Renormalization and Shearless Invariant Tori: Numerical Results

Denis Gaidashev
Department of Mathematics, University of Toronto
Toronto, Ontario, Canada M5S 3G4

Hans Koch
Department of Mathematics, University of Texas at Austin
Austin, TX 78712

Abstract. We present some numerical evidence for universality associated with the breakup of shearless invariant tori, by studying a renormalization group transformation acting on an appropriate space of Hamiltonians.

1. Introduction

Shearless tori appear in many physical systems, where they play an important role as transport barriers. This includes models of the atmosphere, toroidal plasma devices, fluid dynamics, and others [1–6]. In the context of this paper, an invariant torus of a Hamiltonian system with two degrees of freedom is called *shearless*, if its rotation number is a local maximum or minimum. Such tori are observed to be surprisingly stable. Some bifurcations involving shearless tori have been described in [11–18]. At the other end of the spectrum are systems that satisfy a twist condition, which guarantees that the rotation number varies monotonely from one torus to the next. In both types of system, some of the most interesting questions relate to the breakup of invariant tori and the resulting loss of stability.

The twist case has been studied extensively, and for quadratic irrational rotation numbers such as the golden mean $\vartheta = \frac{1}{2} + \frac{1}{2}\sqrt{5}$, all of the evidence indicates that the breakup of invariant tori involves self-similar structures and universal scaling ratios. There has been considerable success in describing the underlying mechanisms with renormalization group methods [19–30]. But shearless tori are different in many respects. Visually, a shearless torus near breakup separates two totally chaotic looking regions, while in the twist case, elliptic islands dominate a non-trivial fraction of phase space. It seems surprising that the breakup of such tori should also be governed by universality and scaling. Nevertheless, this appears to be what happens. In the case of the golden mean and related rotation numbers, numerical investigations of specific two-parameter families [7–10] reveal self-similarity phenomena, with asymptotic scaling ratios (both in parameter space and phase space) that seem to be independent of the family considered. Furthermore, the self-similarity transformation involves as 12-step shift in the sequence of continued fraction approximants for ϑ, as opposed to the ordinary 1-step shift describing the similarity of periodic orbits in the twist case.

Our goal is to provide some evidence of universality for these phenomena, by relating the observed scaling ratios to properties of a renormalization group (RG) transformation,

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near a fixed point of its 12-th iterate. We will use a RG transformation $\mathcal{R}$ of the type introduced in [26]. These transformations have been studied mostly in connection with systems that satisfy a twist condition [27–30]. But as was shown in [18], they apply equally well to the study of isoenergetically degenerate Hamiltonians and shearless invariant tori.

$\mathcal{R}$ will be chosen in such a way that it has a "trivial" period two, consisting of Hamiltonians of the form

$$H^0_\gamma(q,p) = \omega\cdot p + \gamma(\Omega\cdot p)^3\,, \qquad q\in\mathbb{T}^2\,, \quad p\in\mathbb{R}^2\,, \tag{1.1}$$

where $\omega = (\vartheta^{-1},1)$ and $\Omega = (1,-\vartheta^{-1})$. We recall that the flow associated with a Hamiltonian $H(q,p)$ is given by $\dot q = \nabla_2 H$ and $\dot p = -\nabla_1 H$, where $\nabla_n H$ denotes the partial gradient of H with respect to its n-th argument. In the case of H^0_γ, the torus $p = p_0$ is invariant for every value of p_0, and the angular velocity $\dot q = \omega + 3\gamma(\Omega\cdot p)^2\Omega$ is constant on this torus. Thus, the rotation number is simply $\dot q_1/\dot q_2$. The torus for $p = 0$ is shearless: Its rotation number ϑ^{-1} is maximal if $\gamma\le 0$, or minimal if $\gamma\ge 0$, among the rotation numbers of nearby orbits.

In what follows, we consider Hamiltonians on $\mathbb{T}^2\times\mathbb{R}^2$ of the form

$$H(q,p) = \omega\cdot p + \sum_{(\nu,k)\in I} H_{\nu,k}e^{i\nu\cdot q}z^k\,, \qquad z = \Omega\cdot p\,, \tag{1.2}$$

where $I = \mathbb{Z}^2\times\mathbb{N}$. We assume that H takes real values for real arguments. The renormalization of H involves the composition with a "scaling" of phase space $T_\mu(q,p) = \bigl(Tq,\mu T^{-1}p\bigr)$, where μ is some nonzero real number, and $T = \left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]$ is a matrix related to the golden mean: its eigenvalues are ϑ and $-\vartheta^{-1}$, and the corresponding eigenvectors are ω and Ω, respectively. Our RG transformation $\mathcal{R}$ is of the form

$$\mathcal{R}(H) = \frac{\vartheta}{\mu}H'\circ\mathcal{U}_{H'}\,, \quad H' = H\circ T_\mu\,. \tag{1.3}$$

where $\mathcal{U}_{H'}$ is a canonical transformation described below, and where μ is a normalization constant depending on H, which is chosen in such a way that the coefficient of $(\Omega\cdot p)^3$ in the renormalized Hamiltonian $\mathcal{R}(H)$ is equal to $\pm\frac{1}{2}$, if possible. The change of variables $H'\mapsto H'\circ\mathcal{U}_{H'}$ can also be regarded as a normalization procedure, but one that involves infinitely many normalization conditions. This feature of $\mathcal{R}$ arises from the need of working with periodic variables.

