# Two novel methods and multi-mode periodic solutions for the Fermi-Pasta-Ulam model

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**Abstract.** We introduce two novel methods for studying periodic solutions of the FPU $\beta$-model, both numerically and rigorously. One is a variational approach, based on the dual formulation of the problem, and the other involves computer-assisted proofs. These methods are used e.g. to construct a new type of solutions, whose energy is spread among several modes, associated with closely spaced resonances.

## 1. Introduction

The Fermi-Pasta-Ulam model was introduced in [2] in order to study questions related to the principle of equipartition of energy in systems with many degrees of freedom. The model consists of $P$ particles whose dynamics is described by the equations

$$\ddot{q}_m = \phi'(q_{m+1} - q_m) - \phi'(q_m - q_{m-1}), \qquad m = 1, 2, \ldots, P,$$

where $\phi(x) = \frac{x^2}{2} + \alpha\frac{x^3}{3} + \beta\frac{x^4}{4}$.

Fermi, Pasta and Ulam investigated this model numerically, with the aim of establishing the rate of relaxation to equipartition of the distribution of energy among modes. Surprisingly, their numerical experiment yielded the opposite result. They observed that in a low energy regime the energy of the system remained confined among the original modes, instead of spreading towards all modes. This result motivated a large number of further numerical and analytical investigations; see e.g. [12–18] and references therein. A relatively recent review of the subject can be found in [13].

In this paper, we investigate time-periodic solutions for the FPU model with $\alpha = 0$, also known as the $\beta$-model. In particular, we construct a previously unknown type of solutions, associated with closely spaced resonances. This includes both numerical and rigorous results. Our analysis is restricted to periodic boundary conditions, but this covers solutions with free or zero boundary conditions as well, since such solutions can be extended periodically. It is based on two novel methods — one is a variational approach, based on the dual formulation of the problem, and the other involves computer-assisted proofs. They are used here in the context of the $\beta$-model, but it seems clear that they can be extended to a more general class of differential equations. The computer-assisted proof requires

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that the potential be analytic, and for the variational methods, it needs to be convex and coercive. In the FPU model, these conditions are satisfied in some cases where $\alpha$ and $\beta$ are both nonzero. See [4] for details concerning convexity methods.

Since the nonlinear part of the equation is homogeneous, its size relative to the linear part can be changed by rescaling $q$, and thus it suffices to consider the case $\beta = 1$. Similarly, the fundamental period $T$ of a non-constant solution can be normalized to $2\pi$ by a rescaling of time. This leads us to consider the equation

$$\omega^2 \ddot{q} = -\nabla^* \bigl[\nabla q + (\nabla q)^3\bigr] , \tag{1.1}$$

where $\omega = 2\pi/T$, and where $\nabla$ and $\nabla^*$ are the difference operators defined by

$$\bigl(\nabla q\bigr)_m(t) = q_{m+1}(t) - q_m(t) , \qquad \bigl(\nabla^* q\bigr)_m(t) = q_{m-1}(t) - q_m(t) .$$

Here, $m$ and $t$ range over the groups $\mathcal{P} = \mathbb{Z}/(P\mathbb{Z})$ and $S^1$, respectively; and $q$ is regarded as a function $(m,t) \mapsto q_m(t)$ defined on $\mathcal{P} \times S^1$, with values in $\mathbb{R}$.

The best known periodic solutions of the $\beta$-model are near $q = 0$. In this regime, the cubic term in (1.1) is small compared to $q$, and we have $L_\omega q \approx 0$, where

$$L_\omega q = -\omega^2 \ddot{q} - \nabla^* \nabla q . \tag{1.2}$$

The values of $\omega > 0$ for which $L_\omega$ has an eigenvalue zero, and the corresponding nonzero solutions of $L_\omega q = 0$, will be referred to as *resonant frequencies* and *normal modes*, respectively. A theorem by Lyapunov [1] implies the existence of a four-parameter family of solutions near $q = 0$, for each resonant frequency, that is tangent to the corresponding zero eigenspace of $L_\omega$ at $q = 0$. To be more precise, this theorem does not apply directly to the periodic FPU model, but due to a result of Conway and Jones [5], it does apply to the FPU model with zero boundary conditions, and by the observation made earlier, this leads to a branch of solutions for the periodic model. A characteristic property of these "Lyapunov solutions" is that most of their energy is concentrated on a single normal mode. Furthermore, their minimal period is close to that of the corresponding normal mode. A few of these families (up to 5, depending on the remainder of $P$ in the division by 3 and 4) are pure one-mode solutions, and they can be extended away from $q = 0$. These solutions, as well as some two-mode solutions, have been obtained in [13].

More general, but less detailed results have been obtained by variational methods [3–11], using the Lagrangean action

$$A_\omega(q) = \sum_{m\in\mathcal{P}} \int_{-\pi}^{\pi} \Bigl[\tfrac{1}{2}\bigl(\omega \dot{q}_m(t)\bigr)^2 - \phi\bigl(q_{m+1}(t) - q_m(t)\bigr)\Bigr]\, dt . \tag{1.3}$$

As was shown in [10], it is possible to prove that the equation (1.1) has periodic solutions for every $\omega > 0$, by applying general theorems from Critical Point Theory to the Lagrangean functional $A_\omega$. However, this functional is not easy to deal with, since it is not bounded from below or above. In addition, the most interesting periodic solutions turn out to have a large Morse index, which makes it hard to find them numerically.

One of our goals is to develop a technique for finding non-Lyapunov solutions for equation (1.1), that is, solutions whose energy is spread among several modes. To this end, we consider the dual formulation of the problem, on a space of odd functions (in $t$), and define a functional $\Phi_\omega$ that is bounded from above and from below. This functional is described in Section 2. Heuristic considerations indicate that for certain frequencies $\omega$, the maximizer of $\Phi_\omega$ should have most of its energy concentrated on a few resonant frequencies below $\omega$. We confirm this numerically for several of these maximizers, as well as for many other critical points of $\Phi_\omega$. Our findings are described in Section 3, and further numerical data can be found in the Appendix. This includes a detailed bifurcation diagram, for solutions with frequencies $\omega$ near 0.2. In addition, we give some rigorous results for 12 of these solution, confirming e.g. the observed distribution of energy among modes. To our knowledge, solutions with such properties were not known to exist before. Our proof is computer-assisted, and is described in Section 4. Unlike previous computer-assisted proofs with similar goals, our analysis is not based on integrating a system of ODEs in a finite dimensional space. Instead of points in $\mathbb{R}^{\mathcal{P}}$, our primary objects are analytic functions, and (1.1) is solved as an equation in function space. This approach yields much more information on the solution, while being conceptually quite simple; see Section 4. Section 5 is devoted to the proof of two Lemmas concerning solutions with near-resonant frequencies.

## 2. Variational Setting

The equation (1.1) can be rewritten in the form

$$L_\omega q = \nabla^*(\nabla q)^3 \,. \tag{2.1}$$

As mentioned in the introduction, we will relate solutions of this equations to critical points of a dual functional $\Phi_\omega$. Unlike the Lagrangean action (1.3), the new functional is bounded from above (and below). Moreover, for suitably chosen frequencies $\omega$, its absolute maximizers correspond to periodic solutions of (2.1), which exhibit a nontrivial distribution of energies among the modes.

We start by stating some basic properties of the operator $L_\omega$. In order to simplify notation, we identify real-valued functions on $\mathcal{P} \times S^1$ with $\mathbb{R}^P$-valued functions on $S^1$. Furthermore, we restrict our analysis to functions that are odd under time reflection $t \mapsto -t$. This restriction is not required by the dual variational methods that are used here, but it greatly simplifies the analysis, and the desired multi-mode solutions are found already in this case. Given any real number $r$, denote by $H_o^r$ the subspace of all functions in $H^r = H^r(S^1, \mathbb{R}^{\mathcal{P}})$ that are odd in $t$.

**Proposition 2.1.** *Let $\vartheta = 2\pi/P$. Equation (1.2) defines a self-adjoint linear operator $L_\omega$ on $H_o^r$, with domain $H_o^{r+2}$. It has eigenvalues*

$$-\mu_h^2 + \omega^2 k^2 \,, \qquad \text{where} \quad \mu_h = \sin(h\vartheta/2) \,, \quad h \in \mathcal{P} \,, \quad 0 \neq k \in \mathbb{Z} \,, \tag{2.2}$$

*and no other spectrum. Furthermore, as an operator from $H_o^{r+2}$ to $H_o^r$, $L_\omega$ is invertible whenever $\omega$ is such that none of the eigenvalues (2.2) is zero.*

**Proof.** This claim follows immediately from the spectral representation

$$\bigl(\mathcal{F}L_\omega q\bigr)(h,k) = \bigl(-\mu_h^2 + \omega^2 k^2\bigr)\bigl(\mathcal{F}q\bigr)(h,k)\,, \qquad h \in \mathcal{P}\,, \quad k \in \mathbb{Z}\,,$$

where $\mathcal{F}$ denotes the Fourier transform

$$\bigl(\mathcal{F}q\bigr)(h,k) = \frac{1}{\sqrt{2\pi P}} \sum_{m\in\mathcal{P}} e^{-imh\vartheta} \int_{-\pi}^{\pi} e^{-itk} q_m(t)\,dt\,. \tag{2.3}$$

**QED**

This proposition shows e.g. that a frequency $\omega_n > 0$ is resonant, that is, $L_\omega$ has an eigenvalue zero, if and only if

$$\omega_n k = \pm 2\sin(h\theta/2)\,, \tag{2.4}$$

for some nonzero $k \in \mathbb{Z}$ and $h \in \mathcal{P}$. In what follows, we will restrict our attention to nonresonant frequencies $\omega$, so that $L_\omega$ is invertible as a linear operator from $H_o^1$ to $H_o^{-1}$.

