On the Renormalization of Hamiltonian Flows, and Critical Invariant Tori

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Abstract. We analyze a renormalization group transformation $\mathcal{R}$ for partially analytic Hamiltonians, with emphasis on what seems to be needed for the construction of non-integrable fixed points. Under certain assumptions, which are supported by numerical data in the golden mean case, we prove that such a fixed point has a critical invariant torus. The proof is constructive and can be used for numerical computations. We also relate $\mathcal{R}$ to a renormalization group transformation for commuting maps.

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1. Introduction

Renormalization group methods for Hamiltonian flows were introduced first in [1,2], as a tool for explaining the breakup of certain smooth invariant tori. Our aim here is to make these methods precise, in a domain where the breakup is expected to occur, and to obtain information about invariant tori in the borderline (critical) case.

We are particularly interested in invariant tori for rotation vectors $\omega \in \mathbb{R}^d$ whose component ratios span an algebraic number field of degree d. In this case, there exists an integer $d \times d$ matrix T, with determinant ± 1 and $d-1$ simple eigenvalues of modulus less than 1, for which ω is an eigenvector with a real eigenvalue $\vartheta_1 > 1$. Given such a matrix T, its transpose T^*, and a nonzero real number μ, we define $\mathcal{T}_\mu(q,p) = \bigl(Tq, \mu(T^*)^{-1}p\bigr)$. Here, and in what follows, (q,p) denotes a point in the cotangent space of the torus $\mathbb{T}^d$ or its complexification.

A renormalization group (RG) transformation associated with T will be a transformation $\mathcal{R}$, acting on a space of Hamiltonians $H = H(q,p)$, such that

$$\mathcal{R}(H) = H \circ \mathcal{T}_1 \pmod{\mathcal{G}}\,. \tag{1.1}$$

Here, $\mathcal{G}$ is some group (ideally) of transformations that preserves rotation vectors, up to multiplication by a scalar. This includes coordinate changes, $H \mapsto H \circ \mathcal{U}$, with $\mathcal{U}$ canonical (symplectic) and homotopic to the identity. In addition, we will allow a scaling of the action variables, $H \mapsto \mu^{-1} H(\,.\,, \mu\,.\,)$, and a re-normalization of the energy, $H \mapsto \tau H - \epsilon$. This leads to the following general form of $\mathcal{R}$:

$$\mathcal{R}(H) = H'' \circ \mathcal{U}_{H''}\,, \quad H'' = H' \circ U'\,, \quad H' = \frac{\tau}{\mu} H \circ \mathcal{T}_\mu - \epsilon\,, \tag{1.2}$$

where τ, μ and ϵ are allowed to depend on H. In what follows, ϵ is taken to be zero. U' is some fixed canonical transformation, introduced for convenience later on. The canonical transformation $\mathcal{U}_{H''}$ will be described below. It is similar to the one introduced in [11], but differs significantly from those used in earlier renormalization schemes [1–6]. The problem that needs to be dealt with is that the map $H \mapsto H'$ involves a loss of regularity in the direction $(\omega, 0)$, due to the fact that $\mathcal{T}_\mu$ expands this direction. (All other directions are contracted, if $\vartheta_1|\mu| < 1$.)

Denote by $\partial_3 H$ the directional derivative of H in the direction $(0, \omega')$, where ω' is the eigenvector of T^* for the eigenvalue ϑ_1, normalized such that $\omega \cdot \omega' = 1$. In this paper, we will only consider Hamiltonians H that satisfy $\partial_3 H = 1$. In other words, $\omega' \cdot q$ can be regarded as "time". Consequently, we need to choose $\tau = \vartheta_1$ in equation (1.2), which ensures that $\partial_3 H' = 1$. Furthermore, in order for the renormalized Hamiltonian $\mathcal{R}(H)$ to satisfy the same condition, we will restrict the choice of U' and $\mathcal{U}_{H''}$ to canonical transformations that leave $\omega' \cdot q$ invariant.

Before defining $\mathcal{R}$ more precisely, let us mention a connection between the transformation (1.2), and the RG transformations for commuting maps, considered e.g. in [13,14] for the golden mean case, where $d = 2$, and $T = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ as a possible choice. Let H be a fixed but arbitrary Hamiltonian, and denote by $t \mapsto \Phi_t$ the flow associated with H. If x is any vector in $\mathbb{R}^d$, define $V(x)$ to be the translation $(q,p) \mapsto (q-x,p)$. Denote by δ_k the unit vector in $\mathbb{R}^d$ in the direction of the k–th coordinate axis, and consider the maps

$$\Lambda = \mathcal{T}_\mu \circ U' \circ \mathcal{U}_{H''}\,, \qquad F_k = \Phi_{2\pi\omega'_k} \circ V(2\pi\delta_k)\,, \qquad k=1,\ldots,d\,. \tag{1.3}$$

Proposition 1.1. *Assume that H is differentiable on $\mathbb{T}^d \times \mathbb{R}^d$, and that Λ is a diffeomorphism of this space. Then the maps $F_1, F_2, \ldots, F_d$ commute with each other, leave the (symplectic) manifold $\mathcal{M}_H = \bigl\{(q,p) \in H^{-1}(0)\colon \omega' \cdot q = 0\bigr\}$ invariant, and their restriction to $\mathcal{M}_H$ is symplectic. If we denote by $\widetilde{F}_1, \widetilde{F}_2, \ldots, \widetilde{F}_d$ the corresponding maps for $\widetilde{H} = \mathcal{R}(H)$, then*

$$\widetilde{F}_k = \Lambda^{-1} \circ F_1^{T_{1,k}} \circ F_2^{T_{2,k}} \circ \ldots \circ F_d^{T_{d,k}} \circ \Lambda\,, \quad k=1,\ldots d, \tag{1.4}$$

where $T_{j,k}$ are the entries of the matrix T.

Assume now that T has a real eigenvalue ϑ_2 of modulus less than 1. Let Ω and Ω' be eigenvectors of T and T^*, respectively, for the eigenvalue ϑ_2, normalized such that $\Omega\cdot\Omega' = 1$. In what follows, we will restrict our attention to Hamiltonians that are invariant under the reflection $(q,p) \mapsto (-q,p)$, and to canonical transformations U' and $\mathcal{U}_{H''}$ that commute with this reflection. In addition, we impose the following

Restriction. In the remaining part of this paper, every function on $\mathbb{T}^d \times \mathbb{R}^d$ being considered is assumed to depend only on the variables q, $\omega \cdot p$ and $\Omega \cdot p$. Thus, our phase space is effectively $\mathbb{T}^d \times \mathbb{R}^2$.

Let $H_0(q,p) = \omega \cdot p$. Then all of our Hamiltonians are of the form

$$H = H_0 + h\,, \quad h(q,p) = \sum_{(\nu,k)\in I} h_{\nu,k} \cos(\nu \cdot q)(\Omega \cdot p)^k\,, \tag{1.5}$$

where $I = \mathbb{Z}^d \times \mathbb{N}$. We define the *resonant* part of such a Hamiltonian H (or h) to be the function $\mathbb{I}^{+}H$ (or $\mathbb{I}^{+}h$), which is obtained by restricting the sum in (1.5) to the index set

$$I^{+} = \bigl\{(\nu,k) \in I : |\omega \cdot \nu| \le \sigma|\Omega \cdot \nu| \text{ or } |\omega \cdot \nu| \le \kappa k\bigr\}\,, \tag{1.6}$$

where σ and κ are fixed but arbitrary positive real numbers. Notice that a Hamiltonian that depends on p only is resonant, in the sense that it agrees with its resonant part.