To be more precise, we note that the scaling T_μ has an eigenvalue $\vartheta > 1$. (The other eigenvalues are of modulus < 1, for the values of μ that appear in our analysis.) As a result, the rescaled Hamiltonian $H\circ T_\mu$ is less regular than H in the direction of the corresponding eigenvector, which is $(\omega,0)$. It needs to be re-normalized, in order for $\mathcal{R}$ to define a dynamical system on some fixed space of Hamiltonians. The modes $e^{i\nu\cdot q}z^k$ that are negatively affected by the scaling T_μ are those for which n is relatively small and ν has a significant component in the expanding direction ω of T. These modes will be referred to as "non-resonant". To be more precise, let

$$I^+ = \bigl\{(\nu,k)\in I:\ |\omega\cdot\nu|\le\sigma|\Omega\cdot\nu| \text{ or } |\omega\cdot\nu|\le\kappa k\bigr\}\,, \tag{1.4}$$

where σ and κ are fixed positive real numbers that will be specified later. The resonant part $\mathbb{I}^{+}H$ of a Hamiltonian H is defined by restricting the sum in equation (1.2) to the index set I^{+}, and the non-resonant part is defined as $\mathbb{I}^{-}H = H - \mathbb{I}^{+}H$.

In order to compensate for the abovementioned loss of regularity in the scaling $H \mapsto H' = H \circ T_\mu$, we will determine $\mathcal{U}_{H'}$ in such a way that

$$\mathbb{I}^{-}\bigl(H' \circ \mathcal{U}_{H'}\bigr) = 0\,, \tag{1.5}$$

if possible. In other words, the change of variables $H' \mapsto H' \circ \mathcal{U}_{H'}$ is designed to eliminate the non-resonant modes of H'. The solution of equation (1.5) is in general not unique. In our computations, we choose for $\mathcal{U}_{H'}$ a canonical transformation of the form

$$U_\phi(q,p) = \Bigl(q + \partial_2\varphi\bigl(q, z + Z(q,z)\bigr)\Omega\,, p - \nabla_1\varphi\bigl(q, z + Z(q,z)\bigr)\Bigr)\,, \tag{1.6}$$

associated with a non-resonant generating function $\phi(q,p) = \varphi(q,z)$, where $z = \Omega \cdot p$, and where Z is defined implicitly by the equation

$$Z(q,z) = -\Omega \cdot \nabla_1\varphi\bigl(q, z + Z(q,z)\bigr)\,. \tag{1.7}$$

This makes (1.5) an equation for ϕ. Our approach is to solve this equation by Newton's method, starting with 0 as an initial guess.

We note that if H' is purely resonant, then $\mathcal{U}_{H'}$ is the identity. Examples of such Hamiltonians are functions $H(q,p)$ that only depend on the action variable p. Thus, it straightforward to verify e.g. that the two Hamiltonians H^0_γ, with $\gamma = \pm\frac{1}{2}$, form an orbit of period 2 for $\mathcal{R}$. This period is hyperbolic, with one unstable direction (not counting constant Hamiltonians) represented by the function $(q,p) \mapsto \Omega{\cdot}p$, and expanding eigenvalue $-\vartheta^2$.

2. Results

The RG transformation $\mathcal{R}$ can be defined for a more general class of Hamiltonians than the ones considered here; see e.g. [26]. However, the subclass of Hamiltonians of the form (1.2) is invariant under $\mathcal{R}$, and numerical experiments suggest that families of such Hamiltonians exhibit the phenomena that we are interested in. For the same reason, we now restrict our analysis to Hamiltonians H satisfying $\mathcal{J}H = -H$, where $\mathcal{J}H = H \circ J$ and $J(q,p) = (-q,-p)$. Formally, this symmetry also ensures that an invariant torus passing through the origin is shearless. By requiring that our generating functions satisfy $\mathcal{J}\phi = \phi$, we find that U_ϕ and $\mathcal{R}$ commute with J and $\mathcal{J}$, respectively.