Denote by $L_o^r$ the subspace of functions in $L^r = L^r(\mathbb{R}^{\mathcal{P}} \times S^1)$ that are odd under time reflection $t \mapsto -t$. On the space $L_o^{4/3}$, consider the two functionals $U$ and $Q_\omega$,

$$U(p) = \frac{3}{4}\bigl\langle p^{4/3}, \mathbf{1}\bigr\rangle\,, \qquad Q_\omega(p) = \frac{1}{2}\bigl\langle \nabla L_\omega^{-1}\nabla^* p, p\bigr\rangle\,,$$

where $\mathbf{1}$ is the constant function with value 1, and where $\langle .,.\rangle$ denotes the product

$$\langle f,g\rangle = \sum_{m\in\mathcal{P}} \int_{-\pi}^{\pi} f_m(t) g_m(t)\,dt\,.$$

Since $H^1 \subset L^4$, and thus $L^{4/3} \subset H^{-1}$ by duality, these two functionals are well defined. In fact, they are of class $C^1$, as is straightforward to check. Define now

$$\Phi_\omega(p) = \frac{Q_\omega(p)}{U(p)^{3/2}}\,, \qquad \lambda_\omega(p) = \left(\frac{2U(p)}{3Q_\omega(p)}\right)^{3/2}\,. \tag{2.5}$$

**Theorem 2.2.** *Let $p \in L_o^{4/3}$, and assume that $Q_\omega(p) > 0$. Then $p$ is a critical point of $\Phi_\omega$ if and only if $q = \lambda_\omega(p) L_\omega^{-1}\nabla^* p$ is a solution of (2.1).*

**Proof.** Under the given assumption, the equation $D\Phi_\omega(p) = 0$ for a critical point $p$ of $\Phi_\omega$ can be written as

$$3Q_\omega(p) DU(p) = 2U(p) DQ_\omega(p)\,, \tag{2.6}$$

or by homogeneity, $DU(g) = DQ_\omega(g)$ for some nonzero constant multiple $g$ of $p$. This equation for $g$ is equivalent to

$$g^{1/3} = \nabla L_\omega^{-1} \nabla^* g\,. \tag{2.7}$$

The abovementioned constant is easily found to be $\lambda_\omega(p)$. Thus, if we take the third power on both sides of (2.7), we obtain equation (2.1) for $q = L_\omega^{-1}\nabla^* g$. **QED**

Notice that the solutions $q$ described in Theorem 2.2 belong to $H_o^1$, which, by equation (1.1), implies that they are smooth. They are of course also critical points of the Lagrangean (1.3). In addition, we have

$$A_\omega(q) = \tfrac{4}{27}\Phi_\omega(p)^{-2} \qquad \text{(at critical points).} \tag{2.8}$$

This can be seen as follows. Let $g$ be the function described in the proof of Theorem 2.2. Then $A_\omega(q) = \frac{1}{2}\langle L_\omega q, q\rangle - \frac{1}{4}\langle(\nabla q)^3, \nabla q\rangle = \frac{1}{2}\langle\nabla^* g, q\rangle - \frac{1}{4}\langle g, \nabla q\rangle = \frac{1}{2}Q_\omega(g)$. Now we can use that $(3/2)^{3/2}Q_\omega(g)^{1/2}\Phi_\omega(g) = \lambda_\omega(g)^{-1} = 1$, to obtain the identity (2.8).

The functional $\Phi_\omega$ is bounded from above and below, as the following shows.

**Proposition 2.3.** *If $\omega \in \mathbb{R}$ is nonresonant and $p \in L_o^{4/3}$ nonzero, then*

$$\bigl|\Phi_\omega(p)\bigr| \le \left(\frac{8b_\omega}{27\pi P}\right)^{1/2}, \qquad b_\omega = \sum_{h,k}\left[\frac{\mu_h^2}{-\mu_h^2+\omega^2k^2}\right]^2. \tag{2.9}$$

**Proof.** By using the representation

$$2Q_\omega(p) = \sum_{h,k}\frac{\mu_h^2}{-\mu_h^2+\omega^2k^2}\bigl|(\mathcal{F}p)(h,k)\bigr|^2\,,$$

together with the inequalities by Cauchy–Schwarz and Hausdorff–Young, we obtain

$$4Q_\omega(p)^2 \le b_\omega\sum_{h,k}\bigl|(\mathcal{F}p)(h,k)\bigr|^4 \le \frac{b_\omega}{2\pi P}\|p\|_{L^{4/3}}^4 = \frac{32b_\omega}{27\pi P}U(p)^3\,. \tag{2.10}$$

**QED**

Our goal now is to maximize $\Phi_\omega$. For a maximizer $p$ of this functional, $Q_\omega(p)$ is necessarily positive, so that Theorem 2.2 applies, yielding a solution $q$ of equation (2.1). By contrast, if $p$ minimizes $\Phi_\omega$, then $Q_\omega(p)$ is negative, and this would lead to imaginary solutions of (2.1). Thus, we restrict our attention to maximizers of $\Phi_\omega$. Since this functional is homogeneous of degree zero, it suffices to consider its restriction to the unit ball $B$ in $L_o^{4/3}$. Define

$$c_\omega = \sup_{p\in B}\Phi_\omega(p)\,, \tag{2.11}$$

and notice that $c_\omega > 0$.

**Theorem 2.4.** *For every nonresonant $\omega \in \mathbb{R}$, the functional $\Phi_\omega$ has a critical point $p_\omega \in B$ with $\Phi_\omega(p_\omega) = c_\omega$.*

**Proof.** Let $\{p_n\}$ be a sequence of maximizers in $B$. Since the injection $L^{4/3} \to H^{-1}$ is compact, some subsequence $\{g_n\}$ of $\{p_n\}$ converges to a limit $p_\omega$ in $H_o^{-1}$. This limit satisfies $|\langle f, p_\omega\rangle| \le \|f\|_{L^4}$ for all $f \in H_o^1$, and thus it is a function in $L_o^{4/3}$ of norm $\le 1$. Given that $Q_\omega$ is continuous on $H_o^{-1}$, the values $Q_\omega(g_n)$ converge to $Q_\omega(p_\omega)$. Furthermore, $g_n \to p_\omega$ weakly in $L_o^{4/3}$, and since $U$ is weakly lower semicontinuous, $\liminf U(g_n) \le U(p_\omega)$. This shows that $p_\omega$ belongs to $B$ and maximizes $\Phi_\omega$. **QED**

We note that the same argument can be used to find critical points that minimize $\Phi_\omega$ on invariant subspaces. The $\beta$-model has in fact a large number of symmetries, that is, if equation (1.1) is written in the form $F(q) = q$ (see e.g. Section 4), then there is a large group $\mathcal{G}$ of linear transformations on $H^1$ that commute with $F$. These transformations include $q \mapsto -q$, and the maps $q \mapsto q \circ \gamma$, where $\gamma$ is an element of the group $\mathcal{P} \times S^1$, or a reflection of the form $(m, t) \mapsto (\pm m, \pm t)$. All solutions described in this paper are invariant under time reflection and simultaneous sign change, and most are in fact invariant under larger subgroups of $\mathcal{G}$. For reference later on, we define

$$\begin{aligned} \bigl(Rf\bigr)_m(t) &= f_{-m}(t)\,, & \bigl(Uf\bigr)_m(t) &= f_{-m}(\pi - t)\,, \\ \bigl(Tf\bigr)_m(t) &= f_{m+1}(t)\,, & \bigl(V_n f\bigr)_m(t) &= f_m(t + 2\pi/n)\,. \end{aligned} \tag{2.12}$$

## 3. Distribution of energies among modes

We start with the simplest situation, that is, with frequencies $\omega$ near a given resonance $\omega_n$. In what follows, $p_\omega$ denotes an arbitrary maximizer of $\Phi_\omega$ in $L_o^{4/3}$, and $q_\omega = \lambda_\omega(p) L_\omega^{-1} \nabla^* p_\omega$ is the corresponding solution of equation (2.1), described in Theorem 2.2.

Denote by $\mathbb{P}_n$ the orthogonal projection in $H^1$ (or any $H^r$) onto the subspace of functions $f$ whose Fourier coefficients $\bigl(\mathcal{F}f\bigr)(h,k)$ are zero, unless $(h,k)$ is associated with the frequency $\omega_n$ via the relation (2.4).

**Lemma 3.1.** *There exist positive constants $B$ and $C$, such that for $\omega > \omega_n$ sufficiently close to $\omega_n$,*

$$\|q_\omega\|_{H^1} \le B\bigl(\omega - \omega_n\bigr)^{1/2}\,, \qquad \|(\mathbb{I} - \mathbb{P}_n) q_\omega\|_{H^1} \le C\bigl(\omega - \omega_n\bigr)\|q_\omega\|_{H^1}\,. \tag{3.1}$$

A proof of this lemma, and of Lemma 3.2 below, are given in Section 5. The proof also shows that as $\omega \to \omega_n$ from above, a subsequence of the maximizer family $\{p_\omega\}$ converges to a solution of the equation

$$p = (\alpha \mathbb{P}_n p)^3\,, \tag{3.2}$$

where $\alpha$ is some positive constant.

