A Renormalization Group for Hamiltonians: Numerical Results

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Abstract. We describe a renormalization group transformation that is related to the breakup of golden invariant tori in Hamiltonian systems with two degrees of freedom. This transformation applies to a large class of Hamiltonians, is conceptually simple, and allows for accurate numerical computations. In a numerical implementation, we find a nontrivial fixed point and determine the corresponding critical index and scaling. Our computed values for various universal constants are in good agreement with existing data for area–preserving maps. We also discuss the flow associated with the nontrivial fixed point.

[1] Supported in Part by the National Science Foundation under Grant No. DMS–9705095.
[2] Supported in Part by the Swiss National Science Foundation.

1. Introduction

The renormalization group transformation considered in this paper acts on analytic Hamiltonians for systems of two degrees of freedom. It is designed to apply to the stability problem for invariant tori associated with frequency vectors $\omega = (\omega_1, \omega_2)$, for which ω_1/ω_2 is a reduced quadratic irrational. For simplicity, we will focus from the beginning on the case of the golden mean. Our formalism can also be extended to higher dimensions, as in [4], but the results may be qualitatively different; see also [14–16].

We start with some simple facts about the golden mean $\vartheta = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. The numbers ϑ and $-1/\vartheta$ are solutions of a quadratic equation with integer coefficients, namely the equation for the eigenvalues of the matrix $T = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. The corresponding eigenvectors are $\omega = (1/\vartheta, 1)$ and $\Omega = (1, -1/\vartheta)$, respectively. The matrix T can also be used to generate the continued fraction approximants $\vartheta_k = b_k/a_k$ of ϑ. More specifically, we have $(a_k, b_k) = T^k(1,1)$, for $k = 0, 1, 2, \ldots$.

The Hamiltonians considered here are assumed to have a rapidly convergent Fourier–Taylor series of the form

$$H(q,p) = \sum_{(\nu,\alpha)\in I} H_{\nu,\alpha}\, F_{\nu,\alpha}(q,p)\,, \qquad F_{\nu,\alpha}(q,p) = (\omega\cdot p)^{\alpha_1}(\Omega\cdot p)^{\alpha_2} e^{i\nu\cdot q}\,, \tag{1.1}$$

for all q in $\mathbb{R}^2$, and for p in some open neighborhood of the origin in $\mathbb{R}^2$. The sum in this equation ranges over all multi–indices (ν,α) in $I = \mathbb{Z}^2\times\mathbb{N}^2$, with $\mathbb{N}$ being the set of all non–negative integers. We will refer to p and q as the action and (lifted) angle variable, respectively. Recall that the flow defined by H is $(\dot q, \dot p) = \bigl(\nabla_2 H, -\nabla_1 H\bigr)$, where ∇_j denotes the gradient with respect to the j–th variable.

In order to motivate our analysis, let us first consider the integrable Hamiltonian $H_0(q,p) = \omega\cdot p + \frac{1}{2}(\Omega\cdot p)^2$. When restricted to a fixed level set $H_0(q,p) = E$, this Hamiltonian has an invariant torus $\mathcal{O}_\infty$ whose frequency vector is proportional (in fact equal) to ω. In addition, for every sufficiently large integer k, we can find a periodic orbit $\mathcal{O}_k$ whose frequency vector is a constant multiple of (a_k, b_k). The same can be observed (and proved) for analytic Hamiltonians that are sufficiently close to H_0. In fact, the orbits $\mathcal{O}_k$ accumulate at the torus $\mathcal{O}_\infty$. To give a more precise description, assume for simplicity that each of the orbits $\mathcal{O}_k$ contains a unique point (q_k, p_k) with $q_k = 0$. This holds e.g. for many Hamiltonians that satisfy $H(q,p) = H(-q,p)$. Then p_k approaches a limit p_∞ as k tends to infinity, and the distance d_k between p_k and p_∞ tends to zero with an asymptotic ratio $d_{k+1}/d_k \to \vartheta^{-2}$. This should not bee too astonishing, as $|\vartheta_k - \vartheta|$ tends to zero with the same asymptotic ratio. What is more astonishing is what seems to happen in many one–parameter families $\beta \mapsto H_0 + \beta F$, as the strength of the perturbation is increased: By looking for orbits with a given degree of stability (e.g. residue 1), among the orbits $\mathcal{O}_k$ for sufficiently large k, one finds a convergent sequence $\beta_m, \beta_{m+1}, \ldots$ in some parameter interval J, with the following property. The Hamiltonian $H_0 + \beta F$, with $\beta \in J$, has an orbit with the given stability only if $\beta = \beta_k$ for some $k \geq m$, and in this case, the orbit is unique and equal to $\mathcal{O}_k$. Furthermore, the ratios $(\beta_{k-1} - \beta_k)/(\beta_k - \beta_{k+1})$ converge to a limit δ, as k tends to infinity; and the ratios d_{k+1}/d_k, for the Hamiltonian with the parameter value $\beta_\infty = \lim_{k\to\infty}\beta_k$, converge to a limit λ different from ϑ^{-2}. The critical index δ and the critical scaling λ appear to be independent of the family considered, within a large class of Hamiltonians. The observed values are $\delta = 1.627\ldots$ and $\lambda = 0.326\ldots$. A more detailed description of these phenomena can be found in References [1,2,9–12].

