Non-symmetric low-index solutions for a symmetric boundary value problem

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Abstract. We consider the equation $-\Delta u = wu^3$ on a square domain in $\mathbb{R}^2$, with Dirichlet boundary conditions, where w is a given positive function that is invariant under all (Euclidean) symmetries of the square. This equation is shown to have a solution u, with Morse index 2, that is neither symmetric nor antisymmetric with respect to any nontrivial symmetry of the square. Part of our proof is computer-assisted. An analogous result is proved for index 1.

1. Introduction

It is a well known phenomenon that symmetric equations can have non-symmetric solutions. However, "simple" solutions often tend to be symmetric, even in cases where the notion of simplicity is not manifestly related to symmetry. A case in point is the boundary value problem

$$-\Delta u(z) = f(z, u(z)), \quad \forall z \in \Omega\,, \qquad u(z) = 0, \quad \forall z \in \partial\Omega\,, \tag{1.1}$$

on a bounded open domain $\Omega \subset \mathbb{R}^n$ that is symmetric with respect to some codimension 1 hyperplane. If Ω is convex in the direction orthogonal to this plane, and if some monotonicity properties are satisfied, which include the case where f does not depend explicitly on z, then any positive solution u of the equation (1.1) is necessarily symmetric as well. This is a classical result by Gidas, Ni, and Nirenberg [1]. Subsequent extensions include, among other things, classes of solutions that are not necessarily positive [5,7,9,11,12].

We consider the same equation (1.1) but focus on a different class of "simple" solutions, proposed first in [5], namely solutions with fixed Morse index. Recall that solutions of equation (1.1) are critical points of the functional J on $H_0^1(\Omega)$,

$$J(u) = \int_\Omega \Big[\tfrac{1}{2}\big|\nabla u(z)\big|^2 - F(z, u(z))\Big]\, d^2 z\,, \qquad \partial_u F = f\,, \tag{1.2}$$

assuming that F satisfies some growth and regularity conditions; and the Morse index of a critical point u is the number of descending directions of J at u.

The question considered in this paper is motivated by the symmetry results in [11], which cover domains (balls and annuli) and nonlinearities f that are radially symmetric. We refer to [11] for the precise assumptions and results. Roughly speaking, $\partial_u f$ is assumed to be convex in u, but f need not be monotone in $|z|$. Then any solution u of Morse index $\le n$ has an axial symmetry. Given this result, it is natural to ask whether there is an analogue for domains that only have discrete symmetries, such as regular polytopes.

We will give a partial answer by constructing counterexamples with index 1 and 2, in $n = 2$ dimensions. We start with the easier case: a non-symmetric index-1 solution. Let

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Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ with only finitely many (Euclidean) symmetries. A function u on Ω is said to have symmetry σ if $u \circ \sigma = u$.

Theorem 1.1. *There exists a C^∞ function $w \geq 0$ on Ω, possessing all symmetries of Ω, such that (1.1) with $f = wu^3$ admits a positive solution $u \in H_0^1(\Omega)$, with index 1, that has no nontrivial symmetry of Ω.*

This theorem can be proved by standard variational methods; see Section 2.

In what follows, the domain Ω is fixed to be the square $\Omega = (0, \pi)^2$. Our main goal is to prove an analogous result for index 2, using computer-assisted methods. Such a result seems currently outside the scope of other known methods. Similar techniques should apply to a variety of other semilinear elliptic problems, as long as the domain and other quantities involved take a relatively simple form.

When considering solutions u with multiple extrema, the natural question is whether $|u|$ is symmetric; u itself may be antisymmetric with respect to some of the reflections that leave Ω invariant. There is numerical evidence that this is indeed the case for low-index solutions, at least for some standard nonlinearities that do not depend explicitly on the variable z [2,3,4,6,8]. But it is not clear whether this holds more generally. While symmetry results have been proved in many situations, antisymmetry results are available only in some special cases [10], as far as we know.

Our analysis in the index-2 case uses a nonlinearity $f = w_C u^3$, with

$$\begin{aligned} w_C(x,y) = & C_1\Bigl(\tfrac{41}{32}\cos(x)\cos(y)\bigl[\cos(2y)-\cos(2x)\bigr]\Bigr)^2 + (1-C_1) \\ & + C_2\bigl[\sin(x)\sin(y)\bigr]^8 , \end{aligned} \tag{1.3}$$

and $0 \leq C_k \leq 1$. These functions w_C possess all the symmetries of the square Ω, and they are nonnegative.

Theorem 1.2. *Let $f = w_C u^3$ with $C_1 = \frac{97}{128}$ and $C_2 = \frac{145}{256}$. Then the equation (1.1) admits a real analytic solution, that has Morse index 2 and is neither symmetric nor antisymmetric with respect to any nontrivial symmetry of the square.*

Our proof of this theorem is computer-assisted. To be more precise, we first reformulate (1.1) as a fixed point problem $G(u) = u$ and show that the Morse index of u is related to the spectrum of the derivative $DG(u)$. This is done in Section 3. In Section 4, we reduce the proof of Theorem 1.2 to a set of sufficient conditions on G and DG, near an approximate fixed point u_0. Section 5 describes how these conditions (inequalities) can be, and have been, verified with the aid of a computer.

The computation of the approximate solution u_0 is described in Section 6. The graphs of u_0 and of w_C are shown in Figure 1.