The bounds (3.1) agree with those that are obtained for the branches of solutions described in Lyapunov's theorem [1]. This suggests that for $\omega > \omega_n$ close to $\omega_n$, our function $q_\omega$ is in fact one of these Lyapunov solutions.

Unlike a resonance below $\omega$, a resonance above $\omega$ suppresses the corresponding modes of the maximizer $p_\omega$.

**Lemma 3.2.** *There are constants $A, B > 0$, such that for $\omega < \omega_n$ sufficiently close to $\omega_n$,*

$$\|\mathbb{P}_n p_\omega\|_{L^{4/3}} \leq A(\omega_n - \omega)\|p_\omega\|_{L^{4/3}}, \qquad \|q_\omega\|_{H^1} \leq B. \tag{3.3}$$

We note that many qualitative properties of $p_\omega$ are easy to understand from the fact that this function maximizes the quadratic form $Q_\omega$ on the unit ball of $L_o^{4/3}$. In particular, we expect that the largest Fourier coefficients of $p_\omega$ are those that are associated with the resonances closest to and below $\omega$. In order to see what this means for $q_\omega$, it is useful to consider the identity

$$p_\omega = \lambda_\omega(p)^{-1}(\nabla q_\omega)^3,$$

obtained from (2.7). Consider e.g. the first inequality in (3.3). It shows that $\mathbb{P}_n\bigl((\nabla q_\omega)^3\bigr)$ tends to zero as $\omega \to \omega_n$ from below. This does not imply that $\mathbb{P}_n q_\omega$ itself tends to zero. But if the dominant Fourier coefficients of $p_\omega$ come from a few resonances below $\omega$, then we can expect $\mathbb{P}_n q_\omega$ to be small, unless three of the vectors $(h_j, k_j)$ associated with these resonances happen to add up to a vector $(h, k)$ satisfying (2.4).

Consider now an ordering $\omega_1 > \omega_2 > \omega_3 > \ldots$ of all resonant frequencies. This is possible since resonances accumulate only at zero. The lemmas above suggest that if the spacing of resonances is regular near $\omega_n$, then the Fourier coefficients of $q_\omega$ associated with $\omega_n$ will dominate the remaining Fourier coefficients of $q_\omega$, for frequencies $\omega$ between $\omega_n$ and $\omega_{n-1}$. This "Lyapunov property" can indeed be observed numerically,

More interesting solutions $q_\omega$ can be found in places where the resonances are not spaced regularly. If $m > n > 0$ and

$$\frac{\omega_n - \omega_m}{\omega_{n-1} - \omega_n} \ll 1, \qquad \frac{\omega_n - \omega_m}{\omega_m - \omega_{m+1}} \ll 1, \tag{3.4}$$

we will refer to $\omega_n$ as an "almost $N$-fold" resonance, where $N = m + 1 - n$. The following is a rough summary of our numerical results. More details will be given below and in the Appendix.

**Observation 3.3.** *If $\omega_n$ is an almost $N$-fold resonance, then for $\omega < \omega_{n-1}$ close to $\omega_{n-1}$, the maximizers of $\Phi_\omega$ yield solutions $q_\omega$ of (2.1) whose Fourier coefficients associated with each of the frequencies $\omega_n, \ldots, \omega_{n+N-1}$ are much larger than the remaining Fourier coefficients (as one would expect for the maximizers $p_\omega$, but not necessarily for the corresponding solutions $q_\omega$).*

To be more precise, we observed this for several $N$-fold resonances, with values of $N$ ranging from 2 to 4, and with particle numbers $P$ between 11 and 41. In addition, we also constructed critical points of $\Phi_\omega$ that are not maximizers. Their largest Fourier coefficients are still associated with subsets of the frequencies $\{\omega_n, \ldots, \omega_m\}$, and it is possible to find critical points for many different subsets.

Instead of trying to collect data systematically for many values of $N$ and $P$, we have chosen to investigate in more detail the specific case of four closely spaced resonances for the $\beta$-chain with $P = 32$ particles. Using the notation $\omega_{h,k} = 2\sin(h\theta/2)/k$, these four resonances are

$$\begin{aligned} \omega_{14,10} &= 0.19615\ldots, & \omega_{1,1} &= 0.19603\ldots, \\ \omega_{11,9} &= 0.19598\ldots, & \omega_{2,2} &= 0.19508\ldots. \end{aligned} \tag{3.5}$$

Their closest neighbors above and below are $\omega_{15,10} = 0.19903\ldots$ and $\omega_{3,3} = 0.193523\ldots$. For the corresponding gap ratios (3.4) we find approximately the values 0.37 and 0.69, respectively. Although these numbers are not that small, the gaps between the resonances (3.5) are significantly smaller than their distance to other resonances, and we can still observe the phenomenon described above.

For every $h \in \mathcal{P}$, define $\mathbb{P}'_h$ to be the orthogonal projection in $H^1$, onto the subspace of functions $f$ satisfying $(\mathcal{F}f)(h', k') = 0$, unless $h' = \pm h$ modulo $P$. As a measure for the size of the $h$-th spatial mode of $q \in H^1_o$, we consider its "harmonic energy"

$$E_h(q) = E(\mathbb{P}'_h q), \qquad E(q) = \tfrac{1}{2}\langle \dot q, \dot q\rangle + \tfrac{1}{2}\langle \nabla q, \nabla q\rangle. \tag{3.6}$$

These energies are not directly related to the FPU Hamiltonian (unless $\alpha = \beta = 0$), but they have the advantage of being additive, that is, the sum of $E_h(q)$ over all $h \in \mathbb{Z}/(2P\mathbb{Z})$ is equal to $E(q)$.

**Theorem 3.4.** *For $P = 32$ and $\omega = 0.1989$, the equation (2.1) has a set of 11 real analytic solutions, $\{f_A, f_B, \ldots, f_K\}$, with the following properties:*

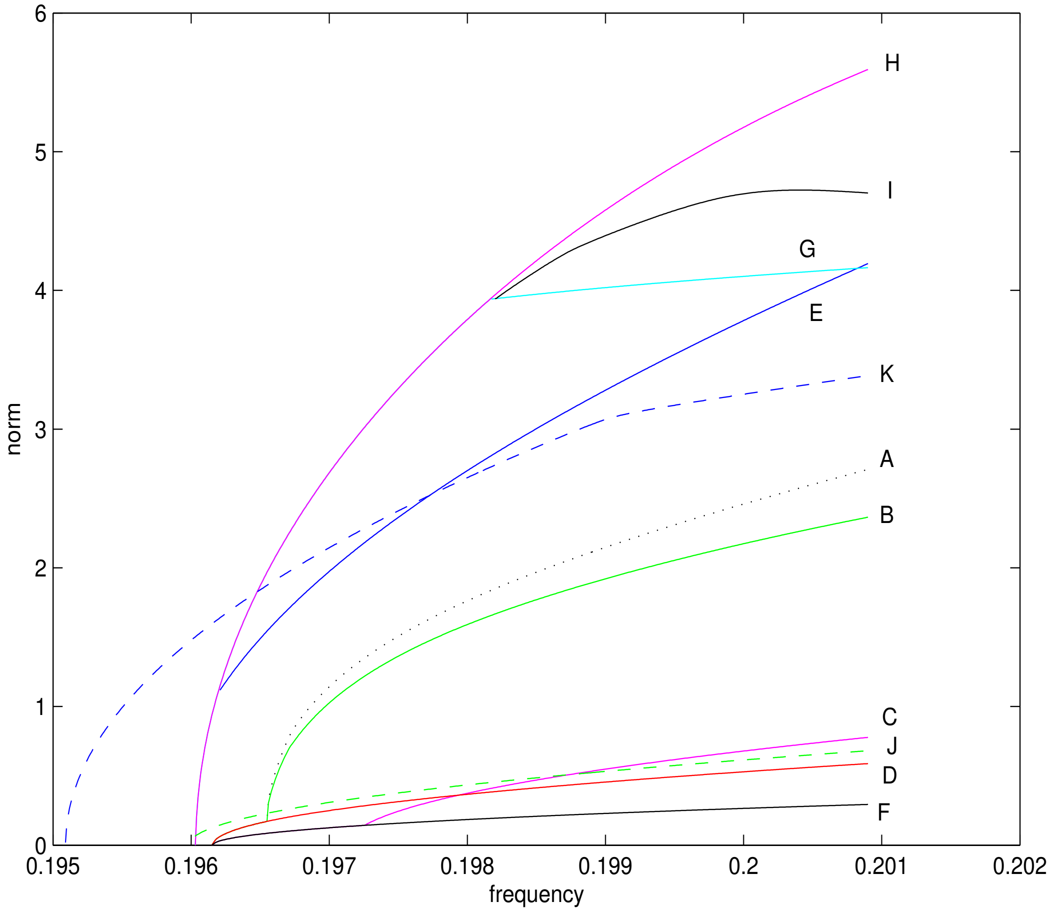
| solution | $\Phi_\omega$ | $E$ | $\mathcal{E}_1$ | $\mathcal{E}_2$ | $\mathcal{E}_{11}$ | $\mathcal{E}_{14}$ | symmetry |
|---|---|---|---|---|---|---|---|
| $f_A$ | $5.71\ldots$ | $0.614\ldots$ | $0.248\ldots$ | $0.109\ldots$ | $0.195\ldots$ | $0.442\ldots$ | $-UT^{17}, RT$ |
| $f_B$ | $5.67\ldots$ | $0.623\ldots$ | $0.185\ldots$ | $0.123\ldots$ | $0.215\ldots$ | $0.470\ldots$ | $UT^{17}, -RT$ |
| $f_C$ | $5.48\ldots$ | $0.645\ldots$ | $\epsilon$ | $0.243\ldots$ | $\epsilon$ | $0.755\ldots$ | $-RT, T^{16}, V_2$ |
| $f_D$ | $5.38\ldots$ | $0.717\ldots$ | $\epsilon$ | $\epsilon$ | $0.375\ldots$ | $0.622\ldots$ | $-UT^{17}, RT$ |
| $f_E$ | $5.21\ldots$ | $0.747\ldots$ | $0.537\ldots$ | $\epsilon$ | $0.458\ldots$ | $\epsilon$ | $R, -T^{16}$ |
| $f_F$ | $5.16\ldots$ | $0.791\ldots$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $0.999\ldots$ | $R, -T^8, V_{10}$ |
| $f_G$ | $5.02\ldots$ | $0.734\ldots$ | $0.814\ldots$ | $0.137\ldots$ | $\epsilon$ | $\epsilon$ | $-U, RT^{16}$ |
| $f_H$ | $4.97\ldots$ | $0.811\ldots$ | $0.974\ldots$ | $\epsilon$ | $\epsilon$ | $\epsilon$ | $-UT^{17}, RT$ |
| $f_I$ | $4.95\ldots$ | $0.833\ldots$ | $0.883\ldots$ | $\epsilon$ | $\epsilon$ | $0.075\ldots$ | $-U, RT^{16}$ |
| $f_J$ | $4.81\ldots$ | $0.864\ldots$ | $\epsilon$ | $\epsilon$ | $0.999\ldots$ | $\epsilon$ | $R, -T^{16}, V_9$ |
| $f_K$ | $3.65\ldots$ | $1.144\ldots$ | $\epsilon$ | $0.996\ldots$ | $\epsilon$ | $\epsilon$ | $R, -T^8, V_2$ |