For every pair $\rho = (\rho_1, \rho_2)$ of positive real numbers, denote by $\mathcal{B}_\rho$ the Banach space of all functions h of the form (1.5), which have finite norm

$$\|h\|_\rho = \sum_{(\nu,k)\in I} |h_{\nu,k}| \cosh(\rho_1 \Omega \cdot \nu)\rho_2^k\,, \tag{1.7}$$

and define $\mathcal{A}_\rho$ to be the affine space $H_0 + \mathcal{B}_\rho$ with the metric given by the norm on $\mathcal{B}_\rho$. The functions $H \in \mathcal{A}_\rho$ are continuous on the subset of $\mathbb{R}^{d+2}$ characterized by $|\Omega \cdot p| < \rho_2$, and $H(q+s\Omega,p)$ can be analytically continued to $|\mathrm{Im}\ s|<\rho_1$ and $|\Omega\cdot p|<\rho_2$. This domain will be denoted by D_ρ.

Remarks.

∘ Our choice of $\mathbb{I}^+$ and norms is different from the one used previously [10–12]. It was motivated by a study [9] of Hamiltonians in $d=2$, near an approximate fixed point of the golden mean RG. The current choice leads to significantly improved estimates (close to optimal for near-resonant Hamiltonians) and at the same time, allows for Hamiltonians that are non-analytic in $d-1$ directions.

∘ One of the important properties of the projection $\mathbb{I}^+$ is that it yields analyticity in the time direction $(\omega,0)$: If the sum in (1.7) is restricted to I^+, then for sufficiently small $\varepsilon>0$, an additional factor $\exp|\varepsilon\omega\cdot\nu|$ can be included in the sum, and compensated for (up to a constant factor) by decreasing the values of ρ_1 and ρ_2 slightly (by some minimal amount of order ε). In the case $d=2$, this implies that resonant Hamiltonians in $\mathcal{A}_\rho$ are analytic. Another important property of $\mathbb{I}^+$ is described in Proposition 2.2.

Proposition 1.2. *Assume $0<\rho_1'<\rho_1/|\vartheta_2|$ and $0<\rho_2'<\rho_2|\vartheta_2/\mu|$. Then the map $H\mapsto H'$ defined in (1.2) is continuous from $\mathcal{A}_\rho$ to $\mathcal{A}_{\rho'}$. In the case $d=2$, the restriction of this map to $\mathcal{A}_\rho^+=\mathbb{I}^+\mathcal{A}_\rho$ is compact.*

The proof of this proposition is straightforward. In particular, the compactness for $d=2$ is due to the abovementioned regularizing property of $\mathbb{I}^+$. This property also suggests that we try to choose the canonical transformation $\mathcal{U}_{H''}$, which appears in the definition (1.2) of $\mathcal{R}$, in such a way that

$$\mathbb{I}^-\bigl(H''\circ\mathcal{U}_{H''}\bigr)=0\,, \tag{1.8}$$

where $\mathbb{I}^-$ is the projection onto nonresonant Hamiltonians: $\mathbb{I}^-H=\mathbb{I}^-h=h-\mathbb{I}^+h$. In order to preserve the form (1.5) of a Hamiltonian under this change of coordinates, we also require that $\mathcal{U}_{H''}$ be admissible, in the following sense. A map U will be called *admissible* if

$$U=\mathrm{I}+u\,,\quad u(q,p)=\bigl(u_2(q,p)\Omega,u_3(q,p)\omega'+u_4(q,p)\Omega'\bigr)\,, \tag{1.9}$$

with u_2 odd and u_3,u_4 even functions of q. Unless defined for $p=0$ only, an admissible map $\mathrm{I}+u$ is also required to satisfy $\partial_3 u=0$.

Denote by $\mathcal{B}_\rho'$ the space of all functions h in $\mathcal{B}_\rho$ whose derivatives in the directions $(\Omega,0)$ and $(0,\Omega')$ belong to $\mathcal{C}_\rho$ and $\mathcal{B}_\rho$, respectively, where $\mathcal{C}_\rho$ is the analogue of $\mathcal{B}_\rho$ for functions that are odd in q; see also Section 2. These directional derivatives will be denoted by $\partial_2 h$ and $\partial_4 h$, respectively. As the norm of a function h in $\mathcal{B}_\rho'$ we take the larger of the two numbers

$$\|h\|_\rho'=\sup_{\substack{f\in\mathcal{C}_\rho:\ \kappa\rho_2\|f\|_\rho\le 1\\ g\in\mathcal{B}_\rho:\ \sigma\|g\|_\rho\le 1}}\|f\partial_2 h-g\partial_4 h\|_\rho \tag{1.10}$$

and $\|h\|_\rho/a$, for some fixed but arbitrary $a>0$. The corresponding affine space $H_0+\mathcal{B}_\rho'$ will be denoted by $\mathcal{A}_\rho'$.

Theorem 1.3. *Let $\rho_1'',\rho_2''>0$ and $0<A<1$. Consider the open ball in $\mathcal{A}_{\rho''}'$ of radius A, centered at H_0, and denote by $B(\epsilon)$ the set of all functions H'' in this ball such that*

$\|\mathbb{I}^{-}H''\|_{\rho''} < \epsilon^3$. *Then for sufficiently small* $\epsilon > 0$, *there exists* $\varrho = \rho'' - \mathcal{O}(\epsilon)$, *and for every* $H'' \in B(\epsilon)$ *an admissible canonical transformation* $\mathcal{U}_{H''} : D_\varrho \to D_{\rho''}$ *that is analytic and solves equation (1.8). If* H'' *takes real values for real arguments, then so does* $\mathcal{U}_{H''}$. *The map* $H'' \mapsto H'' \circ \mathcal{U}_{H''}$ *is analytic from* $B(\epsilon)$ *to* $\mathcal{A}_\varrho^+$, *and*

$$\|H'' \circ \mathcal{U}_{H''} - H''\|_\varrho \leq (1-A)^{-1}\|\mathbb{I}^{-}H''\|_{\rho''} + \mathcal{O}\big(\|\mathbb{I}^{-}H''\|_{\rho''}^2\big) . \tag{1.11}$$

A proof of this theorem is given in Section 2, for a more general class of quasiperiodic Hamiltonians. The solution of (1.8) is based on a Nash-Moser type iteration, and the convergence of this iteration imposes an upper bound on the size of ϵ, as a function of the parameters σ, κ, a and ρ''. Aside from the condition $A < 1$, which appears naturally, Theorem 1.3 imposes no other constraints on these parameters.

By combining Theorem 1.3 with Proposition 1.2, we immediately get the following.