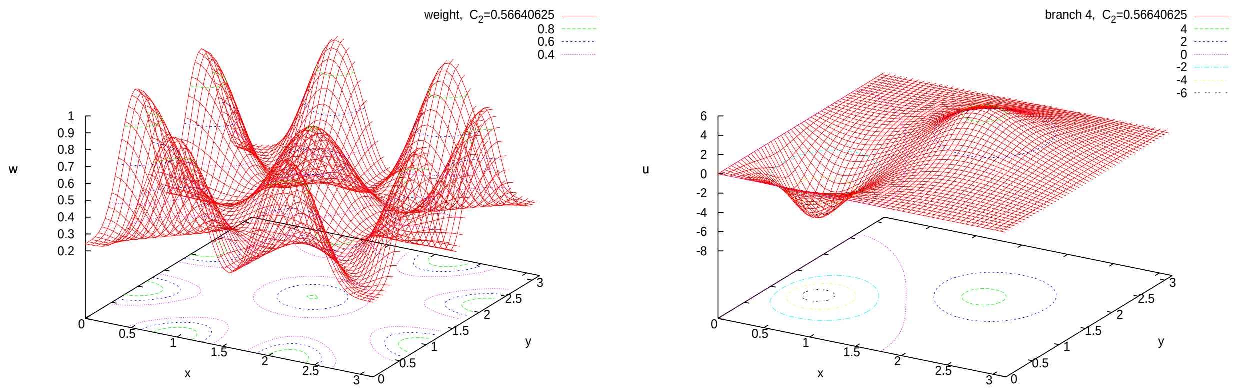

Figure 1. Weight w_C and solution u, for $C_1 = \frac{97}{128}$ and $C_2 = \frac{145}{256}$.

2. Proof of Theorem 1.1

To simplify notation, we write $H_0^1 = H_0^1(\Omega)$ and $L^p = L^p(\Omega)$. Given a nontrivial continuous function $w \ge 0$ on Ω, the functional (1.2) can be written as

$$J(u) = \frac{1}{2}\|u\|_H^2 - F(u)\,, \qquad \|u\|_H^2 = \int_\Omega |\nabla u|^2\,, \qquad F(u) = \frac{1}{4}\int_\Omega wu^4\,.$$

We start by maximizing F on the unit sphere $S = \{u \in H_0^1 : \|u\|_H = 1\}$. Notice that F is well defined and continuous on L^4. Since S is a compact subset of L^4, we can find a sequence (u_n) in S, that converges strongly in L^4 and weakly in H_0^1, such that $\lim_n F(u_n) = \sup_S F$. The limit u cannot be zero, since $\sup_S F > 0$. Furthermore, $\|u\| \le 1$. In fact, we must have $\|u\|_H = 1$, otherwise $\|u\|_H^{-1} u \in S$ satisfies $F\bigl(\|u\|_H^{-1}u\bigr) = \|u\|_H^{-4} F(u) > F(u)$.

Let $u \in S$ be any point where $\max_S F$ is achieved. Then $v = \tau u$ is a critical point of J for some Lagrange multiplier $\tau > 0$. Thus $-\Delta v = wv^3$, implying e.g. that u is continuous. We may assume that $u \ge 0$, since $F(|u|) \ge F(u)$, and $|u| \in S$. The latter follows from the fact that $\bigl|\nabla |u|\bigr| = |\nabla u|$ a.e. [14, Theorem 6.17]. Given that the function u is continuous and vanishes on $\partial\Omega$, it has a maximum at some point $z_1 \in \Omega$. Since u is harmonic outside the support D of w, we must have $z_1 \in D$, and $u(z) < u(z_1)$ for all $z \in \Omega \setminus D$.

Next, we pick a particular weight w. Let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the symmetries of Ω, with σ_1 the identity. We may assume that $n \ge 2$. Let w_1 be a nontrivial nonnegative C^∞ function on Ω, such that the functions $w_j = w_1 \circ \sigma_j$ have mutually disjoint supports $D_j = \mathrm{supp}(w_j)$, where $1 \le j \le n$. Define $w = w_1 + w_2 + \ldots + w_n$.

Assume for contradiction that $u \circ \sigma = u$ for some nontrivial symmetry σ of Ω. Without loss of generality, we may assume that $z_1 \in D_1$ and $\sigma = \sigma_2$. Then u takes its maximum value M at the distinct points $z_1 \in D_1$ and $z_2 = \sigma(z_1) \in D_2$.

The goal is to modify u near z_1 and z_2 in such a way that F increases, while the norm stays the same. To this end, choose $c < M$ such that $B = \{z \in \Omega : u(z) > c\}$

is disconnected, with one connected component B_1 containing z_1, and another connected component B_2 containing z_2. Then we cut u at level c in B_1 and add in B_2 a symmetric copy of the cut-out piece. More specifically, let $u' = u - v_1 + v_2$, where $v_j(z) = u(z) - c$ for all $z \in B_j$, and $v_j(z) = 0$ for all $z \notin B_j$. Using again [14, Theorem 6.17], together with the fact that $v_2 = v_1 \circ \sigma$ has the same norm as v_1, we obtain

$$\begin{aligned} \|u'\|_H &= \|u_0 + 2v_2\|_H = \|u_0\|_H + 2\|v_2\|_H \\ &= \|u_0\|_H + \|v_1\|_H + \|v_2\|_H = \|u_0 + v_1 + v_2\|_H = \|u\|_H = 1, \end{aligned} \tag{2.1}$$

where $u_0 = u - v_1 - v_2$. Thus, u' belongs to S. And $F(u') > F(u)$, since

$$\big[u(z) - v_1(z)\big]^4 + \big[u(\sigma(z)) + v_1(\sigma(z))\big]^4 > \big[u(z)\big]^4 + \big[u(\sigma(z))\big]^4 , \tag{2.2}$$

for all $z \in B_1$. This contradicts the fact that $F(u) = \max_S F$. Thus, u cannot have a nontrivial symmetry of Ω.