*Here, $\Phi_\omega$ denotes the value of the functional (2.5), $E$ is the harmonic energy of the given solution, and $\mathcal{E}_h = E_h/E$. The symbol $\epsilon$ stands for a real number of modulus less than 0.002, which may vary from one instance to the next. The transformations listed in the last column generate a symmetry group under which the given solution is invariant; see equation (2.12) for the definition of $R$, $T$, $U$, and $V_n$.*

Notice that among these 11 solutions, 4 have most of their (harmonic) energy concentrated on one mode, 4 on two modes, and 3 on four modes. It is possible that some of these solutions are maximizers of $\Phi_\omega$ on a subspace of functions with (a subset of) the same symmetries. However, the solutions $f_B, f_C, \ldots f_K$ were not obtained via maximization, but with a Newton type method, starting with functions whose energies are concentrated on a subset of the spatial modes corresponding to the resonances (3.5).

Our computer-assisted proof of Theorem 3.4 is described in Section 4. In this proof, we use the symmetries involving $R$, $T$, and $U$ to reduce the analysis to a system of 8 or 16 particles, with appropriate boundary conditions. Symmetries involving $V_n$ are taken care of by rescaling time and replacing $\omega$ by $n\omega$.

Additional data on the solutions $f_A, f_B, \ldots f_K$ , as well as on the solution $f_L$ described below, can be found in the Appendix. Starting from the solutions described in Theorem 3.4, we have also computed the following bifurcation diagram.




In many cases, it was possible to continue these solutions to high values of the frequency $\omega$. The resulting solutions of (2.1) are characterized by high values of the energy, spread among many modes. One such example is described in the following theorem.

**Theorem 3.5.** *For $P = 41$ and $\omega = 1$, the equation (2.1) has a real analytic solution $f_L$ with symmetry $-RUT^{21}$, critical value $\Phi_\omega(f_L) = 0.01429\ldots$, energy $E = 1844.2\ldots$, and* *harmonic energy ratios* $\mathcal{E}_h = E_h/E$ *given by*

$$
\begin{array}{lllll}
\mathcal{E}_1 = 0.0122\ldots, & \mathcal{E}_2 = 0.0012\ldots, & \mathcal{E}_3 = 0.0425\ldots, & \mathcal{E}_4 = 0.2086\ldots, & \mathcal{E}_5 = 0.0258\ldots, \\
\mathcal{E}_6 = 0.0054\ldots, & \mathcal{E}_7 = 0.0378\ldots, & \mathcal{E}_8 = 0.0440\ldots, & \mathcal{E}_9 = 0.0221\ldots, & \mathcal{E}_{10} = 0.0156\ldots \\
\mathcal{E}_{11} = 0.0537\ldots, & \mathcal{E}_{12} = 0.0217\ldots, & \mathcal{E}_{13} = 0.0478\ldots, & \mathcal{E}_{14} = 0.0404\ldots, & \mathcal{E}_{15} = 0.0990\ldots \\
\mathcal{E}_{16} = 0.2232\ldots, & \mathcal{E}_{17} = 0.0457\ldots, & \mathcal{E}_{18} = 0.0127\ldots, & \mathcal{E}_{19} = 0.0289\ldots, & \mathcal{E}_{20} = 0.0105\ldots.
\end{array}
$$

Another prominent feature of the orbit $f_L$ is that it shows a heavy exchange of "instantaneous harmonic energy" between its modes; see the last figure in the Appendix. This occurs to a much lesser extent in low energy solutions like $f_A, \ldots, f_K$. The path of solutions leading up to $f_L$ involves several symmetry-breaking bifurcations. Such bifurcations, which destabilize invariant subspaces, have been linked by some authors to the onset of chaos in the FPU model. We believe that our methods can be extended to (and are well suited for) an analysis of invariant manifolds, bifurcations and bifurcation diagrams.

## 4. A computer–assisted proof of Theorems 3.4 and 3.5

In order to prove Theorem 3.4 and Theorem 3.5, we rewrite equation (2.1) in the form $F(q) = q$,

$$F(q) = \omega^{-2}\partial^{-2}\nabla^*\bigl(\nabla q + (\nabla q)^3\bigr)\,, \tag{4.1}$$

where $\partial^{-1}$ denotes the antiderivative operator on the space of continuous $2\pi$-periodic functions with average zero. The functions $q : \mathbb{R} \to \mathbb{R}^P$ considered here extend analytically to a strip $\mathcal{D}_\rho = \{t \in \mathbb{C} : |\mathrm{Im}(t)| < \rho\}$. To be more precise, given $\rho > 0$, denote by $\mathcal{F}_\rho$ the vector space of all $2\pi$-periodic analytic functions $f : \mathcal{D}_\rho \to \mathbb{C}$,

$$f(t) = \sum_{k=1}^{\infty} f_k \sin(kt) + \sum_{k=0}^{\infty} f'_k \cos(kt)\,, \qquad t \in \mathcal{D}_\rho\,,$$

which take real values for real arguments (i.e. all Fourier coefficients $f_k$ and $f'_k$ are real), and for which the norm

$$\|f\|_\rho = \sum_{k=1}^{\infty} e^{\rho k}|f_k| + \sum_{k=0}^{\infty} e^{\rho k}|f'_k|$$

is finite. When equipped with this norm, $\mathcal{F}_\rho$ is a Banach space. The subspaces of odd and even functions in $\mathcal{F}_\rho$ will be denoted by $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, respectively. On the direct sum $\mathcal{F}_\rho^P$, we define the norm

$$\|q\|_\rho = \max_{1\le i\le P} \|q_i\|\,. \tag{4.2}$$

We note that $\mathcal{F}_\rho$ is a Banach algebra, that is, $\|fg\|_\rho \le \|f\|_\rho \|g\|_\rho$, for all $f$ and $g$ in $\mathcal{F}_\rho$. Furthermore, $\partial^{-2}$ acts as a compact linear operator on $\mathcal{A}_\rho$, as well as on $\mathcal{A}_\rho^P$. This shows that equation (4.1) defines a differentiable map $F$ on $\mathcal{A}_\rho^P$ with compact derivatives $DF(q)$. Thus, $F$ can be well approximated locally by its restriction to a suitable finite dimensional subspace of $\mathcal{A}_\rho$. This property makes it ideal for a computer-assisted analysis.