Corollary 1.4. *Assume* $0 < \rho_1 < \rho_1'' < \rho_1' < \rho_1/|\vartheta_2|$ *and* $0 < \rho_2 < \rho_2'' < \rho_2' < \rho_2|\vartheta_2/\mu|$. *Let* $U' : D_{\rho''} \to D_{\rho'}$ *be an admissible canonical transformation which is analytic in a complex open neighborhood of* $D_{\rho''}$, *maps real points to real points, and defines a bounded linear operator* $H' \mapsto H' \circ U'$ *from* $\mathcal{A}_{\rho'}$ *to* $\mathcal{A}_{\rho''}$. *Let* H_1 *be Hamiltonian in* $\mathcal{A}_\rho$ *such that* $H_1'' = \mu^{-1}\vartheta_1 H_1 \circ \mathcal{T}_\mu \circ U'$ *lies in within some fixed distance* $A < 1$ *of* H_0 *in* $\mathcal{A}'_{\rho''}$. *If* $\|\mathbb{I}^{-}H_1''\|_\rho$ *is sufficiently small, then there exists an open neighborhood* B *of* H_1 *in* $\mathcal{A}_\rho$ *such that the transformation* $\mathcal{R}$, *given by equation (1.2), with* $H'' \mapsto \mathcal{U}_{H''}$ *as described in Theorem 1.3, is well defined and analytic as a map from* B *to* $\mathcal{A}_\rho^+$. *If* $d = 2$ *and* $\mathbb{I}^{-}H_1 = 0$, *then the restriction of* $\mathcal{R}$ *to* $\mathbb{I}^{+}B$ *is compact.*

As with RG transformations in general, one of the main goals is to find fixed points, and to study the action of $\mathcal{R}$ near these fixed points. The Hamiltonian H_0 is such a fixed point, with $\mathcal{U}_{H_0''} = U' = \mathrm{I}$. The corresponding RG analysis (for Hamiltonians that are not necessarily even in q or satisfy $\partial_3 H = 1$, but are analytic) yields results about invariant ω-tori [11] and sequences of closed orbits accumulating at these tori [12]. There are other integrable fixed points, such as $H^\circ(q,p) = \omega\cdot p + (\Omega\cdot p)^2$, with scaling $\mu = \vartheta_2^2/\vartheta_1$. But more interestingly, numerical results [1–10] suggest that there exist non-integrable fixed points as well, for $d = 2$, and maybe even for $d > 2$. The most extensively studied example is that of the golden mean. Numerically, the non-integrable fixed point in this case determines a scaling $\mu^* = 0.2304601966125\ldots$, which is one of the universal constants that can be observed during the breakup of golden invariant tori. We refer to [10] for more detail, including a discussion of the critical ω-torus.

In [9] we have started to investigate the golden mean version of $\mathcal{R}$, near an approximation for the expected non-integrable fixed point. A suitable domain for $\mathcal{R}$ was determined by first solving the fixed point problem for a numerical implementation of $\mathcal{R}$, using $U' = \mathrm{I}$. This yields an approximate fixed point $H_1 \in \mathcal{A}_\rho^+$, and an approximation ϕ for the generating function of the corresponding canonical transformation $\mathcal{U}_{H_1''}$. Then, for an RG analysis in a neighborhood of H_1, we choose U' to be the canonical transformation generated by ϕ. One of our observations so far is that Theorem 1.3 and Corollary 1.4 apply perfectly well to this situation.

In order to make the connection with observable phenomena, it is necessary to show that a (nontrivial) fixed point of $\mathcal{R}$ has a (critical) invariant ω-torus. We will address this question below and in Section 3. We say that H has a (unique) *invariant ω-torus* in D_ρ if there exists a (unique) continuous admissible map Γ from $\mathbb{T}^d \times \{0\}$ to D_ρ, such that

$$\Phi_t \circ \Gamma = \Gamma \circ \Phi_t^\circ\,, \qquad \Phi_t^\circ(q,0) = (q + t\omega, 0)\,, \tag{1.12}$$

for all $t \in \mathbb{R}$, where Φ denotes the flow for H.

Assume now that the transformation $\mathcal{R}$, as described in Corollary 1.4, has a (non-trivial) fixed point $H^* \in \mathcal{A}_\rho^+$, and denote by Λ^* the corresponding scaling map on D_ρ, as defined by equation (1.3). Both H^* and Λ^* are assumed to take real values when restricted to real arguments.

Theorem 1.5. *If the fixed point $H^* \in \mathcal{A}_\rho^+$ has a real-valued unique invariant ω-torus in D_ρ, then this torus Γ^* satisfies the equation*

$$\Gamma^* = \Lambda^* \circ \Gamma^* \circ \mathcal{T}_1^{-1}\,. \tag{1.13}$$

Conversely, let Γ^* be a continuous admissible map from $\mathbb{T}^d \times \{0\}$ to D_ρ, that satisfies the equation (1.13). Clearly, $\Gamma^*(0,0)$ is a fixed point of Λ^*. Assume that the derivative of Λ^* at this fixed point has exactly one non-contracting direction, and that $t \mapsto \Gamma^*(t\omega, 0)$ is continuously differentiable. Then Γ^* is an ω-torus for H^*.

A proof of this theorem, and of Proposition 1.1, is given in Section 4. In Section 3 we show that, under some additional assumptions, the equation (1.13) has a solution defining an invariant ω-torus for H^*. This solution is of class C^r, for some $r > 0$. If we assume also that H^* is nontrivial, in the sense that ϑ_2 is not an eigenvalue of $D\Lambda^*(\Gamma^*(0,0))$, then it is clear from equation (1.13) that the torus Γ^* cannot be of class C^1, i.e., that it is "critical". All these assumptions are supported by numerical results (and seem to be verifiable) in the golden mean case.

2. Near-resonant quasiperiodic Hamiltonians

In this section we prove Theorem 1.3, for a class of Hamiltonians that are not necessarily periodic in the variables $q_1, \ldots, q_d$.

It is convenient to perform a canonical change of coordinates $(q,p) = \mathcal{L}(x,y)$ with $x_1 = \omega' \cdot q$, $x_2 = \Omega' \cdot q$, …, and $y_1 = \omega \cdot p$, $y_2 = \Omega \cdot p$, …. In these coordinates, admissible maps like $\mathcal{U}_{H''}$ only depend on x and y_2, and only change x_2 and y. Since all but the coordinates (x_1, x_2) and (y_1, y_2) enter in a trivial way, we will only present the case $d = 2$ here. An extension to $d > 2$ is purely a matter of notation, and does not affect any estimate or result. We will simplify (abuse) notation further by writing $h(x,y)$ instead of $(h \circ \mathcal{L})(x,y)$, and $h \in \mathcal{B}_\rho$ instead of $h \circ \mathcal{L} \in \mathcal{B}_\rho$, etc.

In these new coordinates, the Fourier-Taylor series and norm of $h \in \mathcal{B}_\rho$ are given by

$$h(x,y) = \sum_{(v,k)\in I} h_{v,k} \cos(v \cdot x) y_2^k\,, \quad \|h\|_\rho = \sum_{(v,k)\in I} |h_{v,k}| \cosh(\rho_1 v_2) \rho_2^k\,, \tag{2.1}$$

where $I = \mathcal{V} \times \mathbb{N}$, and $\mathcal{V} \subset \mathbb{R}^2$ is an index set that can be determined from the vectors ω and Ω. But in what follows, $\mathcal{V} \subset \mathbb{R}^2$ is allowed to be any vector space over $\mathbb{Z}$. The set

I^+ defining the resonant projection $\mathbb{I}^+$ consists of all pairs $(v,k)\in I$ for which $|v_1|\le\sigma|v_2|$ or $|v_1|\le\kappa k$. If I is uncountable, then a function $h\in\mathcal{B}_\rho$ has (by definition) at most a countable number of nonzero coefficients $h_{v,k}$.

In addition to the spaces $\mathcal{B}_\rho$ of functions that are even in x, we will also need the corresponding spaces $\mathcal{C}_\rho$ of functions that are odd in x. The analogue of (2.1) for these spaces is obtained by replacing the cosine by the sine, and excluding terms with $v=0$ from the sums. Denote by $\mathcal{F}_\rho$ the direct sum of $\mathcal{B}_\rho$ and $\mathcal{C}_\rho$. The functions in $\mathcal{F}_\rho$ are defined in the complex domain D_ρ characterized by $\mathrm{Im}(x_1)=0$, $|\mathrm{Im}(x_2)|<\rho_1$, and $|y_2|<\rho_2$, where they are continuous, and analytic in x_2 and y.