Consider now the function $\mathcal{J} : \mathbb{R} \times S \to \mathbb{R}$ defined by $\mathcal{J}(t, v) = J(tv) = \frac{1}{2}t^2 - t^4 F(v)$. When restricted to $\mathbb{R} \times \{u\}$, it has a maximum at some value $t = \tau > 0$. And the restriction of $\mathcal{J}$ to $\{\tau\} \times S$ has a minimum at (τ, u). Consequently, τu is a critical point of J, with Morse index 1. This completes the proof of Theorem 1.1.

The remaining part of this paper is devoted to the proof of Theorem 1.2.

3. The fixed point equation and Morse index

Solutions of the equation (1.1) can be obtained as fixed points of the map G,

$$G(u) = (-\Delta)^{-1} f(\,\cdot\,, u) . \tag{3.1}$$

In this section we relate the Morse index of a solution u to the spectral properties of the derivative of G at u. For simplicity, we assume that $f(z, u)$ is a polynomial in u with coefficients in $L^\infty(\Omega)$. Then the functional (1.2) is of class C^∞ on $H_0^1(\Omega)$, and its second derivative is given by the quadratic form

$$Q_u(v) = \int_\Omega \big(|\nabla v|^2 - W_u v^2\big) , \tag{3.2}$$

where $W_u(z) = (\partial_u f)(z, u(z))$. The Morse index of u is the number of negative directions of Q_u. The derivative of G at u is given by

$$DG(u)v = (-\Delta)^{-1}(W_u v) . \tag{3.3}$$

Define $v_m(x) = \sin(mx)$ for positive integers m. The functions $v_m \times v_n$ are the eigenfunctions of the Dirichlet Laplacean on Ω, and they constitute an orthogonal basis for $H_0^1 = H_0^1(\Omega)$, with the standard inner product on this space (see below). Thus, every function h in H_0^1 has a convergent sine series expansion

$$h = \sum_{m,n \in \mathcal{K}} h_{m,n}\, v_m \times v_n , \tag{3.4}$$

where $\mathcal{K}$ is the set of all positive integers. Modulo a constant factor, the standard inner product on H_0^1 is given by

$$\langle g,h\rangle_H \stackrel{\rm def}{=} \pi^{-2}\int_\Omega \bigl(\nabla g\bigr)(z)\cdot\bigl(\nabla h\bigr)(z)\,d^2z = \sum_{m,n\in\mathcal{K}}\bigl(m^2+n^2\bigr)g_{m,n}h_{m,n}\,. \tag{3.5}$$

And the inverse Dirichlet Laplacean takes the following simple form

$$-\Delta^{-1}h = \sum_{m,n\in\mathcal{K}}\bigl(m^2+n^2\bigr)^{-1}h_{m,n}\,v_m\times v_n\,. \tag{3.6}$$

Proposition 3.1. *Assume that W_u is of class C^1. Then $DG(u)$ is a compact positive self-adjoint operator on $H_0^1(\Omega)$. Its eigenvalues are strictly positive, if $W_u>0$ almost everywhere on Ω. If u solves equation (1.1), then the Morse index of u agrees with the number of eigenvalues of $DG(u)$ that are larger than 1.*

Proof. The compactness of $DG(u)$ follows from the fact that $-\Delta^{-1}$ is compact and $h\mapsto W_u h$ bounded. The identity

$$\langle g,DG(u)h\rangle_H = \pi^{-2}\int_\Omega g(z)W_u(z)h(z)\,d^2z \tag{3.7}$$

shows that $DG(u)$ is self-adjoint and positive. Furthermore, if $W_u>0$ almost everywhere, then $\langle h,DG(u)h\rangle_H$ is positive, unless $h=0$. Denote by $\lambda_1\ge\lambda_2\ge\ldots\ge 0$ the eigenvalues of $DG(u)$. The corresponding eigenvectors $u_1,u_2,\ldots$ can be chosen to be an orthonormal basis for H_0^1. Then

$$Q_u(v) = \bigl\langle v,[\mathbb{I}-DG(u)]v\bigr\rangle_H = \sum_n(1-\lambda_n)\bigl|\langle v,u_n\rangle_H\bigr|^2\,. \tag{3.8}$$

This shows that the number of negative directions for Q_u agrees with the number of eigenvectors u_n for which $1-\lambda_n<0$. **QED**

Our aim is to solve the fixed point equation $G(u)=u$ on a space $\mathcal{A}^o$ that is much smaller than H_0^1. The following proposition will be used to recover properties of $DG(u): H_0^1\to H_0^1$ from properties of $DG(u):\mathcal{A}^o\to\mathcal{A}^o$.