Such scaling and universality properties suggest the use of renormalization group methods. The idea is to find a transformation $\mathcal{R}$, acting on a space of Hamiltonians, which contracts differences that are irrelevant for the phenomena being considered. According to the description above, $\mathcal{R}$ should therefore have two fixed points: One that is a prototype of a Hamiltonian with trivial asymptotic scaling, attracting all other Hamiltonians of this type, and another one that plays the same role for Hamiltonians with critical scaling. The fixed point equation $\mathcal{R}(H) = H$ should of course imply self–similarity for the set of ϑ_k–orbits of H. A transformation $\Re$ which has the potential of satisfying all of the necessary requirements (modulo the existence of a trivial unstable direction) has been described in [4]. It is of the following form:

$$\Re(H) = \frac{\tau}{\mu} H \circ \mathcal{U}_H \circ T_\mu - \epsilon\,, \qquad T_\mu(q,p) = \bigl(Tq, \mu T^{-1} p\bigr)\,. \tag{1.2}$$

$\mathcal{U}_H$ is a canonical change of coordinates that will be defined later; in particular, $\mathcal{U}_H$ is the identity map if $H(q,p)$ does not depend on q. The constants $\epsilon = \epsilon(H)$, $\tau = \tau(H)$, and $\mu = \mu(H)$ are determined by three normalization conditions for $\widetilde{H} = \Re(H)$. In this paper, we choose these conditions to be $\widetilde{H}_{0,0} = 0$, $\widetilde{H}_{0,(1,0)} = 1$, and $\widetilde{H}_{0,(0,2)} = \frac{1}{2}$. If they cannot be satisfied, then $\Re(H)$ remains undefined. It is easy to check that the integrable Hamiltonian $H_0(q,p) = \omega \cdot p + \frac{1}{2}(\Omega \cdot p)^2$ is a fixed point for $\Re$, with $\epsilon(H_0) = 0$, $\tau(H_0) = \vartheta$, and $\mu(H_0) = \vartheta^{-3}$.

The first step $H \mapsto H \circ \mathcal{U}_H$ in the definition of $\Re(H)$ is similar in spirit to the averaging operation used in KAM theory [17,18]. Its purpose is to eliminate irrelevant (non–resonant) degrees of freedom from H. The next step $H \mapsto H \circ T_1$ represents a "shift" of frequencies, in the sense that if H has a periodic orbit for a frequency vector ν, then $H \circ T_1$ has a (trivially) related orbit for the frequency vector $T^{-1}\nu$. Notice that T_1 is a canonical transformation. In order to re–normalize the resulting Hamiltonian, we perform a rescaling $H \mapsto \mu^{-1} H(.,\mu.)$ of the action variable p, a rescaling $H \mapsto \tau H$ of the energy (or time), and subtract a trivial constant ϵ. We note that the shift and scalings are directly related to the observation of self–similarity near invariant ω–tori. They have also been used in other renormalization transformation [1–8].

The change of coordinates $\mathcal{U}_H$ will be chosen within a restricted class of canonical transformations. Each transformation $U : (q,p) \mapsto (q+Q, p+P)$ in this class is associated with a generating function $\phi(q,p)$ that is 2π–periodic in the components of q, and $U = U_\phi$ is defined implicitly by the equation

$$\begin{aligned} Q(q,p) &= \nabla_2 \phi\bigl(q, p + P(q,p)\bigr)\,, \\ P(q,p) &= -\nabla_1 \phi\bigl(q, p + P(q,p)\bigr)\,. \end{aligned} \tag{1.3}$$

One of the consequences of this restriction is that $\mathcal{U}_H$ cannot include p–translations like $V_s : (q,p) \mapsto (q, p - s\Omega)$. As a result, $\Re$ has a trivial unstable direction. This could be avoided easily by including suitable translations in the definition of our renormalization group transformation. More specifically, we can define $\mathcal{R}(H) = \bigl(\Re(H \circ V_s)\bigr) \circ V_{-s}$, with

$s = s(H)$ determined in such a way that $\widetilde{H} = \mathcal{R}(H)$ satisfies $\widetilde{H}_{0,(0,1)} = 0$. Our numerical result for $\Re$, described in the next section, can be translated into corresponding results for $\mathcal{R}$. They indicate that $\mathcal{R}$ has a trivial attractive fixed point (namely H_0), and a nontrivial fixed point H_* whose stable manifold is of codimension one.