In our numerical implementation of $\mathcal{R}$, we approximate functions like H and ϕ by finite linear combinations of all modes $\cos(\nu \cdot q)z^k$ and $\sin(\nu \cdot q)z^k$ below a certain cut-off, and represent them as arrays of Fourier–Taylor coefficients. (Roughly half of them are zero, due to the abovementioned symmetry.) The cut-off used is of the form

$$|\nu_1| + |\nu_2| \le N_1\,, \quad k \le N_2\,, \quad |\nu_1| + |\nu_2| + k \le N_3\,, \qquad N \in \mathbb{N}^3\,. \tag{2.1}$$

We have written procedures (in the programming language Ada95) for the the addition, differentiation, multiplication, and composition of such functions. Implicit equations like (1.7) and (1.5) are solved numerically by using Newton's method. Our numerical implementation of $\mathcal{R}$ is obtained by combining these steps, according to the definition of $\mathcal{R}$. In what follows, we will not always distinguish between $\mathcal{R}$ (or other objects) and its numerical implementation (or approximation).

As mentioned earlier, $\mathcal{R}^2$ has one unstable direction at the trivial fixed points H^0_γ. (Here, and in what follows, $\gamma = \pm\frac{1}{2}$.) This unstable direction corresponds to a change of rotation number for the orbit passing through the origin. At the nontrivial fixed points of $\mathcal{R}^{12}$, we expect one additional unstable direction, corresponding to a change of stability of this orbit. In order to find these fixed points, it is desirable to use modified RG transformations that have no (or fewer) unstable directions. Our construction of such transformations is based on the following principle. Consider a map $\mathcal{F}$ on $\mathbb{R}^n$, that has a hyperbolic fixed point at x_0, with $d > 0$ unstable directions. Let p be a $\mathbb{R}^d$-valued function on the domain of $\mathcal{F}$, such that the equation $p(\mathcal{F}(x)) = p(x)$ defines a manifold Σ of codimension d near x_0. Let $\mathcal{P}$ be a projection in $\mathbb{R}^n$ onto Σ, leaving points on Σ fixed, and define $\mathcal{M} = \mathcal{F} \circ \mathcal{P}$. If p is chosen in such a way that Σ coincides with the local stable manifold of F at x_0, then x_0 is clearly a stable fixed point for $\mathcal{M}$. But since the property of having a stable fixed point is preserved under small perturbations of the map, we can expect the same to hold for other "reasonable" choices of p, and compatible projections $\mathcal{P}$.

We have used a procedure of this type, with $d = 1$, to define a map $\mathcal{M}_1 = \mathcal{R}^{12} \circ \mathcal{P}$ for which H^0_γ is a stable fixed point. This map was then used in a bisection procedure, along lines of Hamiltonians passing through H^0_γ, to search for points on the boundary of the basin of attraction for H^0_γ. Ideally, this boundary coincides in some regions with the codimension one stable manifold of a non-trivial fixed point for $\mathcal{M}_1$ that is also a fixed point of $\mathcal{R}^{12}$. This task turned out to be extremely tedious. For the vast majority of Hamiltonians considered, the numerical procedure broke down before reaching the 12-th iterate of $\mathcal{R}$. And for most of the others, the same happened during subsequent iterations of $\mathcal{M}_1$, which should bring a "critical" Hamiltonian closer to the expected fixed point. (We also used Hamiltonians whose criticality was determined by investigating their dynamics.) But for some of the better Hamiltonians, $\mathcal{R}^{n+6}(H)$ was noticeably similar to $\mathcal{R}^n(H)$, modulo a rotation $\mathcal{V}_\alpha$, where

$$\mathcal{V}_\alpha H = H \circ V_\alpha\,, \qquad V_\alpha(q,p) = (q + \pi\alpha, p)\,,$$

with α a vector from $A = \{(1,0),(0,1),(1,1)\}$. Thus, we continued our search with $\mathcal{N} = \mathcal{V}_\alpha \circ \mathcal{R}^6$ in place of $\mathcal{R}^{12}$. It should be noted that $\mathcal{N}^2 = \mathcal{R}^{12}$. This follows from an explicit computation [26], which shows that for any α,

$$\mathcal{R} \circ \mathcal{V}_\alpha = \mathcal{V}_{T^{-1}\alpha} \circ \mathcal{R}\,.$$

Our next goal was to get closer to the expected fixed point H_0 of $\mathcal{N}$, by iterating another map $\mathcal{M}_2$ of the form $\mathcal{N} \circ \mathcal{P}$, this time with $d = 2$, designed to have H_0 as a stable fixed point. As starting points for this iteration, we used Hamiltonians obtained with the abovementioned bisection procedure. This task turned out to be much more difficult than expected. (Some possible reasons will be mentioned later.) All computations, including the bisection procedure, were repeated for various choices of the constants σ and κ defining the projections $\mathbb{I}^{\pm}$. The values that produced the best convergence, at the degrees considered, are $\sigma = 1.1$ and $\kappa = \sigma/1.5$. Thus, we will consider these values fixed from now on. One of the intermediate results of this iteration was that certain Fourier–Taylor coefficients of our Hamiltonians approached zero, indicating that each of the Hamiltonians $H_n = \mathcal{R}^n(H_0)$ in the expected period 12 for $\mathcal{R}$ is invariant under a reflection $\mathcal{J} \circ \mathcal{J}_\alpha$,

$$\mathcal{J}_\alpha H = H \circ J_\alpha \,, \qquad J_\alpha(q,p) = (q + \pi\alpha, -p)\,,$$

for some $\alpha = \alpha(n)$ in A. One of the other transformations of this type maps H_n to H_{n+6} .