We start by stating some basic properties of these spaces. In accordance with our previous definition, we denote by $\partial_k h$ the partial derivative of $u\mapsto h((u_1,u_2),(u_3,u_4))$ with respect to u_k, for $k=1,2,3,4$.

Proposition 2.1. *Consider functions $f,g\in\mathcal{F}_\rho$ and $h\in\mathcal{F}_{\rho'}$, and the change of variables $V(x,y)=((x_1,x_2+f(x,y)),(y_1,g(x,y)))$. Let $\alpha=\max\limits_{t\ge0} t[1-\tanh(t)]$.* $(=0.2784\ldots)$ *Then*

(i) $|f(x,y)|\le\|f\|_\rho$, *for all* $(x,y)\in D_\rho$.
(ii) $\|fg\|_\rho\le\|f\|_\rho\|g\|_\rho$.
(iii) $\|\partial_4 h\|_\rho\le(\rho_2'-\rho_2)^{-1}\|h\|_{\rho'}$, *if* $\rho_1'\ge\rho_1$ *and* $\rho_2'>\rho_2$.
(iv) $\|h\circ V\|_\rho\le(\rho_1'/\rho_1)^\alpha\|h\|_{\rho'}$, *if* $\rho_1'\ge\rho_1+\|f\|_\rho$ *and* $\rho_2'\ge\|g\|_\rho$.
(v) $\|h\|_\rho'\le s_0^{-1}(\rho_1'/\rho_1)^\alpha\|h\|_{\rho'}$, *if* $\rho_1'\ge\rho_1+s_0/(\kappa\rho_2)$ *and* $\rho_2'\ge\rho_2+s_0/\sigma$ *and* $s_0>0$.

Proof. The first three statements are straightforward to check. In order to prove (iv), consider first the case $h(x,y)=\cos(v\cdot x)y_2^k$, and let $X(x,y)=x$. Then

$$
\begin{aligned}
\|h\circ V\|_\rho &= \bigl\|\cos(v\cdot X+v_2 f)g^k\bigr\|_\rho\\
&\le \bigl\|\cos(v\cdot X)\cos(v_2 f)-\sin(v\cdot X)\sin(v_2 f)\bigr\|_\rho\|g\|_\rho^k\\
&\le \bigl[\cosh(\rho_1 v_2)\cosh(\|v_2 f\|_\rho)+\cosh(\rho_1 v_2)\sinh(\|v_2 f\|_\rho)\bigr]\bigl(\rho_2'\bigr)^k\\
&= \frac{\cosh(\rho_1 v_2)}{\cosh(\rho_1' v_2)}e^{|v_2|\|f\|_\rho}\cosh(\rho_1' v_2)\bigl(\rho_2'\bigr)^k\le(\rho_1'/\rho_1)^\alpha\|h\|_{\rho'}\,,
\end{aligned}
\tag{2.2}
$$

In the last inequality, we used that for $\lambda=\rho_1/\rho_1'$ and $u\ge0$,

$$
\frac{\cosh(\lambda u)}{\cosh(u)}\,e^{(1-\lambda)u}=\exp\left(\int_\lambda^1 tu\bigl[1-\tanh(tu)\bigr]\frac{dt}{t}\right)\le\lambda^{-\alpha}\,.
\tag{2.3}
$$

An analogous bound holds if $h(x,y)=\sin(v\cdot x)y_2^k$. Since we are dealing with weighted ℓ^1 spaces, the claim (iv) now follows by taking linear combinations and limits.

The bound (v) is obtained from (iv) by estimating the derivative in

$$
\|h\|_\rho'=\sup_{\substack{f\in\mathcal{C}_\rho:\ \kappa\rho_2\|f\|_\rho\le1\\ g\in\mathcal{B}_\rho:\ \sigma\|g\|_\rho\le1}}\left\|\frac{d}{ds}h\circ V_{s,f,g}\,\Big|_{s=0}\right\|_\rho,
\tag{2.4}
$$

where $V_{s,f,g}(x,y) = ((x_1, x_2 + sf(x,y)), (y_1, y_2 - sg(x,y)))$, using Cauchy's formula with contour $|s| = s_0$.

■

The following is one of the main motivations behind our definition of $\mathbb{I}^{\pm}$. Let $h \in \mathcal{B}'_\rho$. If $\psi = \mathbb{I}^{-}\psi$ is a Fourier-Taylor polynomial, then we can define

$$L(h)\psi = (\partial_2 h)\mathcal{D}_4\psi - (\partial_4 h)\mathcal{D}_2\psi\,, \qquad \mathcal{D}_k\psi = \partial_k\partial_1^{-1}\psi\,. \tag{2.5}$$

Proposition 2.2. *$\mathcal{D}_2$, $\mathcal{D}_4$, and $L(h)$ extend to bounded linear operators from $\mathbb{I}^{-}\mathcal{B}_\rho$ to $\mathbb{I}^{-}\mathcal{B}_\rho$, $\mathbb{I}^{-}\mathcal{C}_\rho$, and $\mathcal{B}_\rho$, respectively, with operator norms*

$$\|\mathcal{D}_2\| \le \sigma^{-1}\,, \quad \|\mathcal{D}_4\| \le (\kappa\rho_2)^{-1}\,, \quad \|L(h)\| \le \|h\|'_\rho\,. \tag{2.6}$$

Proof. Consider first $\psi(x,y) = \cos(v\cdot x)y_2^k$ with (v,k) belonging to the complement of I^{+} in I. By using that $v_2/v_1 \le \sigma^{-1}$ and $k/v_1 \le \kappa^{-1}$, we immediately get the bounds $\|\mathcal{D}_2\psi\|_\rho \le \sigma^{-1}\|\psi\|_\rho$ and $\|\mathcal{D}_2\psi\|_\rho \le (\kappa\rho_2)^{-1}\|\psi\|_\rho$. These bounds extend by linearity to all of $\mathbb{I}^{-}\mathcal{B}_\rho$, which proves the first two inequalities in (2.6). The third inequality now follows from the definition (1.10) of $\|h\|'_\rho$.

■

This allows us to solve the "linearized version" of equation (1.8). Denote by $\{H,\phi\}$ the Poisson bracket of H with ϕ.

Corollary 2.3. *Assume $H = H_0 + h$ belongs to $\mathcal{A}'_\rho$. If $\|h\|'_\rho \le A < 1$ then the equation*

$$\mathbb{I}^{-}\bigl(H + \{H,\phi\}\bigr) = 0\,, \qquad \mathbb{I}^{+}\phi = 0\,, \tag{2.7}$$

has a unique solution ϕ such that $\psi = \partial_1\phi$ belongs to $\mathbb{I}^{-}\mathcal{B}_\rho$, and

$$\|\psi\|_\rho \le (1-A)^{-1}\|\mathbb{I}^{-}H\|_\rho\,, \quad \|\{H,\phi\}\|_\rho \le (1-A)^{-1}\|\mathbb{I}^{-}H\|_\rho\,. \tag{2.8}$$

Proof. Let $L^{\pm} = \mathbb{I}^{\pm}L(h)\mathbb{I}^{-}$. By Proposition 2.2, the operator norm of $L^{\pm}$ is $\le A < 1$.