Concerning the notion of relevant or irrelevant degrees of freedom for analytic Hamiltonians, we note that even a critical ω–torus is smooth in the direction of the flow. This suggests that the most relevant perturbations are the ones that distort the ω–torus, and the geometry of nearby orbits, mainly in the directions transversal to the flow. If we assume that (or work in coordinates where) the average velocity on the ω–torus of H is proportional to ω, then these perturbations are given by the functions $F_{\nu,\alpha}$ whose frequency vectors ν are almost perpendicular to ω. They are usually referred to as "resonant" modes. The idea is to eliminate all but these resonant modes. Another way of seeing why this should be done is the following: Assume that, as a function of the angle variable, H is analytic in a strip given by $|\mathrm{Im}\ \omega \cdot q| < r$ and $|\mathrm{Im}\ \Omega \cdot q| < R$. Then $H \circ T_1$ is analytic in a similar strip, but with r and R replaced by $r\vartheta$ and R/ϑ, respectively. Thus, in order to avoid that $\Re$ shrinks the domain of analyticity, we have to regularize H in the direction $(\mathrm{Im}\ q) \propto \omega$. One way of doing this is by eliminating the non–resonant modes of H. To be more precise, we will call $F_{\nu,\alpha}$ a resonant mode if (ν, α) belongs to the set

$$I^+ = \left\{(\nu, \alpha) \in \mathbb{Z}^2 \times \mathbb{N}^2 : |\omega \cdot \nu| \le \sigma\|\nu\| + \kappa\|\alpha\|\right\}, \tag{1.4}$$

or a non–resonant mode if (ν, α) belongs to the complement I^- of this set in $\mathbb{Z}^2 \times \mathbb{N}^2$. Here, σ and κ are positive constants to be chosen later, and $\|.\|$ denotes the ℓ^1 norm on $\mathbb{R}^2$, that is, $\|x\| = |x_1| + |x_2|$. Correspondingly,

$$\mathbb{I}^{\pm} H = \sum_{(\nu,\alpha)\in I^{\pm}} H_{\nu,\alpha} F_{\nu,\alpha} \tag{1.5}$$

will be referred to as the resonant $(+)$ and non–resonant $(-)$ parts of the function H. Notice that the non–resonant part of a function that only depends on the action variable is zero.

Given a Hamiltonian H, we now choose $\mathcal{U}_H$ to be the canonical transformation U_ϕ whose generating function ϕ is defined implicitly by the equation

$$\mathbb{I}^-\big(H \circ U_\phi\big) = 0\,, \qquad \mathbb{I}^+\phi = 0\,, \tag{1.6}$$

provided that a (unique) solution exists. This definition of $\mathcal{U}_H$ is similar in spirit to the one used in [4], but it is easier to implement numerically. Concerning our definition of resonant modes, we note that there are other possible choices that should lead to the same results. But if I^+ is too small (e.g. $\kappa = 0$), then the equation (1.6) becomes singular, and if I^+ is chosen too large, then $\Re$ will not be analyticity improving.

In the remaining part of this paper, we present some numerical evidence for the existence of a nontrivial fixed point of $\Re$ and discuss some of the expected properties of this fixed point.

2. Results

In order to simplify the problem of finding a nontrivial fixed point for $\Re$, we will restrict our search to a suitable subclass of Hamiltonians that is left invariant by $\Re$. In particular, we will only consider Hamiltonians that are even in q, or equivalently, invariant under the reflection $\Gamma : (q,p) \mapsto (-q,p)$. The equations (1.3) and (1.6) will be restricted accordingly, to generating functions ϕ that are odd in q, and to canonical transformations $\mathcal{U}_H = U_\phi$ that commute with Γ. We note that these parity properties of ϕ and U_ϕ follow from H being even in q, if H is close to H_0. As a further restriction, we assume that

$$H(q,p) = \omega \cdot p + h(q, \Omega \cdot p) = \omega \cdot p + \sum_{\nu,n} h_{\nu,n} \cos(\nu \cdot q)(\Omega \cdot p)^n \,, \tag{2.1}$$

with $h_{\nu,n} \in \mathbb{R}$. For the generating function of the canonical transformation $\mathcal{U}_H = U_\phi$ we make the corresponding ansatz $\phi(q,p) = \varphi(q, \Omega \cdot p)$, where $\varphi(q,z)$ is a sum of terms $\varphi_{\nu,n} \sin(\nu \cdot q) z^n$. Given such a function φ, we can turn the second part of (1.3) into an equation for a function $Z = \Omega \cdot P$, that only depends on q and $z = \Omega \cdot p$ and is even in q:

$$Z(q,z) = -\Omega \cdot \nabla_1 \varphi\bigl(q, z + Z(q,z)\bigr) \,. \tag{2.2}$$