Proposition 3.2. *Let H be a Hilbert space. Let X be a Banach space that is continuously and densely embedded in H. Let L be a self-adjoint bounded linear operator on H, that leaves X invariant and defines a compact linear operator L_X on X. Then every eigenvector of L for a nonzero eigenvalue belongs to X.*

Proof. Let λ be a nonzero eigenvalue of L. Denote by $\mathbb{P}$ the spectral projection for L_X, associated with all eigenvalues of modulus $\ge|\lambda|$. Since L is self-adjoint and $\mathbb{P}$ has finite rank, $\mathbb{P}$ defines an orthogonal projection on H that commutes with L.

Consider the self-adjoint operator $T = L(\mathbb{I} - \mathbb{P})$ on H. Assume for contradiction that T has an eigenvalue λ. Let y be a normalized eigenvector for this eigenvalue. Pick $x \in X$ such that $\langle x, y\rangle_H = a > 0$. Then $\|T^n x\|_H \geq a|\lambda|^n$ for all n. This, together with the embedding inequality $\|.\|_H \leq C\|.\|_X$ on X, implies that the operator $L_X(\mathbb{I} - \mathbb{P})$ on X has a spectral radius $\geq |\lambda|$. This is impossible by the definition of $\mathbb{P}$. Thus, every eigenvector of L with eigenvalue λ belongs to $\mathbb{P}H \subset X$. **QED**

Before defining the space $\mathcal{A}^o$ mentioned earlier, we note that the sine series (3.4) extends a function $h \in H_0^1$ to a function on $\mathbb{R}^2$. Denoting the extension again by h, and using the notation $h = h(x, y)$, the function h is 2π-periodic in both variables x and y. Furthermore, $-h(-x, y) = h(x, y) = -h(x, -y)$ for all $x, y \in \mathbb{R}$. A function h with this property will be called an *odd* function. Similarly, a function $h : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $h(-x, y) = h(x, y) = h(x, -y)$ for all $x, y \in \mathbb{R}$ will be called *even*.

Since we will need to estimate both odd and even functions, we consider Fourier series (3.4) with $\mathcal{K} = \mathbb{Z}$, where $v_m(x) = \cos(mx)$ for integers $m \leq 0$. If the series (3.4) for h has only finitely many nonvanishing terms, the function h will be referred to as a Fourier polynomial. Given $\rho > 0$, we define $\mathcal{A}$ to be the completion of the vector space of Fourier polynomials h with respect to the norm

$$\|h\| = \sum_{m,n} |h_{m,n}| e^{\rho|m|+\rho|n|} . \tag{3.9}$$

This space $\mathcal{A}$ is a Banach algebra, that is, $\|gh\| \leq \|g\|\|h\|$, for all $g, h \in \mathcal{A}$. The odd and even subspaces of $\mathcal{A}$ will be denoted by $\mathcal{A}^o$ and $\mathcal{A}^e$, respectively. Clearly, H_0^1 contains $\mathcal{A}^o$ as a dense subspace.

Proposition 3.3. *Assume that W_u belongs to $\mathcal{A}^e$ and is positive on Ω. Then all eigenvectors of $DG(u) : H_0^1 \to H_0^1$ belong to $\mathcal{A}^o$, and the restriction of $DG(u)$ to $\mathcal{A}^o$ defines a compact linear operator on $\mathcal{A}^o$.*

Proof. By using the Banach algebra property of $\mathcal{A}$, and the representation (3.6) for $(-\Delta)^{-1}$, we see that $DG(u)$ defines a compact linear operator on $\mathcal{A}^o$. Clearly, there exists $C > 0$ such that $\langle u, u\rangle_H \leq C\|u\|^2$, for all $u \in \mathcal{A}^o$. The assertion concerning the eigenvectors of $DG(u) : H_0^1 \to H_0^1$ now follows from Proposition 3.1, and from Proposition 3.2, using $X = \mathcal{A}^o$ and $H = H_0^1$. **QED**

4. Estimates used to prove Theorem 1.2

Consider now the fixed point problem for G, in the case where

$$G(u) = (-\Delta)^{-1}\big[wu^3\big] , \tag{4.1}$$

with w some fixed but arbitrary positive function in $\mathcal{A}^e$. Since $\mathcal{A}$ is a Banach algebra, and $\Delta^{-1} : \mathcal{A}^o \to \mathcal{A}^o$ is compact, the equation (4.1) defines a compact C^∞ map G on $\mathcal{A}^o$.

Notice also that $DG(u)$ has a "Nehari eigenvalue" 3 at any fixed point $u \neq 0$ of G, with eigenvector u, due to the fact that G is homogeneous of degree 3.

Let $u_0 \in \mathcal{A}^o$ be fixed, and let A be a linear isomorphism of $\mathcal{A}^o$. If $u \in \mathcal{A}^o$, then $u_0 + Au$ is a fixed point of G if and only if u is a fixed point of $\mathcal{N}$, where

$$\mathcal{N}(h) = G(u_0 + Ah) - u_0 + (\mathbb{I} - A)h\,, \qquad h \in \mathcal{A}^o\,. \tag{4.2}$$

Furthermore, if $DG(u_0)$ does not have an eigenvalue 1, and if we choose A sufficiently close to $[\mathbb{I} - DG(u_0)]^{-1}$, then $\mathcal{N}$ is a contraction near the origin. The equation (3.6) shows that $DG(u_0)$ can be approximated by finite rank operators. This motivates the following.