We can rewrite (2.7) in terms of $\psi = \partial_1\phi$ as $(\mathbb{I} - L^{-})\psi = \mathbb{I}^{-}h$, and this equation can be solved by inverting $\mathbb{I} - L^{-}$ by means of a Neumann series. This yields the first bound in (2.8). The second bound follows from the first, using the fact that $\{H,\phi\} = L(h)\psi - \psi = L^{+}\psi - \mathbb{I}^{-}h$.

■

Our next goal is to define a canonical transformation U_ψ such that $H \circ U_\psi = H + \{H,\phi\}+$ "terms that are quadratic or higher order in ϕ". Our generating functions ϕ are nonresonant, and thus do not depend on the variable y_1. The corresponding transformation $U_\psi : (x,y) \mapsto (x',y')$ is defined implicitly by the equations

$$\begin{aligned} x_1' &= x_1\,, & x_2' &= x_2 + [\mathcal{D}_4\psi](x,y')\,,\\ y_1 &= y_1' + \psi(x,y')\,, & y_2 &= y_2' + [\mathcal{D}_2\psi](x,y')\,. \end{aligned} \tag{2.9}$$

In fact, the only implicit equation that needs to be solved is the last one. We can write it as a fixed point equation for $g = y_2 - y_2'$ as follows:

$$g = \mathcal{K}(g)\,, \quad \bigl(\mathcal{K}(g)\bigr)(x,y) = \bigl[\mathcal{D}_2\psi\bigr]\bigl(x,(y_1, y_2 - g(x,y))\bigr)\,. \tag{2.10}$$

Proposition 2.4. *Let $0 < \sigma^{-1}\varepsilon^3 < \varepsilon r_2 < \rho_2$. Denote by B the closed ball in $\mathcal{B}_{\rho-\varepsilon r}$ of radius $\sigma^{-1}\varepsilon^3$, centered at the zero, with $r = (0, r_2)$. If $\psi \in \mathbb{I}^{-}\mathcal{B}_\rho$ satisfies $\|\psi\|_\rho \le \varepsilon^3$, then the equation (2.10) has a unique solution $g \in B$, and*

$$\|g\|_{\rho-\varepsilon r} \le \sigma^{-1}\|\psi\|_\rho\,. \tag{2.11}$$

Proof. By using Proposition 2.2, Proposition 2.1, and the mean value theorem, we find that for any $g, g' \in B$,

$$\begin{aligned} \|\mathcal{K}(g)\|_{\rho-\varepsilon r} &\le \|\mathcal{D}_2\psi\|_\rho \le \sigma^{-1}\|\psi\|_\rho \le \sigma^{-1}\varepsilon^3\,,\\ \|\mathcal{K}(g') - \mathcal{K}(g)\|_{\rho-\varepsilon r} &\le \|\partial_4\mathcal{D}_2\psi\|_{\rho-\varepsilon r}\|g'-g\|_{\rho-\varepsilon r} \le b\|g'-g\|_{\rho-\varepsilon r}\,, \end{aligned} \tag{2.12}$$

with $b = (\sigma\varepsilon r_2)^{-1}\|\psi\|_\rho < 1$. This shows that $\mathcal{K}$ is a contraction on B, and thus has a unique fixed point in B. The bound (2.11) follows from the first part of (2.12).

■

Now we are ready to compose a Hamiltonian H, satisfying

$$\|H - H_0\|_\rho \le b < b'\,, \quad \|H - H_0\|_\rho' \le A < A' < 1\,, \quad \bigl\|\mathbb{I}^{-}H\bigr\|_\rho \le (1-A')\varepsilon^3\,, \tag{2.13}$$

with the canonical transformation (2.9) that eliminates the nonresonant part $\mathbb{I}^{-}H$ to first order, as described by equation (2.7).

Theorem 2.5. *Assume $0 < \rho' + 2\varepsilon r < \rho$ component-wise, with $r_1 = \sigma r_2/(\kappa\rho_2')$ and $r_2 > \sigma^{-1}\varepsilon^2$. If $H \in \mathcal{A}_\rho'$ satisfies the bounds (2.13), then the abovementioned canonical transformation U_ψ maps $D_{\rho-2\varepsilon r}$ into D_ρ, and $\widetilde{H} = H \circ U_\psi$ belongs to $\mathcal{A}_{\rho-2\varepsilon r}$ and satisfies*

$$\begin{aligned} \bigl\|\widetilde{H} - H\bigr\|_{\rho-2\varepsilon r} &\le \Delta b(\varepsilon) = \bigl[1 + \varepsilon C_2(\varepsilon)\bigr]\varepsilon^3\,,\\ \bigl\|\widetilde{H} - H\bigr\|_{\rho-2\varepsilon r}' &\le \Delta A(\varepsilon) = (\sigma r_2)^{-1}(1 + \varepsilon r_1/\rho_1')^\alpha\bigl[1 + \varepsilon C_1(\varepsilon)\bigr]\varepsilon^2\,,\\ \bigl\|\mathbb{I}^{-}\widetilde{H}\bigr\|_{\rho-2\varepsilon r} &\le C_2(\varepsilon)\varepsilon^4\,, \end{aligned} \tag{2.14}$$

where C_1 and C_2 are two functions defined below.

Proof. Our assumptions guarantee that the function $\psi \in \mathcal{B}_\rho$ described in Corollary 2.3 exists and is bounded in norm by ε^3, and that Proposition 2.4 applies for this ψ, yielding $g \in \mathcal{B}_{\rho-\varepsilon r}$ of norm $\leq \sigma^{-1}\varepsilon^3$.

We will prove the bounds (2.14) by using the following one-parameter family that interpolates between $F(0) = H - H_0$ and $F(1) = \widetilde{H} - H_0$ in such a way that $F'(0) = \{H, \phi\}$:

$$
\begin{aligned}
F(s)(x,y) = &- s\big(\psi \circ V_s\big)(x,y) \\
&+ h\Big(\big(x_1, x_2 + \big(s[\mathcal{D}_4\psi] \circ V_s\big)(x,y)\big), \big(y_1, y_2 - \big(s[\mathcal{D}_2\psi] \circ V_s\big)(x,y)\big)\Big)\,,
\end{aligned}
\tag{2.15}
$$

where $V_s(x,y) = \big(x, (y_1, y_2 - sg(x,y))\big)$. Consider $s \in \mathbb{C}$ with $|s| \leq s_0 = n\sigma r_2 \varepsilon^{-2}$, where n can be either 1 or 2. By using Proposition 2.1 and Proposition 2.2, we have the bounds

$$
\begin{aligned}
\|sg\|_{\rho-n\varepsilon r} &\leq s_0 \|g\|_{\rho-\varepsilon r} \leq n\varepsilon r_2\,,\\
\|s[\mathcal{D}_2\psi] \circ V_s\|_{\rho-n\varepsilon r} &\leq \sigma^{-1} s_0 \|\psi\|_\rho \leq n\varepsilon r_2\,,\\
\|s[\mathcal{D}_4\psi] \circ V_s\|_{\rho-n\varepsilon r} &\leq (\kappa\rho_2')^{-1} s_0 \|\psi\|_\rho \leq n\varepsilon r_1\,.
\end{aligned}
\tag{2.16}
$$