Interestingly, if we assume the existence of such a periodic orbit for $\mathcal{R}$, and rotate H_n by $\frac{1}{2}\pi\alpha(n)$, for each n, we obtain another orbit $\{H^n\}$ of period 12 for $\mathcal{R}$. Each of these new Hamiltonians is of the form

$$H(q,p) = \omega \cdot p \pm \frac{1}{3}z^3 + \sum_{(\nu,k)\in I^+} h_{\nu,k}\cos(\nu \cdot q)z^k\,. \tag{2.2}$$

In addition to being an even function of q, the Hamiltonian H^n satisfies $\mathcal{J}_\alpha H^n = -H^n$, with $\alpha = \alpha(n)$. In terms of its Fourier–Taylor coefficients, this means that $h^n_{\nu,k} = 0$ whenever $\nu \cdot \alpha + k$ is even. We note that the symmetries of a periodic RG orbit like $\{H^n\}$ reflects an asymptotic symmetry of every Hamiltonian in the corresponding universality class. Thus, when studying the breakup of shearless golden tori, such symmetries can be imposed without loss of generality, which greatly simplifies the analysis.

The bisection procedure was performed mostly at degrees $N = (7, 7, 9)$, that is, with a cutoff (2.1) corresponding to this vector N. The subsequent iterations of $\mathcal{M}_2$ were carried out first at the same degree, and later for successively higher degrees, using as input the approximate fixed point from previous computations. Once a certain accuracy was reached, it became possible to replace $\mathcal{M}_2$ by a computationally much more efficient (but less stable) transformation: a Newton-like map, with a fixed linear approximation for the derivative $D\mathcal{N}(H)$.

In addition, we have also computed the largest two eigenvalues δ_1 and δ_2 of the derivative $D\mathcal{N}(H_0)$, by following the orbit of a two dimensional subspace under a difference quotient approximation to $D\mathcal{N}$. Table 1 lists the sixth roots of these eigenvalues, together with the average scaling $\bar{\mu}$, defined by $\bar{\mu}^6 = \mu(H_0)\mu(H_1)\cdots\mu(H_5)$. If the renormalization group picture is correct, then the eigenvalues δ_1 and δ_2 should describe the accumulation of bifurcation points in two-parameter families of Hamiltonians, near a "critical point" at the end of a line of Hamiltonians possessing a shearless invariant torus, with rotation number ϑ. Such a bifurcation analysis was carried out in [8] and [9]. For the ratios corresponding to $\bigl(\delta_1^{1/6}, \delta_2^{1/6}\bigr)$, the values obtained in these two references are (2.683, 1.511) and (2.678, 1.583), respectively.

N	$\bar{\mu}$	$\delta_1^{1/6}$	$\delta_2^{1/6}$
(7, 7, 9)	0.36546	2.6620	1.5850
(12, 8, 14)	0.36594	2.6613	1.5850
(16, 8, 18)	0.36589	2.6612	1.5850

Table 1. Scaling $\bar{\mu}$, and eigenvalues δ_j of $D\mathcal{N}(H_0)$

One notable fact about all these numbers is that they are not nearly as accurate as the corresponding data obtained in the twist case [22,27]. In experiments on two-parameter families, this is largely due to the fact that 12 bifurcations are involved at each level of self-similarity, which limits the number of levels that can be studied at a given numerical precision. An analogous complication in our analysis is that the RG orbits pass through 6 very different regions of function space. In order to renormalize Hamiltonians accurately, it is necessary to adapt the procedure to the region considered. It cannot be expected that the same domains and normal forms (determined by the choice of $\mathbb{I}^{\pm}$) are appropriate along the entire orbit of a critical Hamiltonian. But in the analysis presented here, which is only a first step, the projections $\mathbb{I}^{\pm}$ have been kept fixed. This makes the condition (1.5) effectively much more restrictive in some regions than in others. In two of the six regions (depending on the choice of σ and τ), we obtain canonical transformations $\mathcal{U}_{H'}$ that are clearly incompatible with the domains of the Hamiltonians H', as determined by the decay rate of the Fourier–Taylor coefficients. Furthermore, the determinant of their derivative is around 1.006, instead of 1, independently of the degrees considered. Thus, we estimate that there is a systematic error in our data of the order of 1 to 2 percent. A measure of such an error can be seen in the fourth column of Table 2. We believe that working with fixed projections $\mathbb{I}^{\pm}$ is the main source of these errors, but there may be others. Some of the difficulties may also be due to the fact that the RG transformation considered here has two non-trivial unstable directions, as opposed to one in the twist case. Among other things, this increases the number of near-resonances between eigenvalues, which is a well known source of numerical problems.