Our goal is to find fixed points for $F$ by using a Newton like iteration, starting with a numerical approximation $q_0$ for the desired fixed point. The Newton map $\mathcal{N}$ associated with $F$ is given by $\mathcal{N}(q) = F(q) - \mathcal{M}(q)[F(q) - q]$, with $\mathcal{M}(q) = [DF(q) - \mathbb{I}]^{-1} + \mathbb{I}$. If the spectrum of $DF(q)$ is bounded away from 1, and $q_0$ is sufficiently close to a fixed point of $F$, then $\mathcal{N}$ is a contraction in some neighborhood of $q_0$. Due to the compactness of $DF(q)$, this contraction property is preserved if we replace $\mathcal{M}(q)$ by a fixed linear operator $M$ close to $\mathcal{M}(q_0)$. This leads us to consider the new map $\mathcal{C}$, defined by

$$\mathcal{C}(q) = F(q) - M[F(q) - q]\,, \qquad q \in \mathcal{A}_\rho^P\,. \tag{4.3}$$

To be more specific, $M$ will be chosen to be a "matrix", in the sense that $M = \mathbb{P}_\ell M \mathbb{P}_\ell$ for some $\ell > 0$, where $\mathbb{P}_\ell$ denotes the canonical projection in $\mathcal{A}_\rho^P$ onto Fourier polynomials of degree $k \le \ell$. We also verify that $M - \mathbb{I}$ is invertible, so that $\mathcal{C}$ and $F$ have the same set of fixed points. For the reasons mentioned above, we expect $\mathcal{C}$ to be a contraction on some ball $B(q_0, r)$ in $\mathcal{A}_\rho^P$ of radius $r > 0$, centered at $q_0$. In order to prove that this is indeed the case, it suffices to verify the inequalities

$$\|\mathcal{C}(q_0) - q_0\|_\rho < \varepsilon\,, \qquad \|D\mathcal{C}(q)\| < K\,, \qquad \varepsilon + Kr < r\,, \tag{4.4}$$

for some real numbers $r, \varepsilon, K > 0$, and for arbitrary $q$ in the ball $B(q_0, r)$. These bounds imply that $\mathcal{C}$, and thus $F$, has a unique fixed point in $B(q_0, r)$.

**Theorem 4.1.** *In each of the* 12 *cases described in Theorem 3.4 and Theorem 3.5, there exists a Fourier polynomial* $q_0$ *, and real numbers* $\rho, \varepsilon, r, K > 0$*, such that the inequalities (4.4) hold. Furthermore, the numerical bounds given in these theorems are satisfied for all function in the corresponding ball* $B(q_0, r)$*.*

The proof of this theorem is based on a discretization of the problem, carried out and controlled with the aid of a computer. We will now give an outline of the steps used to generate this discretization, and to control the discretization errors. For details, the reader is referred to the source code of our computer programs [21].

At the trivial level of real numbers, the discretization is implemented by using interval arithmetic. In particular, a number $s \in \mathbb{R}$ is "represented" by an interval $S = [S^-, S^+]$ containing $s$, whose endpoints belong to some finite set of real numbers that are representable on the computer. Such an interval will be called a "standard sets" for $\mathbb{R}$. The collection of all standard sets for $\mathbb{R}$ will be denoted by $\mathrm{std}(\mathbb{R})$, and the same notation will be used below for spaces other than $\mathbb{R}$. Unless mentioned otherwise, $\mathrm{std}(X_1 \times X_2)$ is defined as the collection of all products $S_1 \times S_2$ with $S_1 \in \mathrm{std}(X_1)$ and $S_2 \in \mathrm{std}(X_2)$. In what follows, a "bound" on a function $g : X \to Y$ is a map $G$, from a subset $D_G$ of $\mathrm{std}(X)$ to $\mathrm{std}(Y)$, with the property that $g(s)$ belongs to $G(S)$ whenever $s \in S \in D_G$. Bounds on the basic arithmetic operation like $(r, s) \to rs$ are easy to implement on modern computers. We will use here the procedures defined and described in [19], which also includes bounds on some common functions like $s \mapsto e^s$.

The goal now is to combine these elementary bounds to obtain e.g. a bound $G_1$ on the norm function on $\mathcal{A}_\rho^P$, and a bound $G_2$ on the map $\mathcal{C}$. Then, in order to prove the first inequality in (4.4), it suffices to verify that $G_1(G_2(S)) \subset U$, where $S$ is a set in $\mathrm{std}(\mathcal{A}_\rho^P)$ containing $g_0$, and $U$ is an interval in $\mathrm{std}(\mathbb{R})$ with $U^+ < \varepsilon$.

We start by defining the standard sets for $\mathcal{A}_\rho$. Let $n \geq \ell$ be a fixed integer. Given $U = (U_1, \ldots, U_n)$ in $\mathrm{std}(\mathbb{R}^n)$, and $V = (V_0, \ldots, V_{2n})$ in $\mathrm{std}(\mathbb{R}_+^{2n+1})$, denote by $S(U, V)$ the set of all functions $f$ that can be represented as

$$f(t) = \sum_{k=1}^{n} u_k \sin(kt) + \sum_{m=0}^{2n} v_m(t)\,, \qquad v_m(t) = \sum_{k=m}^{\infty} v_{m,k} \sin(kt)\,, \tag{4.5}$$

with $u_k \in U_k$, and $v_m \in \mathcal{A}_\rho$ with $\|v_m\|_\rho \in V_m$, for all $k$ and $m$. We now define $\mathrm{std}(\mathcal{A}_\rho)$ to be the collection of all such sets $S(U, V)$, subject to the condition that $V_m^- = 0$ for all $m$. The standard sets for $\mathcal{B}_\rho$ are defined analogously, and for product spaces such as $\mathcal{A}_\rho^P$, we use the definition mentioned earlier.

It is now straightforward to implement a bound on the norm function on $\mathcal{A}_\rho^P$, or the operator $\partial^{-2}$, or the sum of two functions in $\mathcal{A}_\rho^P$. In order to obtain a bound on the product of two functions in $\mathcal{A}_\rho$, we simply multiply the representations (4.5) of the two factors term by term, and write the result again as an explicit Fourier polynomial of order $n$, plus a sum of "error terms" of orders $\geq m$, for $m = 0, 1, \ldots, 2n$. The guiding principle here is to keep as much information as possible about the order of each term in the product, since the operator $\partial^{-2}$, which is applied last in the definition (4.1) of $F$, contracts higher order terms more than lower order ones. This principle also motivated our choice of standard sets for $\mathcal{A}_\rho$.

For a bound on the linear operator $M$, we can compute explicitly its restriction to standard sets whose components $S(U_i, V_i)$ have $V_{i,m} = [0, 0]$ whenever $m \leq \ell$. The remaining terms are estimated by using that $\|Mq\|_\rho \leq \|M\|\|q\|_\rho$. The operator norm $\|L\|$ of a continuous linear operator $L$ on $\mathcal{A}_\rho^P$ is given by the following formula. Denote by $h^{j,m}$ the function $(i, k) \mapsto \delta_{ij} e^{-k\rho} \sin(kt)$. Then

$$\|L\| = \max_{1 \leq i \leq P} \sum_{j=1}^{P} \sup_{m \geq 1} \big\| \big(L h^{j,m}\big)_i \big\|_\rho\,. \tag{4.6}$$

In the case where $L$ is the "matrix" $M$, the right hand side of this equation is trivial to estimate. The bounds discussed so far can be combined to yield a bound on the the map $\mathcal{C}$, suitable for proving the first (and last) inequality in (4.4).

In order to prove the second inequality in (4.4), we also need a bound on the map $q \mapsto \|D\mathcal{C}(q)\|$. Its domain only needs to include balls $B(\rho_0, r)$ with positive representable radii, and these balls belong in fact to $\mathrm{std}(\mathcal{A}_\rho^P)$. From the formula (4.6) for the norm of $L = D\mathcal{C}(q)$, we see that it suffices to have a bound on the map $(q, h) \mapsto \|D\mathcal{C}(q)h\|_\rho$, defined on sets $B(q_0, r) \times H^{j,m}$, where $H^{j,m}$ is a suitable standard set in $\mathcal{A}_\rho^P$ containing $h^{j,m}$. In principle, this involves no new bounds, since

$$D\mathcal{C}(q) = DF(q) - M[DF(q) - \mathbb{I}]\,, \quad DF(q)h = \omega^{-2}\partial^{-2}\nabla^*\big(\nabla h + 3(\nabla q)^2 h\big)\,.$$

But there is the issue of choosing infinitely many sets $H^{j,m}$. We note that for small values of $m$, the desired estimate $\|D\mathcal{C}(q)h^{j,m}\|_\rho < K$ relies heavily on cancellations. Thus, the standard sets $H^{j,m}$ containing $h^{j,m}$ have to be small. Such sets are readily available in $\operatorname{std}(\mathcal{A}_\rho^P)$. For larger values of $m$, cancellations are no longer important, since the operator $\partial^{-2}$ contracts terms of high order. Thus, for all $m$ exceeding some value $m' \leq n$, we choose for $H^{j,m}$ the set of all functions $h = \sum_{m>m'} c_m h^{j,m}$ of norm $\leq 1$. This set belongs to $\operatorname{std}(\mathcal{A}_\rho^P)$, if the number 1 is representable (as in our case). As a result, the supremum in equation (4.6) is estimated in a finite number of steps.

The precise definition of all these bounds, down to the level of inequalities between (sums and products of) representable numbers, has been written in the programming language Ada95. A computer (Intel Pentium class PC) was then used to translate these definitions to machine code (with the public version 3.14p of the GNAT compiler [20]) and to verify the actual inequalities. The computer programs and input data can be found in [21].