Proposition 2.1 thus shows that $F(s)$ belongs to $\mathcal{A}_{\rho-n\varepsilon r}$ whenever $|s| \leq s_0$. Furthermore, $s \mapsto F(s)$ is analytic, not only in the interior of the disk $|s| \leq s_0$, but in an open neighborhood of it, since the norm of ψ is strictly less than ε^3, due to the fact that $A < A'$. Thus, using again Proposition 2.1, we have the bound

$$
\begin{aligned}
\big\|\widetilde{H} - H - \{H,\phi\}\big\|_{\rho-n\varepsilon r} &= \big\|F(1) - F(0) - F'(0)\big\|_{\rho-n\varepsilon r}\\
&= \left\| \frac{1}{2\pi i} \oint\limits_{|s|=s_0} \frac{ds}{s^2(s-1)} F(s) \right\|_{\rho-n\varepsilon r}\\
&\leq \frac{1}{s_0(s_0-1)} \Big[s_0\|\psi\|_\rho + \big(1 + n\varepsilon r_1/\rho_1'\big)^\alpha \|h\|_\rho \Big]\\
&\leq C_n(\varepsilon)\varepsilon^4\,,
\end{aligned}
\tag{2.17}
$$

where

$$
C_n(\varepsilon) = \frac{1}{n\sigma r_2(n\sigma r_2 - \varepsilon^2)} \Big[n\sigma r_2 \varepsilon + \big(1 + n\varepsilon r_1/\rho_1'\big)^\alpha b' \Big]\,.
\tag{2.18}
$$

The third inequality in (2.14) now follows from (2.17), together with the fact that $\mathbb{I}^-\widetilde{H} = \mathbb{I}^-(\widetilde{H} - H - \{H,\phi\})$. The first inequality in (2.14) is obtained by adding the bound on $\{H,\phi\}$ from Corollary 2.3, to the $n = 2$ version of the bound (2.17). An analogous inequality is obtained for $\|\widetilde{H} - H\|_{\rho-\varepsilon r}$ by using (2.17) with $n = 1$. When combined with part (*iv*) of Proposition 2.1 (taking $s_0 = \sigma r_2$) this yields the second inequality in (2.14). Finally, the fact that U_ψ maps $D_{\rho-2\varepsilon r}$ into D_ρ follows from the bounds (2.16) and part (*i*) of Proposition 2.1.

■

Proof of Theorem 1.3. To synchronize the notation of Theorem 1.3 with the one used in this section, we will replace ρ'' and ϵ by ρ and the third root of $(1-A')\varepsilon^3$, respectively. Let $0 < A < 1$ and $b = aA$, where a is the positive constant that was used to define the norm in $\mathcal{B}'_\rho$.

Choose $r_2 > 0$, $b' > b$, $A' > A$ less than 1, and $\rho' < \rho$. Our goal is to iterate the map $H \mapsto \widetilde{H}$, described in Theorem 2.5. To this end, it is useful to compare the third inequality in (2.14) with the third inequality in (2.13), by writing the former as

$$\|\mathbb{I}^{-}\widetilde{H}\|_{\rho-2\varepsilon r} \le (1-A')[f(\varepsilon)]^3, \qquad f(\varepsilon) = \bigl(\varepsilon C_2(\varepsilon)/(1-A')\bigr)^{1/3}\varepsilon. \tag{2.19}$$

Clearly, if $\varepsilon > 0$ is sufficiently small, then the sequence of numbers $\varepsilon_n = f^n(\varepsilon)$, for $n = 0, 1, 2\ldots$, converges to zero, and the sum $\sum 2\varepsilon_n r$ converges to a limit $\delta = \mathcal{O}(\varepsilon)$. The sums of the values $\Delta b(\varepsilon_n)$ and $\Delta A(\varepsilon_n)$, described in (2.14), converge as well, to some positive real numbers $S_b = \mathcal{O}(\varepsilon^3)$ and $S_A = \mathcal{O}(\varepsilon^2)$, respectively. Thus, if $\varepsilon > 0$ is sufficiently small, and H satisfies the bounds (2.13), then the map $(\varepsilon, \rho, H) \mapsto (f(\varepsilon), \rho - 2\varepsilon r, \widetilde{H})$ can be iterated indefinitely, and the iterates converge in $\mathbb{R}^3 \times \mathcal{B}_{\rho-\delta}$ to a limit $(0, \rho-\delta, \widehat{H})$. By construction, we have $\mathbb{I}^{-}\widehat{H} = 0$.

We note that Δb, ΔA, and f are all decreasing functions of ρ', and increasing functions of b' and A'. In practice, this means that the results δ, S_b, and S_A from one choice of parameters, can be used to obtain a new choice $(\rho', b', A') \to (\rho - \delta, b + S_b, A + S_A)$ that improves the bounds.

Denote by U_{ψ_n} the canonical transformation used at step n of our iteration, and by ϕ_n its generating function. Then the equation (2.9), together with the inequalities (2.16) and part (i) of Proposition 2.1, shows that $U_{\psi_n} - \mathrm{I}$ is uniformly bounded on $D_{\rho-\delta}$ by a constant times ε_n. Given that the ε_n's are summable, this implies that

$$\mathcal{U}_H = U_{\psi_1} \circ U_{\psi_2} \circ U_{\psi_3} \circ \ldots \tag{2.20}$$

converges uniformly on $D_{\rho-\delta}$, and that $\widehat{H} = H \circ \mathcal{U}_H$.

The analyticity of the map $H \mapsto \widehat{H}$ follows from the uniform convergence of our iteration schemes, including the ones used to solve the equations (2.7) and (2.10). The estimate (1.11) can be obtained e.g. by differentiating the map $F : H \mapsto \widehat{H} - H$ and checking that $DF(\mathbb{I}^{+}H)\mathbb{I}^{-}H = \{\mathbb{I}^{+}H, \phi_1\}$. The desired bound then follows from the second inequality in (2.8).

■

Remark. The results obtained in this section do not change if we include an additional factor $\exp|\delta v_1|$ in the definition of our norms (2.1). If $\delta > 0$, then the domain D_ρ includes complex values of x_1 with $|\mathrm{Im}(x_1)| < \delta$, and the functions in $\mathcal{F}_\rho$ are analytic in this domain. Concerning the transformation $\mathcal{R}$, the main change is a loss of analyticity in the step $H \mapsto H'$, where the domain shrinks to $|\mathrm{Im}(x_1)| < \vartheta_1^{-1}\delta$. But since the step $H'' \mapsto H'' \circ \mathcal{U}_{H''}$ generates analyticity in the variable x_1, as noted in the introduction, Corollary 1.4 remains true for small $\delta > 0$.

3. Construction of an invariant ω-torus

The discussion in this section is based on the assumption that the transformation $\mathcal{R}$, described in Corollary 1.4, has a (nontrivial) fixed point $H^* \in \mathcal{A}_\rho^+$. The function H^* and the corresponding canonical transformation $U^* = U' \circ \mathcal{U}_{(H^*)''}$ are assumed to take real values when restricted to real arguments. We will use the same notation as in the previous section. In particular, we set $d = 2$ to simplify notation; this does not affect any estimate or result.