From a solution of this equation we obtain

$$U_\phi(q,p) = \Bigl(q + \partial_2 \varphi\bigl(q, z + Z(q,z)\bigr)\Omega \,, p - \nabla_1 \varphi\bigl(q, z + Z(q,z)\bigr)\Bigr) \,, \qquad z = \Omega \cdot p \,, \tag{2.3}$$

where $\partial_2 \varphi$ denotes the partial derivative of φ with respect to the second argument. It is easy to check that if H is of the form (2.1), then the same is true (formally) for $H \circ U_\phi$.

Within this restricted class of Hamiltonians, the transformation $\Re$ involves the following steps: Given H, we first solve the equation $\mathbb{I}^-(H \circ U_\phi) = 0$ for a generating function $\phi = \mathbb{I}^- \phi$ of the type described above. After composing H with the canonical transformation $\mathcal{U}_H = U_\phi$, the new Hamiltonian is again of the form (2.1), and made up entirely of resonant modes. Some of these resonant modes are turned into non–resonant ones in the next step: $H \mapsto H \circ T_1$. After this step, we perform the scaling $H \mapsto \tau \mu^{-1} H(.,\mu.) - \epsilon$ described after (1.2). The resulting function $\Re(H)$ belongs again to our restricted class of Hamiltonians. We note that $\tau(H) = \vartheta$ for all Hamiltonians H in this class, i.e., the renormalization of time is trivial.

In our numerical implementation of these steps, we approximate functions like $h(q,z)$ or $\phi(q,z)$ by finite linear combinations of modes $\cos(\nu \cdot q) z^n$ or $\sin(\nu \cdot q) z^n$, respectively, and represent them by arrays of Fourier–Taylor coefficients. The cutoff is taken to be of the form $\|\nu\| + n \leq N$, and all intermediate results are truncated to the same fixed "degree" N. We have written subroutines for the addition, differentiation, multiplication, and composition of such functions; and similarly for the inverses of these operations, using Newton's method when necessary. We also use Newton's method to solve (1.6). The projections $\mathbb{I}^\pm$ are fixed once and for all, by choosing $\sigma = 2/5$ and $\kappa = \sigma^2$.

In order to find a good approximation for the expected nontrivial fixed point H_* of $\Re$, we started with some initial guess, and iterated (the numerical implementation of) a contraction that has the same fixed point as $\Re$. The contraction was chosen of the form $\Re \circ \mathcal{P}$, with $\mathcal{P}$ a projection onto a codimension two manifold passing through H_*, defined by the equation $p(\Re(H)) = p(H)$, for a suitably $\mathbb{R}^2$–valued function p. For the degrees $N = 7, 9, 11, 13, 15, 18$, we iterated the contraction, until reaching a Hamiltonian $H_*^{(N)}(q,p) = \omega \cdot p + h_*^{(N)}(q, \Omega \cdot p)$ that satisfies the fixed point equation for $\Re_{\text{numeric}}^{(N)}$, up to an error less than 2×10^{-15} in the norm

$$\|h\| = \sup_{\nu,n} |h_{\nu,n}| e^{2\|\nu\|} . \tag{2.4}$$

At degree $N = 7$, the initial guess was obtained by a bisection procedure, as described e.g. in [6–8]. Then $H_*^{(7)}$ was used as a starting point at degree 9, and similarly for higher degrees. As a result of this iteration procedure, we get approximate values for the Fourier–Taylor coefficients of H_*, and for the scaling parameter $\mu(H_*)$. In particular, the computed coefficients satisfy $\|h_*^{(N)}\| = 0.92\ldots$ for all N, indicating that H_* has a decent domain of analyticity. Our results for $\mu(H_*^{(N)})$ are listed in the second column of Table 1. The $N = 18$ value for the energy scaling factor ϑ/μ, when truncated to 8 digits, agrees with the value 7.0208826 obtained from a renormalization group analysis of area–preserving maps [9–12].