Let p be an invertible map from $\mathbb{N} = \{1, 2, \ldots\}$ onto $\mathbb{N} \times \mathbb{N}$. For every positive integer k, define $\boldsymbol{v}_k = v_m \times v_n$, with $(m, n) = p(k)$. Furthermore, denote by h_k the coefficient of $\boldsymbol{v}_k$ in the expansion (3.4) of a function $h \in \mathcal{A}^o$. Then, to any real $N \times N$ matrix M, we can associate a linear operator $\hat{M}$ on $\mathcal{A}^o$, by setting

$$\hat{M}h = \sum_{k,j=1}^{N} M_{k,j} h_j \boldsymbol{v}_k\,, \qquad h \in \mathcal{A}^o\,. \tag{4.3}$$

From now on, we fix w to be the function w_C defined in (1.3), for the parameter values described in Theorem 1.2. In addition, we fix the space $\mathcal{A}$ by choosing $\rho = \ln(1 + 2^{-60})$ in the equation (3.9).

Given $r > 0$ and $g \in \mathcal{A}^o$, define $B_r(g) = \{h \in \mathcal{A}^o : \|h - g\| \leq r\}$.

Lemma 4.1. *There exists an odd Fourier polynomial u_0, a real square matrix M, and real numbers $\delta, \varepsilon, K > 0$, satisfying $\varepsilon + K\delta < \delta$, such that the following holds. M has no eigenvalue 1, and the map $\mathcal{N}$, defined by (4.2), with $A = \mathbb{I} - \hat{M}$, satisfies*

$$\|\mathcal{N}(0)\| \leq \varepsilon\,, \qquad \|D\mathcal{N}(h)\| \leq K\,, \quad \forall h \in B_\delta(0)\,. \tag{4.4}$$

The proof of this lemma is computer-assisted and will be described in Section 5.

By the contraction mapping principle, the given bounds imply that $\mathcal{N}$ has a unique fixed point h_* in the ball $B_\delta(0)$. In what follows, $u_* = u_0 + Ah_*$ denotes the corresponding fixed point of G. Notice that u_* belongs to $B_r(u_0)$, if $r \geq \|A\|\delta$.

The following lemma shows that u_* is not symmetric or antisymmetric with respect to any symmetry of the square. Let $E = \{(\pi/4, \pi/2), (\pi/2, \pi/4), (3\pi/4, \pi/2), (\pi/2, 3\pi/4)\}$. Clearly, each nontrivial symmetry of Ω acts as a nontrivial permutation on E.

Lemma 4.2. *There exists $r \geq \|A\|\delta$, such that for every $u \in B_r(u_0)$, the function $z \mapsto |u(z)|$ takes 4 distinct values on E.*

The proof of this lemma is computer-assisted and will be described in Section 5.

Recall that, by Proposition 3.1, all eigenvalues of $DG(u)$ are positive. Our next goal is to prove that all but two eigenvalues of $DG(u_*)$ are smaller than 1. To this end, we approximate $DG(u_*)$ numerically by an operator $\hat{T}$ associated with an $N \times N$ matrix T. In what follows, T^* denotes the adjoint of T with respect to the inner product on $\mathbb{R}^N$ induced by (3.5).

Lemma 4.3. *With A, δ, r, u_0 as in Lemma 4.1 and Lemma 4.2, there exists a square matrix $T = T^*$ with eigenvalues $\mu_1 > \mu_2 > 1 > \mu_3 > \ldots > 0$, such that*

$$\Bigl\| \bigl[DG(u) - \hat{T} \bigr] \bigl(\hat{T} - \mathbb{I} \bigr)^{-1} \Bigr\| < 1 \,, \qquad \forall u \in B_r(u_0) \,. \tag{4.5}$$

The proof of this lemma is computer-assisted and will be described in Section 5. Combining the last three lemmas we arrive at the following.

Proof of Theorem 1.2. By Lemma 4.1 and the contraction mapping principle, the map $\mathcal{N}$ defined by (4.2) has a unique fixed point h_* in $B_\delta(0)$. If $r > \|A\|\delta$ then the corresponding fixed point $u_* = u_0 + Ah$ of G belongs to the ball $B_r(u_0)$. Clearly, u_* is a real analytic solution of (1.1). Furthermore, u_* is not symmetric or antisymmetric with respect to any symmetry of the square Ω, as Lemma 4.2 shows.

Consider the operators $\mathcal{L}_s = sDG(u_*) + (1-s)\hat{T}$, for $0 \le s \le 1$, with $\hat{T}$ as described in Lemma 4.3. They all have the following properties. $\mathcal{L}_s$ is compact, symmetric with respect to the inner product (3.5), and positive, in the sense that $\langle h, \mathcal{L}_s h \rangle_H \ge 0$ for all $h \in \mathcal{A}^o$. Furthermore, $\mathcal{L}_s - \mathbb{I}$ has a bounded inverse,

$$(\mathcal{L}_s - \mathbb{I})^{-1} = (\hat{T} - \mathbb{I})^{-1} (\mathbb{I} + sV)^{-1} \,, \qquad V = \bigl[DG(u_*) - \hat{T} \bigr] (\hat{T} - \mathbb{I})^{-1} \,, \tag{4.6}$$

since $\|V\| < 1$ by Lemma 4.3. In other words, $\mathcal{L}_s$ has no eigenvalue 1. Since the positive eigenvalues of $\mathcal{L}_s$ vary continuously with s, this implies that the operators $\hat{T} = \mathcal{L}_0$ and $DG(u_*) = \mathcal{L}_1$ have the same number of eigenvalues (counting multiplicities) in the interval $[1, \infty)$ and its interior. By Lemma 4.3, this number is 2. This, together with Proposition 3.1, Proposition 3.3, and Proposition 3.2 with $X = \mathcal{A}^o$ and $H = H_0^1$, implies that u_* has Morse index 2. This completes the proof of Theorem 1.2. **QED**