Solving equation (1.13) for Γ^* is equivalent (via the relation $\Gamma^* = \mathrm{I} + \gamma^*$) to finding a fixed point γ^* for the transformation $\mathcal{N}$, defined by the equation

$$\mathcal{N}(\gamma) = \Lambda^* \circ \Gamma \circ \mathcal{T}_1^{-1} - \mathrm{I} = \mathcal{T}_\mu \circ \bigl[u^* \circ (\mathrm{I} + \gamma) + \gamma\bigr] \circ \mathcal{T}_1^{-1}, \tag{3.1}$$

where $\Lambda^* = \mathcal{T}_\mu \circ U^*$ and $u^* = U^* - \mathrm{I}$. This transformation is restricted to functions $\gamma = (0, \gamma_2, \gamma_3, \gamma_4)$ that are admissible, meaning that γ_2 is odd, and γ_2, γ_3 are even. We note that, due to the admissibility property $\partial_3 u^* = 0$, the fixed point equation $\gamma = \mathcal{N}(\gamma)$ can be solved for γ_2 and γ_4 first, independently of γ_3, and then γ_3 can be determined by iteration:

$$\gamma_3 = \sum_{n=1}^{\infty} (\mu/\vartheta_1)^n u_3^* \circ (\mathrm{I} + \gamma) \circ \mathcal{T}_1^{-n}. \tag{3.2}$$

The right hand side of this (formal) equation is independent of γ_3.

Given that u^* and γ have zero component in the expanding direction of $\mathcal{T}_\mu$, we expect $\mathcal{N}$ to be a contraction, if the derivative of u^* is not too large. To be more precise, we also have to take into account the composition with $\mathcal{T}_1^{-1}$. Since $\mathcal{T}_1^{-1}$ is expanding in the x_2 direction, we should restrict $\mathcal{N}$ to a space of functions $\gamma(x, 0)$ that are of class C^r in the variable x_2, with $r < 1$.

Given $r > 0$, denote by $\mathcal{B}_0$ the space of all functions f on $D_0 = \mathbb{R}^d \times \{0\}$, of the form

$$f(x, 0) = \sum_{v \in \mathcal{V}} f_v \cos(v \cdot x), \qquad \|f\|_0 = \sum_{v \in \mathcal{V}} |f_v| \bigl(1 + |v_2|\bigr)^r < \infty. \tag{3.3}$$

The corresponding space of odd functions will be denoted by $\mathcal{C}_0$, and $\mathcal{F}_0$ is defined to be the direct sum of $\mathcal{B}_0$ and $\mathcal{C}_0$. Let

$$c_\rho(s) = \sup_{t \ge 0} \frac{(1+t)^r}{\cosh(\rho_1 t)} e^{st}, \qquad |s| < \rho_1. \tag{3.4}$$

Proposition 3.1. *Consider functions $f, g \in \mathcal{F}_0$ and $h \in \mathcal{F}_\rho$, and the change of variables $V(x, 0) = ((x_1, x_2 + f(x, 0)), (0, g(x, 0)))$. Then*

(i) $|f(x, 0)| \le \|f\|_0$, *for all* $x \in \mathbb{R}^2$.
(ii) $\|f \circ \mathcal{T}_1^{-1}\|_0 \le |\vartheta_2|^{-r} \|f\|_0$.
(iii) $\|fg\|_0 \le \|f\|_0 \|g\|_0$.
(iv) $\|h \circ V\|_0 \le c_\rho(\|f\|_0) \|h\|_\rho$, *if* $\rho_1 \ge \|f\|_0$ *and* $\rho_2 \ge \|g\|_0$.

The proof of these inequalities is straightforward; see also the proof of Proposition 2.1.

If $u = (u_1, u_2, u_3, u_4)$ is a vector-valued function on $D_{\rho'}$ whose second and fourth components belong to $\mathcal{F}_{\rho'}$, and if $\rho < \rho'$ component-wise, then we define

$$\begin{aligned} \|u\|_{\rho'} &= \max\bigl\{ \|u_2\|_{\rho'}\,, b\|u_4\|_{\rho'} \bigr\}\,, \\ \|u\|'_{\rho} &= \max\bigl\{ \|\partial_2 u_2\|_{\rho} + b^{-1}\|\partial_4 u_2\|_{\rho}\,, b\|\partial_2 u_4\|_{\rho} + \|\partial_4 u_4\|_{\rho} \bigr\}\,, \end{aligned} \tag{3.5}$$

where b is some fixed positive constant. A seminorm $\|\gamma\|_0$ is defined analogously for vector-valued functions γ on D_0, whose second and fourth components belong to $\mathcal{F}_0$.

Lemma 3.2. *Let $\rho_0 = \min\{\rho_1, b\rho_2\}$ and $|\mu| < \vartheta_1|\vartheta_2|^r$. Assume that u_3^* belongs to $\mathcal{B}_\rho$, that $\|u^*\|'_\rho$ is finite, and that $\mathcal{N}$ has an admissible approximate fixed point γ' with seminorm $\|\gamma'\|_0$ less than $|\vartheta_2|^r\rho_0$. Choose a positive real number $R < |\vartheta_2|^r\rho_0 - \|\gamma'\|_0$, and define*

$$\varepsilon = \|\mathcal{N}(\gamma') - \gamma'\|_0\,, \qquad K = c_\rho(|\vartheta_2|^{-r}R)|\vartheta_2|^{-r}\|\Lambda^*\|'_\rho\,. \tag{3.6}$$

If $\varepsilon < (1-K)R$, then $\mathcal{N}$ has a unique admissible fixed point γ^* in $\mathcal{F}_0^4$ which satisfies $\|\gamma^* - \gamma'\|_0 \le R$.

Proof. As explained earlier, we consider first the restricted fixed point problem for $\pi\circ\mathcal{N}\circ i$ on $\mathcal{C}_0 \times \mathcal{B}_0$, where $\pi\gamma = (\gamma_2, \gamma_4)$. and $i(f,g) = (0,f,0,g)$. In order to simplify notation, we will omit π and i.

The restricted fixed point problem is solved by using the contraction mapping principle, on the closed ball B of radius R in $\mathcal{C}_0 \times \mathcal{B}_0$, centered at γ'. The constant K is an upper bound on the operator norm of the derivative $D\mathcal{N}(\gamma)$, for γ in B: From the right-composition with $\mathcal{T}_1^{-1}$ we get a factor $|\vartheta_2|^{-r}$, and the factor $c_\rho(|\vartheta_2|^{-r}R)\|\Lambda^*\|'_\rho$ is a bound on the tangent map of Λ^*; both are obtained directly from Proposition 3.1. Finally, the given inequalities involving R ensure that $\mathcal{N}$ maps the ball B into itself, and that $K < 1$. Thus, the restricted fixed point problem has a unique solution (γ_2^*, γ_4^*) in B.

The missing component γ_3^* is now given by the right hand side of equation (3.2), with $\gamma = i(\gamma_2^*, \gamma_4^*)$. The sum in this equation converges in $\mathcal{B}_0$, due to our assumption on μ; this follows from part (ii) of Proposition 3.1.

■

We note that the results obtained in this section do not change if we include an additional factor $\exp|\delta v_1|$ in the definition of our norms (2.1) and (3.3). This can be used to prove the differentiability assumption in Theorem 1.5 on the curve $t \mapsto \Gamma^*(t\omega, 0)$.

Using methods of the type described in [10], we have computed a numerical approximation H_1 (over 17000 Fourier-Taylor coefficients) for the expected non-integrable fixed point H^* for the golden mean version of $\mathcal{R}$, with $U' = \mathrm{I}$. The same computation also yields an approximation for the canonical transformation U^*. Then we iterated the transformation $\mathcal{N}$ a large number of times, starting with $\gamma = 0$, to obtain a numerical approximation γ' (over 4000 Fourier coefficients) for the invariant ω-torus for H^*. The parameters used in this computation were $\rho = (0.9, 0.15)$, $\sigma = 0.85001$, $\kappa = 2.5 * \sigma$, $b = 6.5$, and μ the approximate value of μ^* given in the introduction. Additional data can be found in [15]. Numerically, the assumptions of Lemma 3.2 are satisfied with a comfortable margin (e.g. $\|\gamma'\|_0 < 0.4$ and $K < 0.82$). We have used $r = \delta = 0$, but since the norm of a Fourier-Taylor polynomial is continuous in these parameters, the same holds for small $r, \delta > 0$.