The fact that $\Re \circ \mathcal{P}$ was observed to be a contraction, indicates that in the space of Hamiltonians considered, $\Re$ has (at most) two unstable directions at H_*. In order to determine these directions, we computed the image of a two dimensional subspace under iteration of a numerical approximation to $D\Re(H_*)$. From the action of this map on the limiting "invariant" subspace, we computed approximate values for the two expanding eigenvalues δ_1 and δ_2 of $D\Re(H_*)$; see columns 3 and 4 of Table 1. The value of δ_2 obtained in [12] is 1.6279500. As we mentioned in the introduction, one of the eigenvalues δ_j is associated with an irrelevant degree of freedom corresponding to a translation in the action variable. Indeed, our numerical results are compatible with the expected relation $\delta_1\mu = -1/\vartheta$, involving the scaling parameter μ for the action variables.

We note that our approximate values for δ_1 and δ_2 are not as accurate as our other numerical results. The reason is that we did not actually write a program for the derivative of $\mathcal{R}$, but used instead a difference quotient with fixed increment. This introduces an error that is independent of N.

N	μ	$\delta_1 + 1/(\mu\vartheta)$	δ_2
7	0.2303767475	1.1×10^{-5}	1.6289294
9	0.2304547219	4.8×10^{-7}	1.6280322
11	0.2304600766	7.1×10^{-9}	1.6279539
13	0.2304601795	7.8×10^{-9}	1.6279508
15	0.2304601940	1.1×10^{-8}	1.6279504
18	0.2304601966	1.1×10^{-8}	1.6279503

Table 1. Scaling μ, and eigenvalues δ_j of $D\Re(H_*)$

We do not expect $D\Re(H_*)$ to have additional eigenvalues of modulus ≥ 1, when considered on a space of general Hamiltonians of the form (1.1). Support for this comes from earlier numerical work [5], where a renormalization group transformation similar to $\mathfrak{R}$ was implemented, without the restrictions considered in this section, except for the condition that $H(q,p)$ be an even function of q. At the approximate nontrivial fixed point of this transformation, no eigenvalues of modulus ≥ 1 were found, besides δ_1 and δ_2.

Next, we consider the scaling transformation $\mathcal{S}_H = \mathcal{U}_H \circ T_{\mu(H)}$ which maps orbits for $\mathfrak{R}(H)$ to orbits for H. We will assume that $\mathfrak{R}$ has a fixed point H_* of the form (2.1), close to our Hamiltonians $H_*^{(N)}$. Since $\mathcal{S}_{H_*}$ commutes with the reflection Γ and maps the energy surface $H_*(q,p) = E$ to the surface of of constant energy $(E+\varepsilon)\mu/\vartheta$, with $|\mu/\vartheta| < 1$, we expect that $\mathcal{S}_{H_*}$ has a fixed point of the form $(0, p_\infty)$. Our numerical results indicate that this is the case, and that $p_\infty \approx (0.05616861, 0.01352928)$. Since $\mathcal{U}_{H_*} \circ T_1$ is a canonical transformation and commutes with Γ, the derivative of $\mathcal{S}_{H_*}$ at the point $(0, p_\infty)$ has to be of the form

$$D\mathcal{S}_{H_*}(0,p_\infty) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \qquad A^T B = \mu \mathrm{I}, \tag{2.5}$$

where A and B are 2×2 matrices. By comparing derivatives of $\mathfrak{R}(H_*)$ and H_* at the point $(0, p_\infty)$, one also finds that A has two eigenvalues $\lambda_1 = \vartheta$ and λ_2, with eigenvectors $v_1 = \nabla_2 H_*(0, p_\infty)$ and $v_2 = \Omega$, respectively. Thus, since $A^T B = \mu \mathrm{I}$, the eigenvalues of B are $\lambda_3 = \mu/\lambda_2$ and $\lambda_4 = \mu/\vartheta$, and the corresponding eigenvectors are v_3 and $v_4 = \omega$, with v_3 perpendicular to v_1. We note that the triviality of λ_1 and v_4 is related to the fact that the geometry in the direction of the flow is trivial, and that the vector field for H_* is invariant under translations $(q,p) \mapsto (q, p - s\omega)$. The approximate values for λ_2, obtained from our approximate fixed points $H_*^{(N)}$, are listed in column 3 of Table 2. At $N = 15$ and $N = 18$, our value for $1/\lambda_2$, when truncated to 8 digits, agree with the value found in [12] for area–preserving maps.