There are additional universal quantities, associated with $\mathcal{R}$ and its orbit $\{H_n\}$ of period 12, that can be observed in experiments with two-parameter families. (We assume of course that such an orbit exists, close to our approximate orbit and with the same symmetries.) From the fixed point equation for $\mathcal{R}^{12}$ we have $(\vartheta/\bar{\mu})^{12} H_n \circ \Lambda_n = H_n$, where Λ_n is a composition of 12 maps of the form $T_\mu \circ \mathcal{U}_{H'}$. Notice that the scaling transformations Λ_n are all conjugate to each other. Under the action of Λ_n, an orbit of H_n with frequency vector w is mapped to an orbit of H_n whose frequency vector is a constant multiple of $T^{-12}w$. Thus, assuming that H_n has a unique invariant torus (at some fixed energy) with frequency vector parallel to ω, this torus has to pass through the fixed points of Λ_n. In addition, the set of periodic orbits whose rotation numbers are the diophantine approximants for ϑ^{-1}, should accumulate at any fixed point of Λ_n, at rates determined by the eigenvalues of $D\Lambda_n$ at that fixed point.

Some fixed points of $D\Lambda_n$ can be determined by symmetries alone and thus will be referred to as “symmetry points”. One of them is the origin, since all of our canonical transformations commute with the reflection $J : x \mapsto -x$ and leave $\omega \cdot q$ invariant. We have computed one of the the matrices $D\Lambda_n(0)$ and its four eigenvalues λ_j. Table 2 lists our numerical value for the 12-th root of λ_2, and for three quantities that should be equal to zero, due to known relations among the eigenvalues. Two of these relations come from the fact that Λ_n is the composition of canonical transformations and a scaling $(q,p) \mapsto (q, \bar{\mu}^{12}p)$. As a consequence, the eigenvalues of $D\Lambda_n(0)$ come in pairs whose product is equal to $\bar{\mu}^{12}$. The identity $\lambda_1 = \vartheta^{12}$ expresses the fact that an invariant torus is smooth in the direction of the flow. The relations involving λ_1 are satisfied accurately in our approximations, due to the special form (1.6) of the transformations $\mathcal{U}_{H'}$. But as commented on earlier, we find a 1.7% deviation from the identity $(\lambda_2\lambda_3)^{1/12} = \bar{\mu}^{12}$, which is essentially independent of N (we checked this for higher degrees as well).

N	$\lambda_1^{1/12}/\vartheta - 1$	$\lambda_2^{1/12}$	$(\lambda_2\lambda_3)^{1/12}/\bar{\mu} - 1$	$\vartheta\lambda_4^{1/12}/\bar{\mu} - 1$
(7, 7, 9)	5.2×10^{-12}	0.65643	0.016	-4.1×10^{-11}
(12, 8, 14)	5.2×10^{-12}	0.65704	0.017	-2.2×10^{-11}
(16, 8, 18)	5.2×10^{-12}	0.65694	0.017	5.6×10^{-11}

Table 2. Scaling parameters λ_j at a symmetry point

We note that J has three other fixed points on $\mathbb{T}^2 \times \mathbb{R}$ besides the origin: the vectors $(\pi\alpha, 0) = J_\alpha(0)$ with $\alpha \in A$. They are all fixed points of Λ_n as well. The eigenvalues of $D\Lambda_n$ at these points agree, since $J_\alpha^{-1} \circ \Lambda_n \circ J_\alpha$ is either Λ_n or Λ_{n+6}, due to a symmetry mentioned earlier.

In addition, Λ_n has most likely one additional fixed point (and its image under J) that depends on n. This case is described best in the coordinates in which H_n is represented by a Hamiltonian H^n of the form (2.2). The scaling Λ_n in these coordinates will be denoted by S_n. Since $H^n(q,p)$ is invariant under the reflection $q \mapsto -q$, a closed orbit for H^n whose rotation number is a ratio of odd integers, if unique, will have to intersect the symmetry plane $q = 0$. But if such an orbit exists close to the golden invariant torus, then there exists an infinite sequence of them, with rotation numbers approaching $1/\vartheta$. Their intersection with the plane $q = 0$ should accumulate at a point $(0, p_n)$ on the golden invariant torus, and if S_n is analytic as expected, then $(0, p_n)$ has to be a fixed point of S_n. Here, we have also used that S_n leaves the plane $q = 0$ invariant, which follows from the fact that Hamiltonians of the form (2.2) lead to generating functions $\phi(q,p)$ that are odd in q. This situation is in fact similar to the one encountered in the twist case in [27]. It would be interesting to determine the point $(0, p_n)$ and the eigenvalues of $DS(0, p_n)$ from our numerical data, but we have not yet done so. The analogue of λ_2 is expected to be the accumulation rate of orbits along a "symmetry plane". The numerical value determined in [9] for this rate is 0.6035.