5. The computer-assisted part

What remains to be proved are the Lemmas 3.1, 3.2, and 3.3. Given the Fourier polynomial u_0 and the matrices M and T (obtained from purely numerical computations), this task is clearly a sequence of trivial estimates, assuming that there are no fundamental obstructions. The sequence is finite, since Δ^{-1} can be approximated to arbitrary accuracy by finite rank operators. But the steps are much too numerous to be carried out by hand, so we enlist the help of a computer. For the types of operations needed here, the techniques are quite standard by now. Thus, we will restrict our description mainly to the problem-specific parts.

As with any lengthy task, proper organization is crucial. We start by associating to a space X a collection $\mathrm{std}(X)$ of subsets of X, that are representable on the computer. These sets will be referred to as "standard sets" for X. A "bound" on an element $s \in X$ is then a set $S \in \mathrm{std}(X)$ containing s. Each collection $\mathrm{std}(X)$ corresponds to a data type in our programs. Unless stated otherwise, $\mathrm{std}(X \times Y)$ is taken to be the collection of all sets $S \times T$ with $S \in \mathrm{std}(X)$ and $T \in \mathrm{std}(Y)$.

Our standard sets for $\mathbb{R}$ are associated with a type `Ball`, which consists of pairs `S=(S.C,S.R)`, where `S.C` is a representable number (`Rep`) and `S.R` a nonnegative representable number (`Radius`). The standard set defined by a `Ball S` is the interval $\mathcal{B}(\texttt{S}) =$

$\{s \in \mathbb{R} : |s - \mathtt{S.C}| \leq \mathtt{S.R}\}$. Our standard sets for $\mathcal{A}^o$ are represented by a type `Fourier2` consisting of a triple `F=(F.T,F.C,F.E)`, where `F.T` is a record identifying the space $\mathcal{A}^o$, `F.C` is an `array(0..K,0..K) of Ball`, and `F.E` is an `array(0..2*K,0..2*K) of Radius`. The corresponding set $\mathcal{B}(\mathtt{F})$ in $\mathrm{std}(\mathcal{A}^o)$ is the set of all function $u = p + h \in \mathcal{A}^o$,

$$p = \sum_{m,n=1}^{K} p_{m,n}\, v_m \times v_n\,, \quad h = \sum_{m,n=1}^{2K} h^{m,n}\,, \quad h^{m,n} = \sum_{i \geq m, j \geq n} h^{m,n}_{i,j}\, v_i \times v_j\,, \tag{5.1}$$

with $p_{M,N} \in \mathcal{B}(\mathtt{F.C(M,N)})$ and $\|h^{M,N}\| \leq \mathtt{F.E(M,N)}$, for all $M, N \geq 1$. The type `Fourier2` is also used to define our standard sets for the space $\mathcal{A}^e$, and for some other subspaces of $\mathcal{A}$. In our programs, the "maximal degree" K is either 100 or 125.

For the representable numbers, we choose a data type (renamed to `Rep`) for which elementary operations are available with controlled rounding. This makes it possible to implement a bound `Sum` on the function $(s,t) \mapsto s + t$ on $\mathbb{R} \times \mathbb{R}$, as well as bounds on other elementary functions on $\mathbb{R}$ or $\mathbb{R}^n$, including things like the matrix product or the Gram-Schmidt orthogonalization map.

Here, a bound on a map $f : X \to Y$ is a map $F : D_F \to \mathrm{std}(Y)$, with domain $D_F \subset \mathrm{std}(X)$, such that $f(s) \in F(S)$ whenever $s \in S \in D_F$. Such bounds are implemented as procedures or functions in our programs. This can be done hierarchically. Using e.g. the `Sum` for the type `Ball`, it is straightforward to implement a bound `Sum` on the map $(g,h) \mapsto g + h$ from $\mathcal{A}^o \times \mathcal{A}^o$ to $\mathcal{A}^o$. Similarly for maps like $u \mapsto \|u\|$ or $-\Delta^{-1}$. Implementing a bound on the product $(g,h) \mapsto gh$ is a bit more tedious, but straightforward.

A bound on $\|\mathcal{N}(0)\|$ is now obtained by composing the basic bounds mentioned above. In order to estimate $\|D\mathcal{N}(h)\|$, as required for a proof of Lemma 4.1, we use the following fact. If L is a continuous linear operator on $\mathcal{A}^o$, then

$$\|L\| = \sup_k \|L\boldsymbol{e}_k\|\,, \qquad \boldsymbol{e}_k = \|\boldsymbol{v}_k\|^{-1}\boldsymbol{v}_k\,, \tag{5.2}$$

where $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots$ are the functions described before equation (4.3). This explicit expression for $\|L\|$ is our main reason for working with a weighted ℓ^1 norm. For the operator $L = D\mathcal{N}(h)$, it is easy to determine k_0, given $c > 0$, such that $\|L\boldsymbol{e}_k\| \leq c$ whenever $k \geq k_0$. Thus, estimating the norm of $D\mathcal{N}(h)$ reduces to a finite computation. Choosing $\delta > 0$ to be a representable number, this estimate can be carried out simultaneously for all functions $h \in B_\delta(0)$, since $B_\delta(0)$ belongs to $\mathrm{std}(\mathcal{A}^o)$.