N	$\lambda_1 - \vartheta$	λ_2	$\lambda_3 - \mu/\lambda_2$	$\lambda_4 - \mu/\vartheta$
7	1.5×10^{-16}	-0.7074647886	-1.2×10^{-4}	5.2×10^{-5}
9	1.5×10^{-16}	-0.7068366347	-7.8×10^{-6}	3.4×10^{-6}
11	1.5×10^{-16}	-0.7067965675	-1.7×10^{-7}	7.3×10^{-8}
13	1.5×10^{-16}	-0.7067957946	-2.1×10^{-8}	9.0×10^{-9}
15	1.5×10^{-16}	-0.7067956886	-2.4×10^{-11}	3.0×10^{-11}
18	1.5×10^{-16}	-0.7067956694	-2.2×10^{-10}	9.6×10^{-11}

Table 2. Scaling parameters λ_j

In order to visualize the critical scaling $\mathcal{S}_{H_*}$, we computed some trajectories for the reduced flow in the variables q and $z = \Omega \cdot p$,

$$(\dot{q}, \dot{z}) = \bigl(\omega + \partial_2 h(q,z)\Omega \,, -\Omega \cdot \nabla_1 h(q,z)\bigr)\,, \tag{2.6}$$

for the Hamiltonian $H_*^{(15)}$. The reduced phase space is considered to be $\mathbb{T}^2 \times \mathbb{R}$, that is, q is identified with $q + 2\pi\mathbb{Z}^2$. Figure 1 shows 14 orbits of the return map for a rectangle in the plane $\Pi = \{(sv, z) : s, z \in \mathbb{R}\}$, where $v = (-0.899, 1)$. The coordinates in the horizontal and vertical direction are $v \cdot q$ and z, respectively. In these coordinates, the scaling fixed point is at $(0, z_\infty)$, where $z_\infty = \Omega \cdot p_\infty = 0.0478\ldots$. The curve $\mathcal{C}$ through this point represents the intersection of the critical ω–torus with Π. We note that v is not an eigenvector of A. The vector v has been chosen in such a way that $\mathcal{C}$ has zero curvature at the point $(0, z_\infty)$. In order words, we used the flow to linearize $\mathcal{S}_{H_*}$ approximately, in the angle variable. Figure 2, which will be discussed later, indicates that for most other choices of v, $\mathcal{C}$ would have a nonzero curvature at $q = 0$. As a result, the vertical scaling off the line $v \cdot q = 0$ would be dominated by λ_2^2 instead of λ_3. The points in Figure 1, that lie on the symmetry line $v \cdot q = 0$ and are surrounded by circles, belong to periodic orbits with periods $\vartheta_5 = 13/8$, $\vartheta_6 = 21/13$, ... If a similar picture holds for the first few ϑ_k–orbits of H_*, then the corresponding points $(0, z_k)$ have to accumulate at $(0, z_\infty)$, with an asymptotic ratio $(z_{k+1} - z_\infty)/(z_k - z_\infty) \to \lambda_3 = -0.326\ldots$. And in the direction $(v, 0)$, the orbits have to scale with an asymptotic ratio equal to λ_2. These properties follow from the fact that $H_* \circ \mathcal{S}_{H_*}$ is a constant multiple of H_*, which implies (since $\mathcal{S}_{H_*}$ is canonical modulo a scaling of the actions) that if an orbit for H_* is periodic with frequency vector ν, then its image under $\mathcal{S}_{H_*}$ is another periodic orbit for H_*, with frequency vector $T^{-1}\nu$. The observed accumulation of ϑ_k–orbits also suggests that the point $(0, p_\infty)$ lies on an invariant ω–torus for H_*, and that this torus is mapped onto itself by $\mathcal{S}_{H_*}$. For corresponding results on area–preserving map, we refer to [9–12].

The results obtained here and in [5] suggest that, as far as the breakup of smooth ω–tori is concerned, the unstable manifold of the transformation $\mathcal{R}$ described after (1.3) represents the "typical" one–parameter family of Hamiltonians. On the other hand, these Hamiltonians are all of the form (2.1). Their ω–tori at different energies have to break up simultaneously, since they are just translates of each other. In order to see in what sense this may be typical, consider some "arbitrary" Hamiltonian $H(q, p)$ that is even in q and of the form (1.1). Suppose that H allows us to define a differentiable one–parameter family of area–preserving maps $E \mapsto \Psi_E$, by restricting the flow for H to surfaces $H(q, p) = E$ and choosing an appropriate Poincaré section. Assume now that H has a critical ω–torus for an energy E_∞, and that the family $E \mapsto \Psi_E$ intersects the critical manifold $\mathcal{W}$ for the area–preserving renormalization group fixed point transversally, at $E = E_\infty$. Then, according to the renormalization group picture for area–preserving maps, there is an asymptotic scaling that relates the orbits of Ψ_E to orbits of $\Psi_{f(E)}$, where $f(E_\infty + \varepsilon) = E_\infty + \delta_2\varepsilon + \mathcal{O}(\varepsilon^2)$. Since these orbits all stem from a single Hamiltonian H, just considered on different energy surfaces, one would expect to find an asymptotic scaling rate $\delta_2^{-1} = 0.614\ldots$ in the phase space of H, in a direction transversal to the surface $H(q, p) = E_\infty$. But our results yield a scaling rate $\lambda_4 = 0.142\ldots$ in this direction. This suggests that the family $E \mapsto \Psi_E$ cannot be transversal to $\mathcal{W}$. In addition, if $E \mapsto \Psi_E$ and $\mathcal{W}$ are smooth, then the fact that δ_2^{-1} is not an integer power of λ_4, seems to imply that the distance between Ψ_E and its projection onto $\mathcal{W}$ tends to zero faster than any power of $E - E_\infty$, as E approaches E_∞. In the analytic case, this would lead to the conclusion that H has either no critical torus, or else an entire family of critical tori. We have not done any numerical investigations to confirm such behavior, or to find a counter–example.