## 5. Proofs of Lemma 3.1 and Lemma 3.2

Consider the decomposition

$$\Phi_\omega(p) = \Phi'_\omega(p) + a_\omega \Phi''(p)\,, \qquad a_\omega = \frac{\omega_n^2}{\omega^2 - \omega_n^2}\,, \tag{5.1}$$

where

$$\Phi'_\omega(p) = \frac{Q_\omega((\mathbb{I} - \mathbb{P}_n)p)}{U(p)^{3/2}}\,, \qquad \Phi''(p) = \frac{\langle \mathbb{P}_n p, p\rangle}{2U(p)^{3/2}}\,.$$

Since all of our functionals are homogeneous of degree zero, we will consider only functions in $L_o^{4/3}$ of norm one. The idea now is to use that $a_\omega$ diverges as $\omega \to \omega_n$, while $\Phi'_\omega$ remains bounded: The analogue of (2.10) for the functional $Q_\omega \circ (\mathbb{I} - \mathbb{P}_n)$ shows that $\Phi'_\omega$ is bounded in modulus by a fixed constant $c$, for all $\omega$ in an open neighborhood of $\omega_n$.

**Proof of Lemma 3.1.** First, we note that for every nonresonant $\omega$,

$$\|\mathbb{P}_n q_\omega\|_{H^1}^2 = \left(\frac{\lambda_\omega(p_\omega) a_\omega}{\omega_n}\right)^2 \theta \langle \mathbb{P}_n p_\omega, p_\omega\rangle\,, \tag{5.2}$$

with $1 \leq \theta \leq 2$.

Consider now an interval $W = (\omega_n, \omega_n + \varepsilon)$ whose closure contains no resonances besides $\omega_n$. Denote by $p''$ a maximizer of $\Phi''$. By using equation (5.1), we have

$$\Phi''(p'') - \frac{c}{a_\omega} \leq \frac{1}{a_\omega}\Phi_\omega(p'') \leq \frac{1}{a_\omega}\Phi_\omega(p_\omega) \leq \Phi''(p_\omega) + \frac{c}{a_\omega}\,,$$

for all $\omega$ in $W$. This shows that $\Phi''(p_\omega)$ approaches the maximum of $\Phi''$, as $\omega \to \omega_n$ from above. Since $\Phi'(p_\omega)$ remains bounded, we have

$$\lambda_\omega^2(p_\omega) = \left(\frac{2U(p_\omega)}{3Q_\omega(p_\omega)^2}\right)^3 \leq \frac{1}{25}\left(\frac{2U(p_\omega)}{a_\omega \langle \mathbb{P}_n p_\omega, p_\omega\rangle}\right)^3\,,$$

for all $\omega$ in $W$. Here, and in what follows, we assume that $\varepsilon > 0$ has been chosen sufficiently small. By substituting this bound into (5.2), we find that

$$\|\mathbb{P}_n q_\omega\|_{H^1}^2 \le \bigl(\omega^2 - \omega_n^2\bigr) \frac{\theta}{25\omega_n^4}\, \Phi''(p_\omega)^{-2}\,.$$

This, together with the second inequality in (3.1), implies the first inequality in (3.1).

An estimate analogous to (2.10) yields

$$\|(\mathbb{I} - \mathbb{P}_n) q_\omega\|_{H^1}^2 = \lambda_\omega(p_\omega)^2 \bigl\|(\mathbb{I} - \mathbb{P}_n) L_\omega^{-1} \nabla^* p_\omega\bigr\|_{H^1}^2 \le \lambda_\omega(p_\omega)^2 c' U(p_\omega)^{3/2}\,, \tag{5.3}$$

for all $\omega$ in $W$, where $c'$ is some fixed constant. By combining this bound with (5.2), we find that

$$\frac{\|(\mathbb{I} - \mathbb{P}_n) q_\omega\|_{H^1}^2}{\|\mathbb{P}_n q_\omega\|_{H^1}^2} \le \left(\frac{\omega^2 - \omega_n^2}{\omega_n^2}\right)^2 \frac{c'\omega_n^2}{\theta}\, \Phi''(p_\omega)^{-1}\,.$$

for all $\omega$ in $W$. This proves the second inequality in (3.1). **QED**

By arguments analogous to those used in the proof of Theorem 2.4 and Theorem 2.2, a subsequence of the family $\{p_\omega\}$ converges in $H_o^{-1}$ to a maximizer of $\Phi''$ in $L_o^{4/3}$. Furthermore, any maximizer $p$ of $\Phi''$ satisfies $DU(p)u = \alpha\langle \mathbb{P}_n p, u\rangle$, for all $u \in L_o^{4/3}$, where $\alpha$ is some positive constant. This equation for $p$ is equivalent to (3.2).

**Proof of Lemma 3.2.** Consider the equation (2.6) for the critical point $p_\omega$ of $\Phi_\omega$. By applying both sides of this equation to $\mathbb{P}_n p_\omega$, and dividing by $2U(p_\omega)$, we obtain

$$a_\omega \langle p_\omega, \mathbb{P}_n p_\omega\rangle = \lambda_\omega(p_\omega)^{-2/3} \langle p_\omega^{1/3}, \mathbb{P}_n p_\omega\rangle\,.$$

Since $E$ is a finite dimensional projection, this and Hölder's inequality imply that

$$\|\mathbb{P}_n p_\omega\|_{L^{4/3}}^2 \le C\langle \mathbb{P}_n p_\omega, \mathbb{P}_n p_\omega\rangle \le C|a_\omega|^{-1} \lambda_\omega(p_\omega)^{-2/3} \|p_\omega\|_{L^{4/3}}^{1/3} \|\mathbb{P}_n p_\omega\|_{L^{4/3}}\,,$$

for some positive constant $C$. The first inequality in (3.3) now follows from the fact that $\Phi_\omega(p_\omega)$, and thus $\lambda_\omega(p_\omega)$, stays away from zero as $\omega \to \omega_n$ from below.

From equation (5.2) we now see that $\mathbb{P}_n q_\omega$ stays bounded in $H_o^1$, as $\omega \to \omega_n$ from below. The boundedness $(\mathbb{I} - \mathbb{P}_n) q_\omega$ follows from the fact that $L_\omega^{-1}(\mathbb{I} - \mathbb{P}_n)$ is bounded, as a linear operator from $H_o^{-1}$ to $H_o^1$, for all $\omega$ near $\omega_n$. This proves the second inequality in (3.3). **QED**

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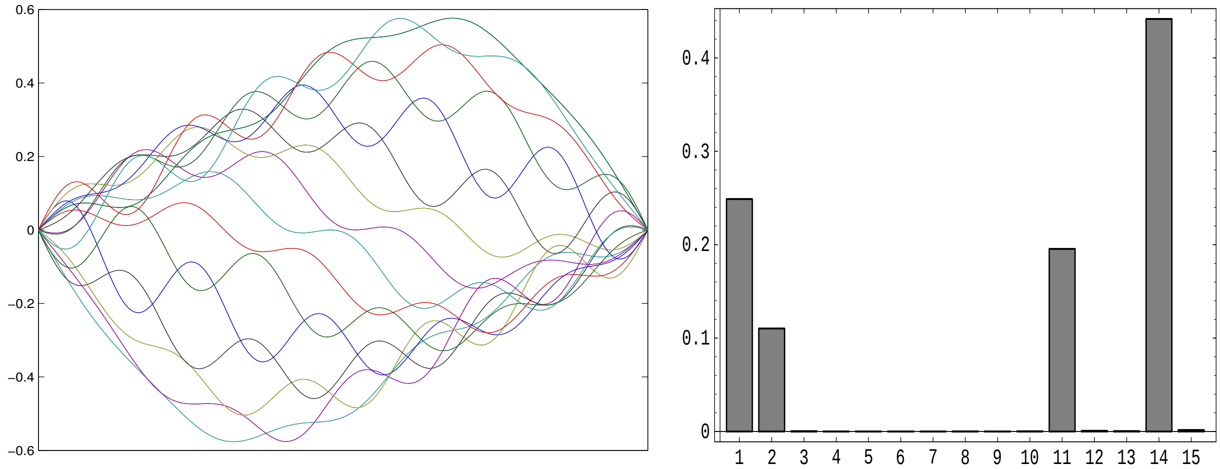
[20] The GNU NYU Ada 9X Translator, available at `ftp://cs.nyu.edu/pub/gnat`

[21] Ada files and data are included with the preprint `mp_arc 03--552`; see also `http://www1.mate.polimi.it/~gianni/fpu.html`

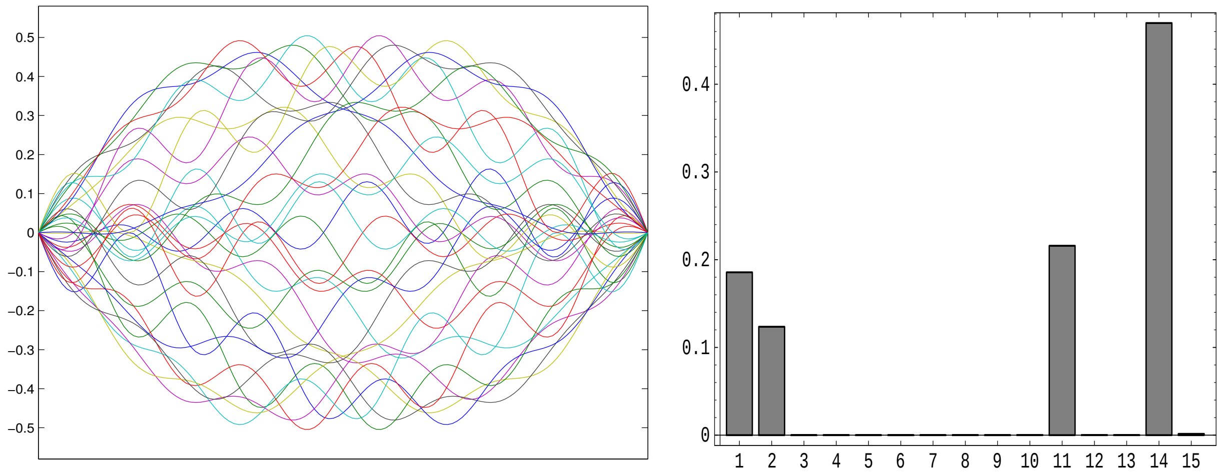
## Appendix

The following figures show the time evolution, and the distribution of harmonic energy among the spatial modes, for each of the 11 periodic solutions described in Theorem 3.4. In particular, the curves in the figure labeled "$f_A$" describe the positions $q_1, q_2, \ldots q_{32}$ of the 32 particles represented by $q = f_A$, as a function of time (by symmetry, two of these positions always coincide). The time interval shown is $[0, \pi]$, which corresponds to half the minimal period of $f_A$. The second figure (to the right) represents the distribution of harmonic energy among the spatial modes of $f_A$. Analogous data for the solutions $f_B, f_C, \ldots, f_K$, are given in the subsequent figure pairs.