The same approach is used to estimate the operator norm in equation (4.5). The $N \times N$ matrix T is taken to be of the form

$$T = UMU^*\,, \qquad M = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N)\,, \tag{5.3}$$

where $\mu_1, \mu_2, \ldots, \mu_N$ are positive numerical approximations for the largest N eigenvalues of $DG(u_*)$, and where U is an orthogonal $N \times N$ matrix. To be more precise, U is orthogonal for the inner product on $\mathbb{R}^N$ induced by (3.5), and U^* is the corresponding adjoint matrix, so that U^* is the inverse of U. This ensures not only that $T = T^*$, but it also makes it easy to compute the inverse of $\hat{T} - \mathbb{I}$. The size N used in our programs is 250.

Verifying the claim in Lemma 4.2 is comparatively simple. All we need is a bound on the evaluation function $(z,u)\mapsto|u(z)|$ on $\mathbb{R}^2\times\mathcal{A}^o$. For the "higher order" part h in the decomposition $u=p+h$, we use the fact that $|h(z)|\le\|h\|$, for all $z\in\mathbb{R}^2$.

For a precise and complete description of all definitions and estimates, we refer to the source code and input data of our computer programs [18]. The source code is written in Ada2005 [15]. For the type `Rep` we use a MPFR floating point type, with 128 or 256 mantissa bits, depending on the program. MPFR is an open source multiple-precision floating-point library that supports controlled rounding [17]. Our programs were run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [16].

6. Some numerical results

Our approximate solution u_0 was obtained by starting with a symmetric solution for $C_1 = C_2 = 0$, where $w_C = 1$, and following solution branches where either C_1 or C_2 is fixed. The symmetry breaking occurs in two steps, as we will now describe.

Consider first $C_2 = 0$. In this case, and for $C_1 > 0$, the weight function w_C looks similar to the function shown in Figure 1, except that the center peak is missing: w_C has a local minimum at the center of Ω. The other peaks increase as C_1 increases.

For $C_1 \ge 0$, we find a branch (referred to as "branch 1") of solutions that are symmetric with respect to the diagonal $x = y$ and antisymmetric with respect to the diagonal $x + y = \pi$. At a value $C_1 \approx 0.66$, we observe a pitchfork bifurcation. As C_1 is increased past this value, the Morse index on branch 1 changes from 2 to 3.

On the intersecting branch (called "branch 2"), for $C_1 \gtrsim 0.66$, the solutions no longer have the two reflection symmetries mentioned above, but they are still antisymmetric with respect to the composition of these symmetries: a rotation by π about the center of Ω. The Morse index is 2, and no bifurcation is observed up to $C_1 = 0.85$.

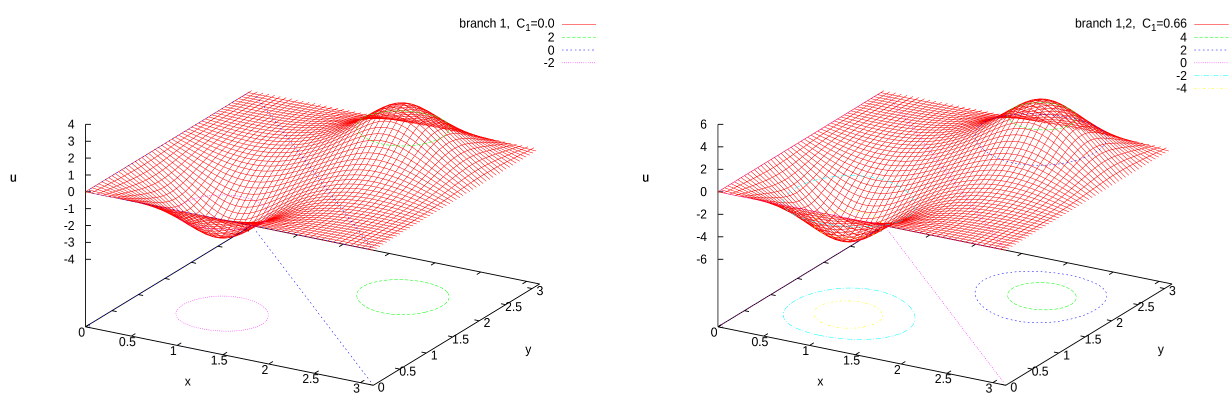

Figure 2. Starting point and bifurcation point on branch 1.

Now we fix $C_1 = \frac{97}{128} = 0.7578125$ and start increasing C_2 . This causes the weight w_C to develop a peak in the center. The goal is to make it favorable for the solution u to have a nonzero value at the center of Ω. And the other 8 peaks of w_C should make it difficult to achieve this goal while keeping a rotation symmetry.