Our approximations for the Hamiltonians H_n can be found in [31]. For a more qualitative impression, we have visualized some orbits of two Hamiltonians. The first Hamiltonian, H, is our numerical approximation to the fixed point H_0 of $\mathcal{R}^{12}$. Figure 1 shows orbits for the return map for H to the cylinder $q_2 = 0$ at energy zero. Only orbits in the lower "half" of this cylinder are shown. The topmost (shearless) orbit passes thorough the origin and has rotation number $0.604\ldots$, instead of $\vartheta^{-1} = 0.618\ldots$, as expected for H_0. Also shown is the golden curve with rotation number ϑ, and adjacent elliptic periods 5/8 and 8/13. The orbits in the lower part of Figure 1 appear to be chaotic. Clearly, H is still quite far from having a critical golden torus with rotation number ϑ. For comparison, Figure

2 shows the orbits for such a critical Hamiltonian. The Hamiltonian used is of the form $H+f$, where f is a linear combination of the two expanding eigenvectors of $D\mathcal{R}^{12}(H)$, determined in such a way that $H+f$ has a critical golden torus with rotation number ϑ. The necessary correction f is quite large: roughly a quarter of the size of H, measured in some appropriate norm.

Some readers may wonder why the number 12 appears in connection with shearless golden invariant tori. We would like to argue that this is related to an asymptotic symmetry near a shearless invariant torus, which exchanges the two regions separated (for fixed energy) by this torus. If H is a fixed point of $\mathcal{R}^m$ for some $m>0$, then H should be odd under composition with such a symmetry S. Another plausible symmetry of H is reversibility of its flow; so for definiteness, let us assume that H is an even function of q. In order to keep the symmetry group as small as possible, assume furthermore that S commutes with $R: q \mapsto -q$. Formally, the golden invariant torus Γ of H satisfies $\Gamma \circ T^m = \Lambda \circ \Gamma$, where Λ is composed of m maps of the form $\mathcal{T}_\mu \circ \mathcal{U}_{H'}$. With our definition of $\mathcal{R}$, the scaling Λ commutes with R. If Γ is assumed to be one-to-one, then R leaves precisely four torus-points invariant: $\Gamma(0)$, and $\Gamma(\pi\alpha)$, for $\alpha \in A$. The first is also a fixed point of Λ, and the other three are either fixed points or a period 3, depending on whether m is a multiple of 3 or not. Here, we have used that on the invariant torus, $\Lambda = \Gamma \circ T^m \circ \Gamma^{-1}$. Assume now that Λ commutes with S as well. (This seems reasonable in the critical case, where Λ has to leave a nontrivial set of periodic orbits invariant.) Then $S\Gamma(0)$ is a fixed point of R, and thus agrees with $\Gamma(0)$ or one of the points $\Gamma(\pi\alpha)$. The first possibility would imply trivial scaling at $\Gamma(0)$, and thus can be ruled out. In the remaining cases, each $\Gamma(\pi\alpha)$ has to be fixed point of Λ. This requires that m is a multiple of 3. In addition, m has to be even, since the shearless golden tori of H and $\mathcal{R}(H)$ cannot both have a locally maximal (or maximal) rotation number. Thus, m should be a multiple of 6. Another factor of 2 might be obtained with assumptions that bring us closer to the symmetries of our Hamiltonians H^n, given that 12 is the order of the group generated by the symmetries J_α and R.

Similar arguments can be used to guess the length of the critical periodic orbit (assuming it exists) for other quadratic irrational rotation numbers, using the integer matrix that replaces T in the corresponding RG transformation. In particular, for $\vartheta = (\sqrt{n^2+4}+n)/2$ we would expect to find a period 6 or 12 if n is odd, and a period 4 or 8 if n is even. This applies of course to any rotation number $\eta \in \mathbb{Q}[\vartheta]$, since we can always perform a change of variables, associated with a matrix in $GL(2,\mathbb{Z})$ that maps $(\eta,1)$ to a constant multiple of $(\vartheta^{-1},1)$.

3. Conclusion

The scaling phenomena observed in [7–9] suggest that the breakup of shearless invariant tori is governed by a periodic RG orbit of length 12. Our goal was to lend support to this hypothesis, by constructing such a periodic orbit and analyzing its properties. To this end, we chose an RG transformation $\mathcal{R}$ of the type introduced in [26], which acts on Hamiltonians with two degrees of freedom. By extending techniques used in the case of shear flows [27], it was indeed possible to find an approximate period 12 for $\mathcal{R}$. The analysis of the shearless case turned out to be much more difficult, and the numerical results are correspondingly less accurate. Still, our data are comparable in accuracy, and consistent, with the ones obtained in [8,9]. Thus, our results represent the first real evidence that the breakup of shearless golden tori is universal, in the sense that the observed asymptotic ratios are independent of the system considered. They also indicate that the period 12 for $\mathcal{R}$ is invariant under a certain group of symmetries. This symmetry should be universal as well, and thus reflected in the structure of shearless golden tori and nearby orbits, at criticality. Further investigations will be needed to resolve some of the accuracy problems, and to develop a better understanding of the mechanisms underlying the breakup of shearless invariant tori.