We conclude with some observations concerning the critical ω–torus for the fixed point

H_*, on the energy surface containing the point $(0, p_\infty)$. A numerical approximation of this torus, in the coordinates of the reduced flow (2.6), is shown in Figure 2. These data suggest e.g. that the critical torus of H_* is the graph of a differentiable function $p = G(q)$, with $G(0) = p_\infty$, and that 0 is a "saddle point" of the corresponding function $z = \Omega \cdot G(q)$. Notice that by symmetry, $\Omega \cdot G$ has to have a critical point at $q = 0$. The fact that the unstable eigenvector of $D\mathcal{S}_{H_*}(0, p_\infty)$ is tangent to the flow at $(0, p_\infty)$, suggests that the orbit of the point $(0, p_\infty)$ coincides with the unstable manifold of $\mathcal{S}_{H_*}$ at this point, and this in turn suggests that the critical ω–torus is an attractor for the scaling map $\mathcal{S}_{H_*}$. In order to obtain information about the regularity of this torus, it is useful to look at periodic points of $\mathcal{S}_{H_*}$. Consider e.g. the fixed point $(0, p_*)$. Our earlier discussion of $\mathcal{S}_{H_*}$ suggests that the degree of differentiability of the critical ω–torus at this point is no more than $\ln|\lambda_3|/\ln|\lambda_2| = 3.229\ldots$. The same holds of course for every point on the orbit of $(0, p_*)$ under the flow defined by H_*. But there are other (dense sets of) points where the degree of differentiability is expected to be less than 2. Consider e.g. the set of points on $\mathbb{T}^2$ that are invariant under the reflection $q \mapsto -q$. This set consists of $\gamma_0 = (0, 0)$, and three other points $\gamma_1, \gamma_2, \gamma_3$, whose coordinates are both either 0 or π, modulo 2π. Notice that, according to Figure 2, these are precisely the critical points of $\Omega \cdot G$. It is easy to see that $\mathcal{U}_{H_*}$ leaves the planes $\Gamma_i = \{\gamma_i\} \times \mathbb{R}^2$ invariant. Thus, since T acts as a permutation on $\{\gamma_1, \gamma_2, \gamma_3\}$, the corresponding planes Γ_i are permuted by $\mathcal{S}_{H_*}$. Assuming that the critical ω–torus is invariant under $\mathcal{S}_{H_*}$, this implies that $\{G(\gamma_1), G(\gamma_2), G(\gamma_3)\}$ is an orbit of period 3 for $\mathcal{S}_{H_*}$. An estimate on the degree of differentiability of the critical torus at $G(\gamma_1)$ can now be obtained by linearizing $\mathcal{S}^3_{H_*}$ at this point, and then computing the analogue of the ratio $\ln|\lambda_3|/\ln|\lambda_2|$. Based on observations for area–preserving maps [10,11], we expect the value of this ratio to be $1.790\ldots$, but we did not verify this numerically. We note that for every periodic orbit of T on the torus $\mathbb{T}^2$, there is a corresponding periodic orbit of the transformation $\mathfrak{R}$. This is a consequence of the following property of $\mathfrak{R}$: Assuming that the equation (1.6) has a unique solution, it is easy to verify that $\mathfrak{R} \circ \mathcal{J}_\gamma = \mathcal{J}_{T^{-1}\gamma} \circ \mathfrak{R}$, where $\mathcal{J}_\gamma$ denotes the rotation $H \mapsto H(. - \gamma, .)$.