The resulting "branch 3" is observed to undergo a pitchfork bifurcation at a value $C_2 \approx 0.095$, where the Morse index changes from 2 to 3. (It appears that there is another bifurcation later, where the solutions become symmetric with respect to $x = y$ and antisymmetric with respect to $x + y = \pi$.)

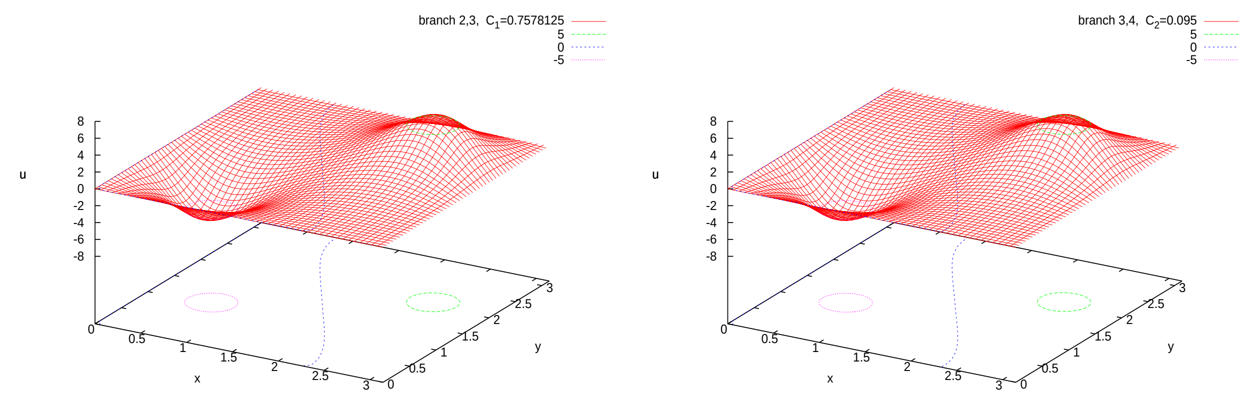

Figure 3. Two points on branch 3.

On the intersecting branch (called "branch 4"), for $C_2 \gtrsim 0.095$, the solutions are neither symmetric or antisymmetric with respect to any of the symmetries of the square. Along this branch, the third largest eigenvalue first decreases from 1 down to about 0.857, and then it increases again (reaching 1 around $C_2 = 1.22$). The minimum is reached near the value of C_2 used in Theorem 1.2.

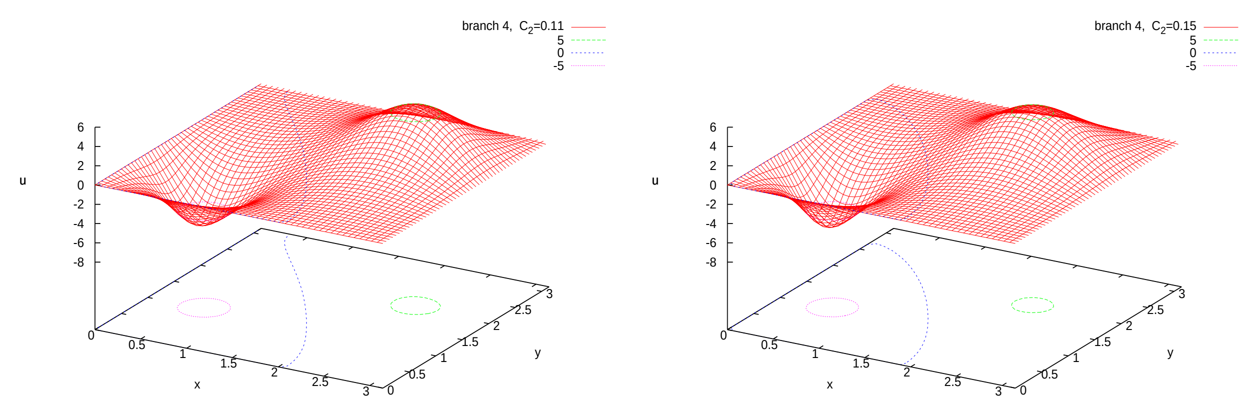

Figure 3. Two points on branch 4.

The "basic" procedure that was used to follow a branch is to gradually change parameter values, and using a Newton-type map $\mathcal{N}$ associated with G, to find an accurate fixed point at each step. Near a bifurcation point u, where $DG(u)$ has an eigenvalue close to 1, we compute the corresponding eigenvector h. The new branch is found by starting with $v = u + \varepsilon h$ and adjusting the parameter to minimize the norm of $G(w) - w$, where $w = \mathcal{N}^k(v)$ for some appropriate k. Then the map $u \mapsto w$ is iterated until the eigenvalues of $DG(u)$ are far enough from 1 for the basic branch-following procedure to work. This approach can of course be improved, but that was not our goal here.

The equation (1.1) for the disk, with nonlinearities that depend explicitly on z, is being investigated in [13]. Other numerical studies on related equations can be found in the references [2,3,4,6,8].

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available e.g. at `http://www.adaic.org/standards/05rm/html/RM-TTL.html`.

[16] A free-software compiler for the Ada programming language, which is part of the GNU Compiler Collection; see `http://gcc.gnu.org/`.

[17 The MPFR library for multiple-precision floating-point computations with correct rounding; see `http://www.mpfr.org/`.

[18] Ada files and data are included with the preprint `mp_arc 10-141`.