact that composition with Φ_∞^s is an isometry on $\mathcal{A}_0$.

Finally, notice that $\lambda_n \|DX_n\|_{\varrho/2}$ is an upper bound on the modulus of the Lyapunov exponent for the flow of $\lambda_n X_n$ on the range of Γ_n. Since X_0 is obtained from $\lambda_n X_n$ by a change of variables, and Γ_0 is the corresponding invariant torus for X_0, the same upper bound applies to the flow for X_0 on the torus Γ_0. But by Theorem 3.4, $\lambda_n \|DX_n\|_{\varrho/2} \to 0$ as $n \to \infty$. This shows that the torus Γ_0 is elliptic. **QED**

In what follows, the torus Γ_0 associated with a vector field $X \in \mathcal{W}$ will be denoted by Γ_X. For convenience, we extend the map $X \mapsto \Gamma_X$ to an open neighborhood of K, by setting $\Gamma_X = \Gamma_{X'}$, where $X' = (\mathbb{I} + W)(X - \mathbb{P}X)$.

Theorem 4.5. *Let $\rho > \varrho + \delta$ with $\delta > 0$. Under the same assumptions as in Proposition 4.2, there exists an open neighborhood B of K in $\mathcal{A}_\rho(\mathcal{V}_0)$, such that Γ_X has an analytic continuation to $\|\mathrm{Im}\, x\| < \delta$, for each $X \in B$. With this continuation, $X \mapsto \Gamma_X$ defines a bounded analytic map from B to $\mathcal{A}_\delta^0(\mathcal{V}_0)$.*

A proof of this theorem is completely analogous to the proof of Theorem 4.5 in [17]. Thus, we will just give a sketch here.

Consider the translations $R_u(x,y) = (x+u, y)$. By examining the construction of $\mathcal{W}$ and Γ_X, one verifies that for any $u \in \mathbb{R}^d$, the translated vector field $R_u^* X$ belongs to $\mathcal{W}$ whenever X does, and that

$$\Gamma_X(u,0) = \bigl(R_u \circ \Gamma_{R_u^* X}\bigr)(0,0)\,. \tag{4.13}$$

The idea now is to use the analyticity of map $X \mapsto \Gamma_X$, to extend the right hand side of equation (4.13) to the complex domain $\|\mathrm{Im}\, u\| < \delta$. This yields the desired analytic continuation of Γ_X. The remaining parts of Theorem 4.5 are proved by using that the right hand side of (4.13) is jointly analytic in X and u.

This theorem, together with Theorem 3.4, implies Theorem 1.1.

References

[1] D.F. Escande, F. Doveil, *Renormalisation Method for Computing the Threshold of the Large Scale Stochastic Instability in Two Degree of Freedom Hamiltonian Systems.* J. Stat. Phys. **26**, 257–284 (1981).

[2] S.J. Shenker, L.P. Kadanoff, *Critical behavior of a KAM surface. I. Empirical results.* J. Stat. Phys. **27**, 631–656 (1982).

[3] R.S. MacKay, *Renormalisation in Area Preserving Maps.* Thesis, Princeton (1982). World Scientific, London (1993).

[4] A. Stirnemann, *Towards an Existence Proof of MacKay's Fixed Point.* Comm. Math. Phys. **188**, 723–735 (1997).

[5] C. Chandre, M. Govin, H.R. Jauslin, *KAM–Renormalization Group Analysis of Stability in Hamiltonian Flows.* Phys. Rev. Lett. **79**, 3881–3884 (1997).

[6] J.J. Abad, H. Koch, and P. Wittwer, *A Renormalization Group for Hamiltonians: Numerical Results.* Nonlinearity **11**, 1185–1194 (1998).

[7] C. Chandre and H.R. Jauslin, *Renormalization–group analysis for the transition to chaos in Hamiltonian systems.* Physics Reports **365**, 1–64, (2002).

[8] H. Koch, *A renormalization group for Hamiltonians, with applications to KAM tori.* Erg. Theor. Dyn. Syst. **19**, 1–47 (1999).