References

[1] W.T. Kyner, *Rigorous and formal stability of orbits about an oblate planet.* Mem. Am. Math. Soc. **81**, 1–27 (1968).

[2] D. Del-Castillo-Negrete and P.J. Morrison, *Chaotic transport by Rossby waves in shear flow.* Phys. Fluids A **5**, 948–965 (1993).

[3] D. Del-Castillo-Negrete, *Dynamics and Transport in Rotating Fluids and Transition of Chaos in Area Preserving Non-twist Maps.* Ph.D. Thesis, University of Texas, Austin, TX (1995).

[4] R. Balescu, *Hamiltonian nontwist maps for magnetic field lines with locally reversed shear in toroidal geometry.* Phys. Rev. E **58**, 3781–3792 (1998).

[5] W. Horton, H.-B. Park, J.-M. Kwon, D. Strozzi, P.J. Morrison and D.-I. Choi, *Drift wave particle transport in reversed shear profile.* Phys. Plasma **5**, 3910–3917 (1998).

[6] C. Chandre, D. Farrelly and T. Uzer, *Threshold to chaos and ionization for the hydrogen atom in rotating fields.* Phys. Rev. A **65**, 053402 1–10 (2002).

[7] D. Del-Castillo-Negrete, J.M. Greene, and P.J. Morrison, *Area preserving non-twist maps: Periodic orbits and transition to chaos.* Physica D **91**, 1–23 (1996).

[8] D. Del-Castillo-Negrete, J.M. Greene, and P.J. Morrison, *Renormalization and transition to chaos in area preserving non-twist maps.* Physica D **100**, 311–329 (1997).

[9] A. Apte, A. Wurm and P.J. Morrison, *Renormalization and destruction of* $1/\gamma^2$ *tori in the standard nontwist map.* Chaos **13**, 421–433 (2003).

[10] A. Apte, A. Wurm and P.J. Morrison, *Renormalization for breakup of invariant tori.* Preprint `mp_arc 03-466` (2003).

[11] J.P. Van der Weele, T.P. Valkering, H.W. Kapel and T. Post, *The birth of twin Poincaré–Birkhoff chains near* $1:3$ *resonance.* Physica A **153**, 283–294 (1988).

[12] J.P. Van der Weele and T.P. Valkering, *The birth process of periodic orbits in non-twist maps.* Physica A **169**, 42–72 (1990).

[13] C. Simó, *Invariant curves of analytic perturbed nontwist area preserving maps.* Reg. Chaot. Dyn. **3**, 180–195 (1998).

[14] H.R. Dulling, J.D. Meiss and D. Sterling, *Generic twistless bifurcations.* Nonlinearity **13**, 203–224 (2000).

[15] A. Delshams and R. de la Llave, *KAM theory and a partial justification of Greene's criterion for non-twist maps.* SIAM J. Math. Anal. **31**, 1235–1269 (2000).
[16] E. Petrisor, *Nontwist area preserving maps with reversing symmetry group.* Int. J. Bifurcation Chaos Appl. Sci. Eng. **11**, 497–511 (2001).
[17] J. Franks and P. Le Calvez, *Regions Of Instability For Non-Twist Maps.* Erg. Theor. Dyn. Syst. **23**, 111–141 (2003).
[18] D. Gaidashev, *A Renormalization Group Fixed Point Associated with the Breakup of Golden Invariant Tori,* Preprint `mp_arc 03-542` (2003).
[19] D.F. Escande and F. Doveil, *Renormalisation Method for Computing the Threshold of the Large Scale Stochastic Instability in Two Degree of Freedom Hamiltonian Systems,* J. Stat. Phys. **26**, 257–284 (1981).
[20] S.J. Shenker and L.P. Kadanoff, *Critical Behavior of a KAM Surface. I. Empirical Results,* J. Stat. Phys. **27**, 631–656 (1982).
[21] R.S. MacKay, *Renormalisation in Area Preserving Maps,* Thesis, Princeton (1982). World Scientific, London (1993).
[22] R.S. MacKay, *Renormalisation Approach to Invariant Circles in Area–Preserving Maps,* Physica D **7**, 283–300 (1983).
[23] A. Mehr and D.F. Escande, *Destruction of KAM Tori in Hamiltonian Systems: Link with the Destabilization of Nearby Cycles and Calculation of Residues,* Physica D **13**, 302–338 (1984).
[24] M. Govin, C. Chandre, and H.R. Jauslin, *KAM–Renormalization Group Approach to the Breakup of Invariant Tori in Hamiltonian Systems.* Phys. Rev. E **57**, 1536–1543 (1998).
[25] C. Chandre and H.R. Jauslin, *Renormalization-group analysis for the transition to chaos in Hamiltonian systems.* Physics Reports **365**, 1–64 (2002).
[26] H. Koch, *A Renormalization Group for Hamiltonians, with Applications to KAM Tori.* Erg. Theor. Dyn. Syst. **19**, 1–47 (1999).
[27] J.J. Abad, H. Koch, and P. Wittwer, *A Renormalization Group for Hamiltonians: Numerical Results.* Nonlinearity **11**, 1185–1194 (1998).
[28] J.J. Abad and H. Koch, *Renormalization and Periodic Orbits for Hamiltonian Flows,* Commun. Math. Phys. **212**, 371–394 (2000).
[29] H. Koch, *On the renormalization of Hamiltonian flows, and critical invariant tori.* Discrete Contin. Dynam. Systems A, **8**, 633–646 (2002).
[30] H. Koch, *A Renormalization Group Fixed Point Associated with the Breakup of Golden Invariant Tori.* preprint `mp_arc 02-175` (2002), to appear in Discrete Contin. Dynam. Systems (special volume on Hamiltonian systems).
[31] Degree $(7, 7, 9)$ Hamiltonians are included with the preprint `mp_arc 03-545` of this paper; additional data can be found at `ftp://ftp.ma.utexas.edu/pub/papers/koch/shearless/`