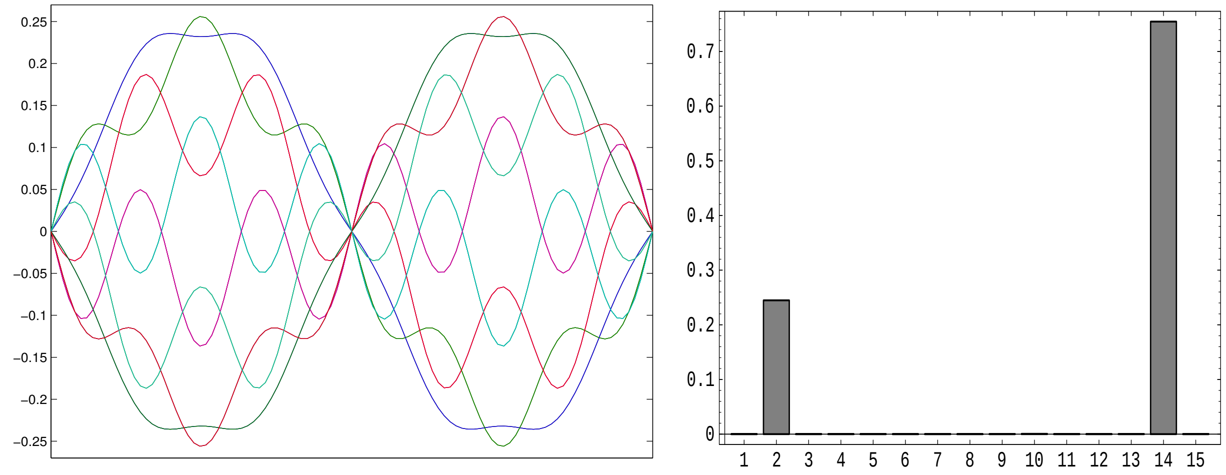
$f_A$




$f_B$



$f_C$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0.2
0.15
0.1
0.05
0
−0.05
−0.1
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−0.25
0.7
0.6
0.5
0.4
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0.1
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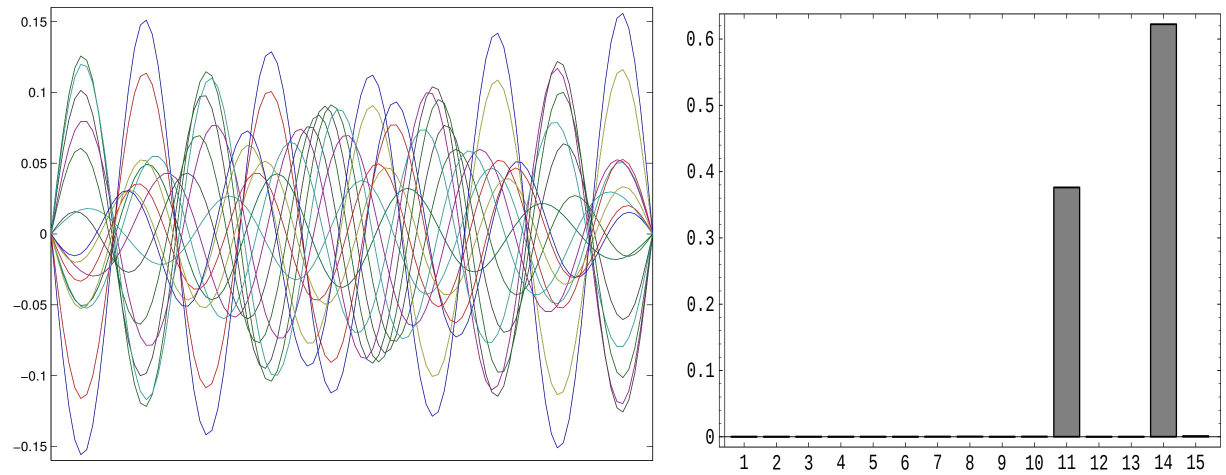
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_D$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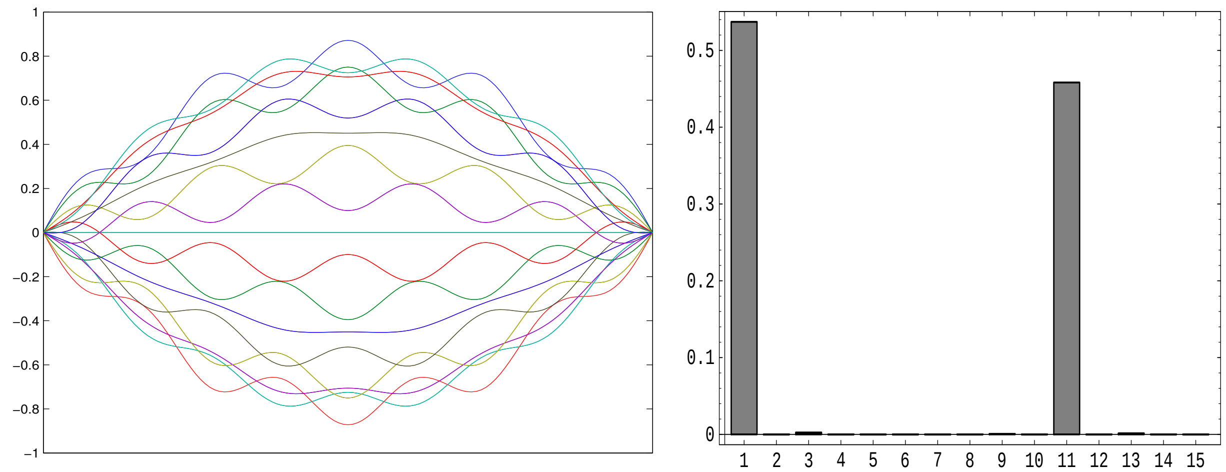
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_E$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0.1
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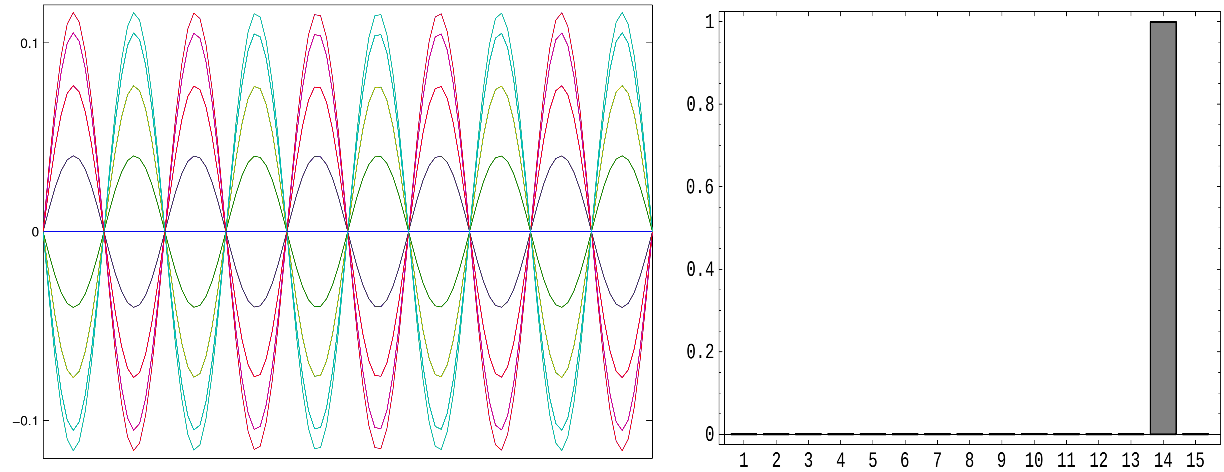
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_F$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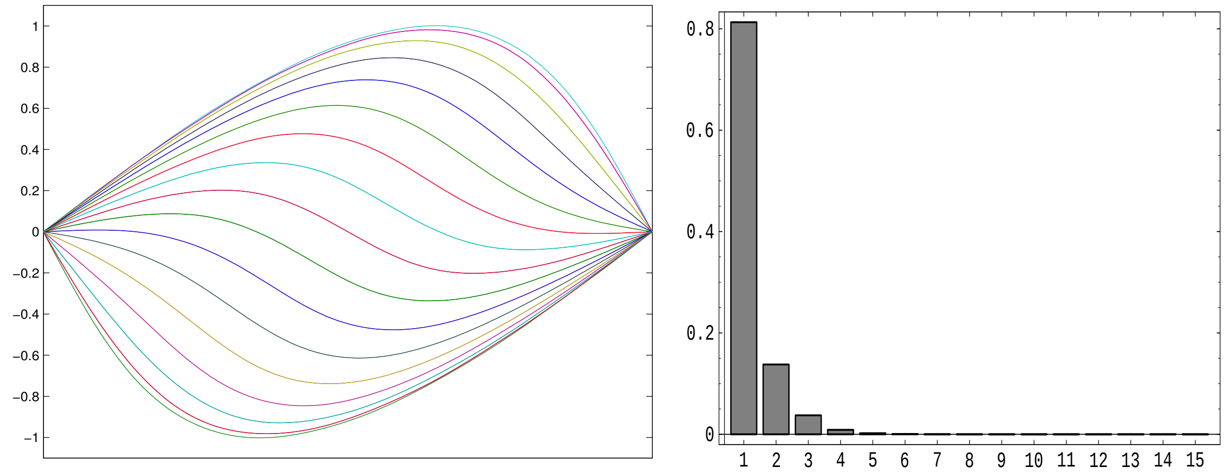
$f_G$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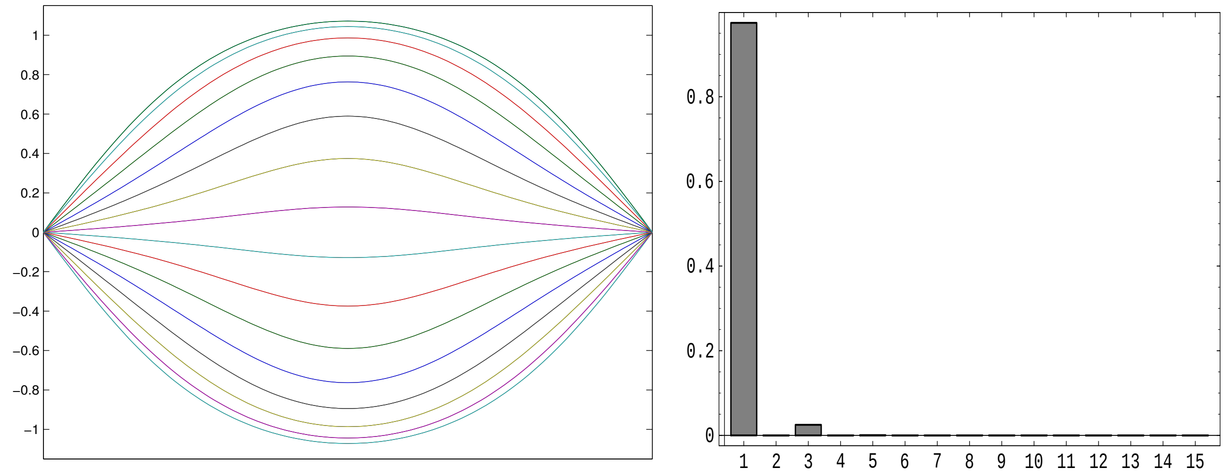