[9] H. Koch, *On the renormalization of Hamiltonian flows, and critical invariant tori.* Discrete Contin. Dynam. Systems A, **8**, 633–646 (2002).

[10] H. Koch, *A Renormalization Group Fixed Point Associated with the Breakup of Golden Invariant Tori.* Discrete Contin. Dynam. Systems **11**, 881–909 (2004).

[11] H. Koch, *Existence of Critical Invariant Tori.* Erg. Theor. Dyn. Syst., to appear.

[12] D.G. Gaidashev, *Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori.* Discrete Contin. Dynam. Systems A **13**, 63–102 (2005).

[13] J. Lopes Dias, *Renormalisation scheme for vector fields on T^2 with a Diophantine frequency.* Nonlinearity **15**, 665–679, (2002).

[14] S. Kocić, *Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori.* Nonlinearity **18**, 2513–2544 (2005).

[15] K. Khanin, J. Lopes Dias, J. Marklof, *Multidimensional continued fractions, dynamic renormalization and KAM theory.* Preprint `mp_arc 05-304`, Commun. Math. Phys., to appear.

[16] H. Koch, *Renormalization of vector fields.* Preprint `mp_arc 06-89`, Fields Institute Communications, to appear.

[17] H. Koch, S. Kocić, *Renormalization of vector fields and Diophantine invariant tori.* Preprint `mp_arc 06-220`.

[18] H. Koch, J. Lopes Dias, *Renormalization of Diophantine skew flows, with applications to the reducibility problem.* Preprint `mp_arc 05-285` (2005).

[19] J. Lopes Dias, *Brjuno condition and renormalization for Poincaré flows.* Discrete Contin. Dynam. Systems, **15**, 641–656 (2006).

[20] J. Lopes Dias, *A normal form theorem for Brjuno skew-systems through renormalization.* J. Differential Equations, to appear.

[21] G. Gentile, V. Mastropietro, *Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications.* Rev. Math. Phys. **8**, 393–444 (1996)

[22] A.D. Brjuno, *Analytic form of differential equations. I.* Trudy Moskov. Mat. Obshch. **25**, 119–262 (1971). Trans. Moscow Math. Soc. **25**, 131–288 (1973).

[23] A.D. Brjuno, *Analytic form of differential equations. II.* Trudy Moskov. Mat. Obshch. **26**, 199–239 (1972). Trans. Moscow Math. Soc. **26**, 199–239 (1974).

[24] J. Pöschel, *Integrability of Hamiltonian systems on Cantor sets.* Comm. Pure Appl. Math. **35**, 653–696 (1989).

[25] H. Rüssmann, *On the frequencies of quasi periodic solutions of analytic nearly integrable Hamiltonian systems.* In "Seminar on Dynamical Systems", Euler Int. Math. Inst. St. Petersburg 1991, S. Kuksin, V. Lazutkin, J. Pöschel (eds). PNLDE **12**, 160–183, Birkhäuser Verlag (1994).

[26] J. Ecalle, B. Valet, *Bruno correction and linearization of resonant vector fields and diffeomorphisms.* Math. Z. **229**, 249–318 (1998).

[27] A. Berretti, G. Gentile, *Bryuno function and the standard map.* Commun. Math. Phys **220**, 623–656 (2001).

[28] H. Rüssmann, *Invariant tori in non-degenerate nearly integrable Hamiltonian systems.* Regul. Chaotic Dynam. **6**, 119–204 (2001).

[29] G. Gentile, *Degenerate lower-dimensional tori under the Bryuno condition.* Preprint `mp_arc 05-256`.

[31] G. Gentile, M.V. Bartuccelli, J.H.B. Deane, *Quasi-periodic attractors, Borel summability and the Bryuno condition for strongly dissipative systems.* Preprint `mp_arc 06-110`.

[32] J.C. Lagarias, *Geodesic multidimensional continued fractions*, Proc. London Math. Soc. **69**, 464–488 (1994).

[33] D.Y. Kleinbock, G.A. Margulis, *Flows on homogeneous spaces and Diophantine approximation on manifolds.* Ann. of Math. (2) **148**, 339–360 (1998).