Figure 1

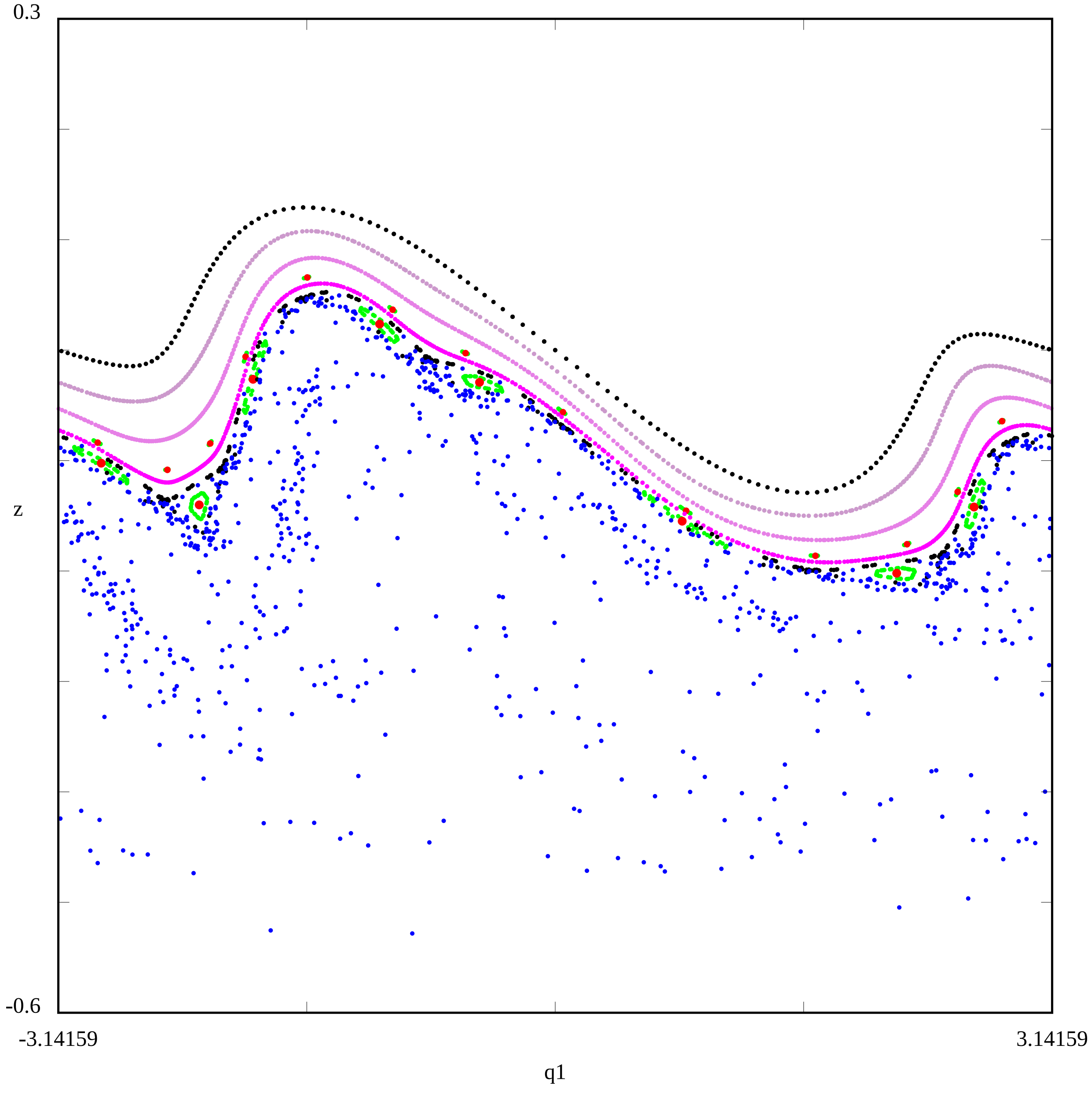

Figure 2