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_H$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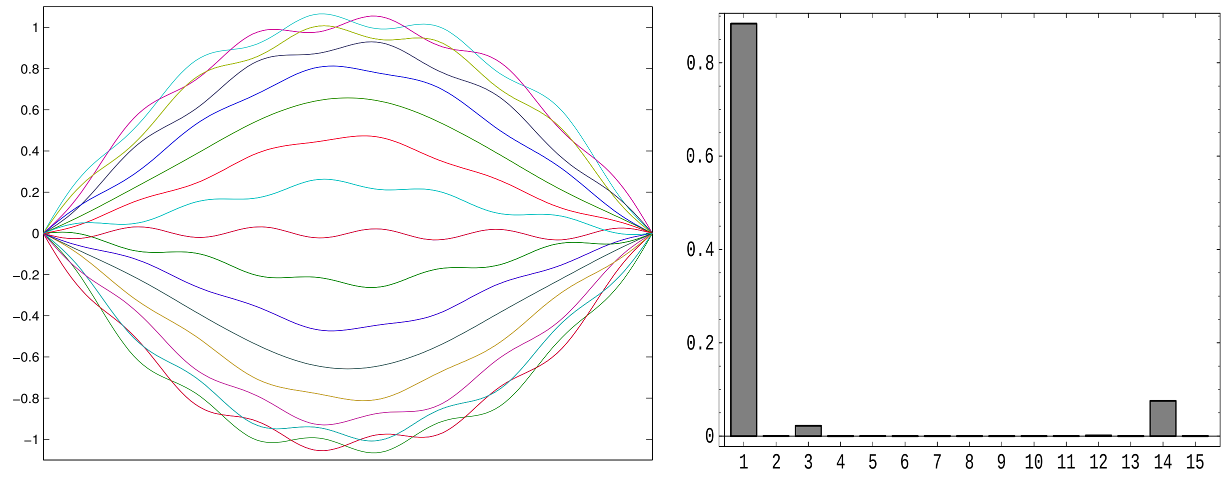
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_I$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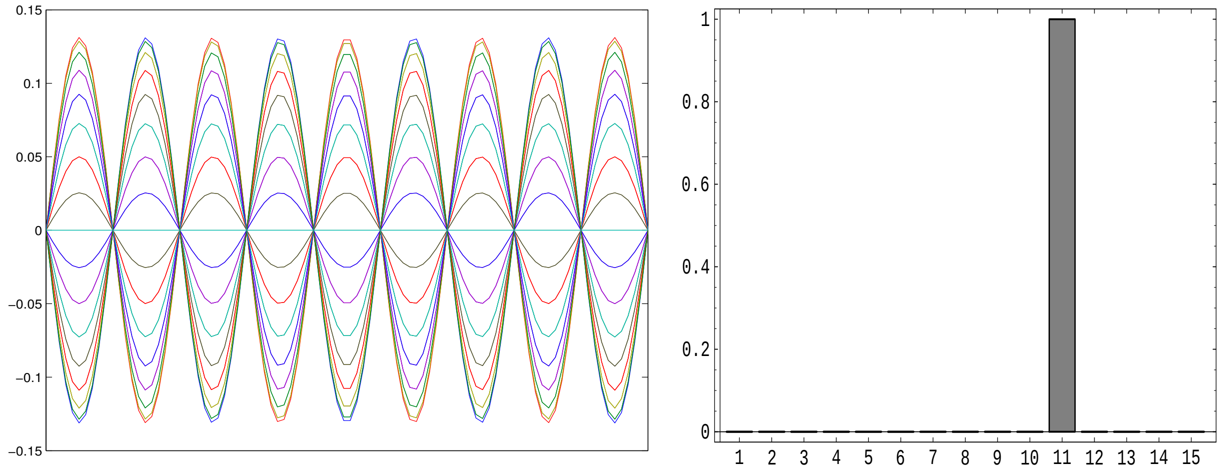
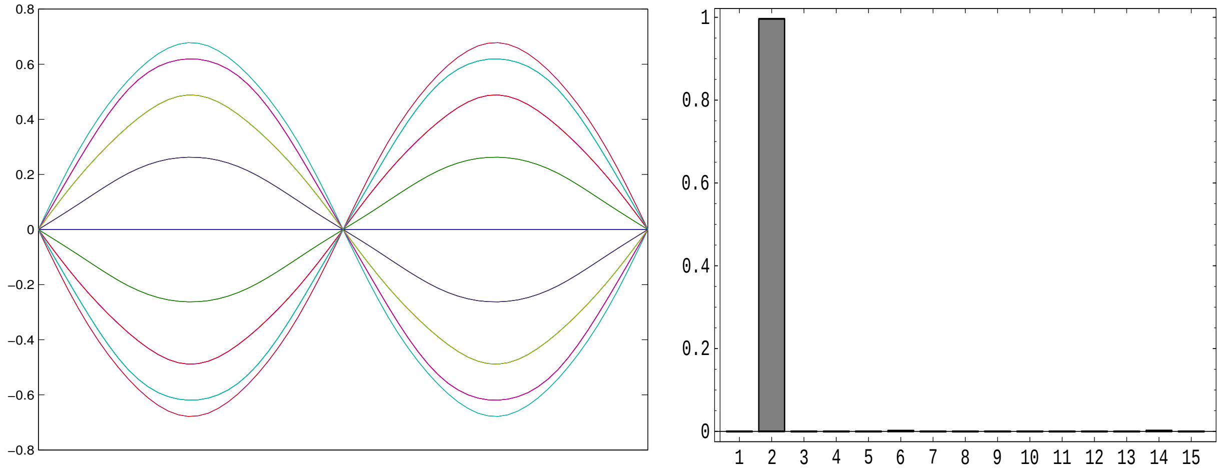
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_J$


0.15
0.1
0.05
0
−0.05
−0.1
−0.15
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

$f_K$


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15

The same data for the periodic solution $f_L$ described in Theorem 3.5 are shown in the first two figures below. The third and last figure graphs the "instantaneous harmonic energy" $t \mapsto E_h(f_L, t)$ for each spatial mode $h$ of $f_L$ , where $E_h(q, t)$ is defined in the same way as the harmonic energy $E_h(q)$ in equation (3.6), but without integrating over time $t$.

$f_L$