3. Conclusion

The renormalization group transformation $\mathfrak{R}$ introduced in this paper can be used to study, and possibly explain, the breakup of golden invariant tori in Hamiltonian systems with two degrees of freedom. We find an approximate numerical fixed point for $\mathfrak{R}$, whose critical indices and scaling parameters are in excellent agreement with the corresponding values for area–preserving maps. The same methods are expected to work for other reduced quadratic irrationals, with similar results. Two distinguishing features of our transformation are that it applies to Hamiltonians of a general type, and that it seems to be analyticity improving. On a conceptual level, this indicates that all irrelevant degrees of freedom are being eliminated under the iteration of $\mathfrak{R}$. Numerically, the analyticity improving property allows for high accuracy, as truncation errors decrease exponentially in the degree of the truncation. Our results not only indicate that Hamiltonians with critical invariant ω–tori are asymptotically self–similar, but they also provide detailed geometric information about the relevant scalings and normal forms. Further investigations will be necessary in order to determine the implications of these findings on the mechanisms underlying the breakup of invariant tori. Another project for future research is to carry out a computer–assisted proof for some of the results presented in this paper.

Acknowledgments
We would like to thank R. de la Llave for helpful discussions.

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Figure 1

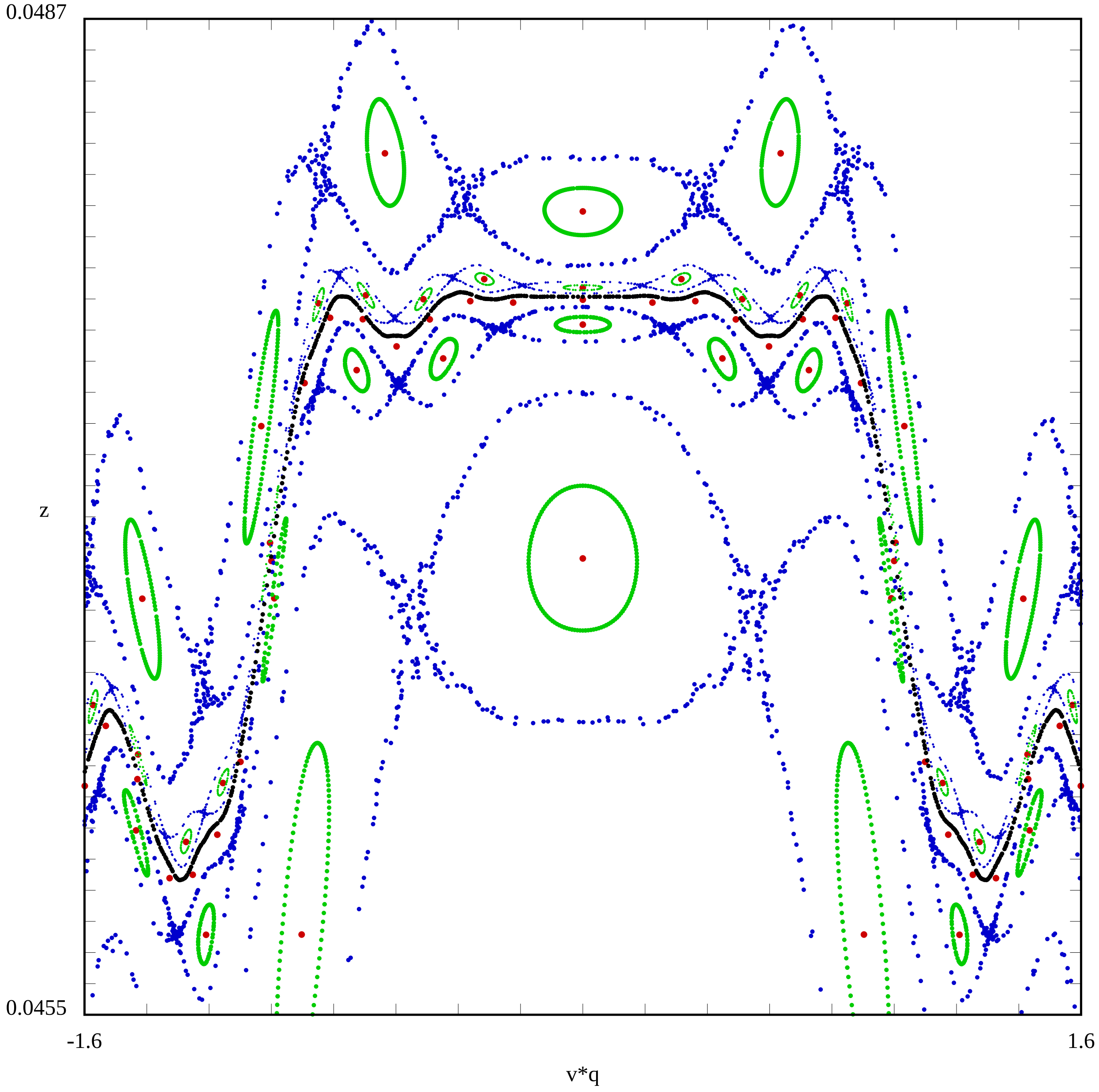

Figure 2