4. Remaining proofs

Proof of Proposition 1.1. It should be noted that the functions considered here are not subject to the restrictions imposed after Proposition 1.1.

Clearly, the maps $\Phi_{2\pi\omega'_k}$ commute with each other and leave $H^{-1}(0)$ invariant. Thus, by the periodicity of H, the same holds for the maps F_k. In addition, since $q \cdot \omega'$ plays the role of time, its value changes by an amount $2\pi\omega'_k$ under the map $\Phi_{2\pi\omega'_k}$. The change under $V(2\pi\delta_k)$ is exactly the opposite: $-2\pi\delta_k \cdot \omega' = -2\pi\omega'_k$. This shows that F_k leaves $\mathcal{M}_H$ invariant.

Since there is a linear symplectic change of coordinates on $\mathbb{R}^{2d}$ which makes $t = q \cdot \omega'$ one of the coordinates, the Hamiltonian H admits a global "flow box" chart with the canonical symplectic form $dt \wedge dH + dq'_2 \wedge dp'_2 + \ldots dq'_d \wedge dp'_d$. Thus, given that the maps F_k are symplectic on $\mathbb{R}^{2d}$, and leave $t = H = 0$ invariant, their restriction to $\mathcal{M}_H$ are symplectic as well.

If f and g are differentiable functions on $\mathbb{T}^d \times \mathbb{R}^d$, denote by $\{f, g\}$ their Poisson bracket. By using the identity $\{\mu^{-1} f \circ \mathcal{T}_\mu, \mu^{-1} g \circ \mathcal{T}_\mu\} = \mu^{-1}\{f, g\} \circ \mathcal{T}_\mu$, we find that

$$\begin{aligned} \frac{d}{dt} f \circ \Phi_{\vartheta_1 t} \circ \Lambda \Big|_{t=0} &= \vartheta_1 \{f, H\} \circ \Lambda \\ &= \mu^{-1}\vartheta_1 \{f \circ \Lambda, H \circ \Lambda\} = \{f \circ \Lambda, \mathcal{R}(H)\}. \end{aligned} \tag{4.1}$$

This shows that the flow for $\widetilde{H} = \mathcal{R}(H)$ is given by $\widetilde{\Phi}_t = \Lambda^{-1} \circ \Phi_{\vartheta_1 t} \circ \Lambda$. Thus, we have

$$\begin{aligned} \widetilde{F}_k &= \widetilde{\Phi}_{2\pi\omega'_k} \circ V(2\pi\delta_k) = \Lambda^{-1} \circ \Phi_{2\pi\vartheta_1\omega'_k} \circ \Lambda \circ V(2\pi\delta_k) \\ &= \Lambda^{-1} \circ \Phi_{2\pi(T^*\omega')_k} \circ \Lambda \circ \Lambda^{-1} \circ V(2\pi T\delta_k) \circ \Lambda \\ &= \Lambda^{-1} \circ \Phi_{2\pi T_{1,k}\omega'_1} \circ \ldots \circ \Phi_{2\pi T_{d,k}\omega'_d} \circ V(2\pi T_{1,k}\delta_1) \circ \ldots \circ V(2\pi T_{d,k}\delta_d) \circ \Lambda \\ &= \Lambda^{-1} \circ F_1^{T_{1,k}} \circ \ldots \circ F_d^{T_{d,k}} \circ \Lambda \end{aligned} \tag{4.2}$$

as claimed.

■

Proof of Theorem 1.5. We note that, since H^* belongs to the domain of $\mathcal{R}$, the scaling transformation Λ^* maps D_ρ into itself. Recall that D_ρ is $d + 2$ dimensional: we only consider (functions of) the variables q, $\omega \cdot p$ and $\Omega \cdot p$.

Since $H^* \in \mathcal{A}_\rho^+$ is real analytic in the variable $\omega \cdot q$, so is the map Λ^*; see the remark at the end of Section 2. Thus, we can use equation (4.1) to obtain

$$\Lambda^* \circ \Phi_t^* = \Phi_{\vartheta_1 t}^* \circ \Lambda^* \tag{4.3}$$

on an open neighborhood of any fixed compact set in D_ρ, for times t in some open interval containing 0.

Assume now that H^* has a real-valued unique invariant ω-torus Γ^* in its domain. By restricting the identity (4.3) to this torus, and using that the left hand side is defined for all times t, we can extend it to all $t \in \mathbb{R}$. When combined with equation (1.12), this yields

$$\begin{aligned}\Lambda^* \circ \Gamma^* \circ \mathcal{T}_1^{-1} \circ \Phi_t^\circ &= \Lambda^* \circ \Gamma^* \circ \Phi^\circ_{\vartheta_1^{-1} t} \circ \mathcal{T}_1^{-1} \\ &= \Lambda^* \circ \Phi^*_{\vartheta_1^{-1} t} \circ \Gamma^* \circ \mathcal{T}_1^{-1} = \Phi_t^* \circ \Lambda^* \circ \Gamma^* \circ \mathcal{T}_1^{-1} .\end{aligned} \tag{4.4}$$

In other words, $\Gamma = \Lambda^* \circ \Gamma^* \circ \mathcal{T}_1^{-1}$ satisfies the equation (1.12) defining an invariant ω-torus. It is straightforward to check that Γ is admissible. Thus, if H^* has a unique invariant ω-torus, then we must have $\Gamma = \Gamma^*$. This proves the first part of the theorem.

Concerning the second part, notice that the unstable manifold of $\mathcal{T}_1$ (acting on $\mathbb{R}^d \times \{0\}$) at the fixed point $(0,0)$ is the orbit of this fixed point under the flow Φ°. Similarly, as equation (4.3) shows, the unstable manifold of Λ^* (acting on $\mathbb{R}^d \times \mathbb{R}^2$) at the fixed point $\Gamma^*(0,0)$ is the orbit of this fixed point under the flow Φ^*. Here, and in what follows, Γ^* is assumed to satisfy the conditions given after (1.13). The identity (1.13) implies that Γ^* maps the unstable manifold of $\mathcal{T}_1$ to the unstable manifold of Λ^*. Thus, for every real number t there exists a real number τ, such that for $q = 0$,

$$\Gamma^*\big(\Phi_t^\circ(q,0)\big) = \Phi_\tau^*\big(\Gamma^*(q,0)\big) . \tag{4.5}$$

By taking the dot product of each side of this equation with the vector $(\omega, 0)$, we find that $t = \tau$. The identity (4.5) with $t = \tau$ extends immediately from $q = 0$ to $q = s\omega$, for arbitrary real numbers s. Thus, we have $\Gamma^* \circ \Phi_t^\circ = \Phi_t^* \circ \Gamma^*$ when restricted to the orbit of $(0,0)$ under the flow Φ°. But since both sides of this equation are well defined as maps from the torus $\mathbb{T}^d \times \{0\}$ to $\mathbb{T}^d \times \mathbb{R}^2$, and are continuous, and since the orbit of $(0,0)$ under the flow Φ° is dense in this torus, the identity $\Gamma^* \circ \Phi_t^\circ = \Phi_t^* \circ \Gamma^*$ holds on all of $\mathbb{T}^d \times \{0\}$. This shows that Γ^* defines an invariant ω-torus for H^*, as claimed.

■

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