# Universal supercritical behavior for some skew-product maps

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**Abstract.** We consider skew-product maps over circle rotations $x \mapsto x+\alpha \pmod 1$ with factors that take values in $\mathrm{SL}(2,\mathbb{R})$. In numerical experiments with $\alpha$ the inverse golden mean, Fibonacci iterates of almost Mathieu maps with rotation number $1/4$ and positive Lyapunov exponent exhibit asymptotic scaling behavior. We prove the existence of such asymptotic scaling for "periodic" rotation numbers and for large Lyapunov exponent. The phenomenon is universal, in the sense that it holds for open sets of maps, with the scaling limit being independent of the maps. The set of maps with a given periodic rotation number is a real analytic manifold of codimension 1 in a suitable space of maps.

## 1. Introduction and main results

We consider the asymptotic behavior of skew products

$$\begin{aligned} A_\circ^{*q}(x) &= A^{*q}\bigl(x-\tfrac{q}{2}\alpha\bigr)\,,\\ A^{*q}(x) &\stackrel{\text{def}}{=} A(x+(q-1)\alpha)\cdots A(x+2\alpha)A(x+\alpha)A(x)\,, \end{aligned} \tag{1.1}$$

as $q\to\infty$ along certain subsequences, where $\alpha$ is an irrational number and $A$ is a real analytic function from the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ to the group $\mathrm{SL}(2,\mathbb{R})$. Products for negative $q$ are defined by replacing the factors $A$ in the above equation by $A(\,\boldsymbol{.}-\alpha)^{-1}$.

The typical growth of such products is described by the Lyapunov exponent. The Lyapunov exponent of a pair $G=(\alpha,A)$ with $A:\mathbb{T}\to\mathrm{SL}(2,\mathbb{R})$ continuous is defined as

$$L(G)=\lim_{q\to\infty}\frac{1}{q}\log\bigl\|A^{*q}(x)\bigr\|\,. \tag{1.2}$$

Assuming that $\alpha$ is irrational, this limit exists by ergodicity and is a.e. constant in $x$.

The work presented in this paper was motivated in part by the observation in [29] of systematic slower-than-typical growth of certain products, as will be described below. We restrict to factors that are reversible in the following sense.

**Definition 1.1.** *The symmetric factor $A_\circ$ associated with a pair $G=(\alpha,A)$ is defined by setting $A_\circ(x)=A\bigl(x-{}^{\alpha}\!/_2\bigr)$ for all $x$. We say that $G$ is reversible if $S^{-1}A_\circ(x)S=A_\circ(-x)^\dagger$ for all $x$, where*

$$S=\begin{bmatrix}0&1\\1&0\end{bmatrix},\qquad C^\dagger=\begin{bmatrix}d&-b\\-c&a\end{bmatrix}\quad\text{if}\quad C=\begin{bmatrix}a&b\\c&d\end{bmatrix}. \tag{1.3}$$

*Notice that $C^\dagger C=\det(C)\mathbf{1}$. We will refer to $C^\dagger$ as the quasi-inverse of $C$.*

To be more precise about the type of products being considered, let $\alpha=\frac{\sqrt{5}-1}{2}$ be the inverse golden mean and $p_n$ the $n$-th Fibonacci number. Then the following holds.

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**Theorem 1.2.** *Let $A : \mathbb{T} \to \mathrm{SL}(2,\mathbb{R})$ be real analytic and reversible. Assume that $G = (\alpha, A)$ has a positive Lyapunov exponent. Then there exist real numbers $M_1, M_2, M_3, \ldots$ such that every subsequence of*

$$n \mapsto M_n \, A_\circ^{*p_n}\bigl(\alpha^n \boldsymbol{.}\bigr) \tag{1.4}$$

*has a subsequence that converges to a nonzero function $z \mapsto b_\diamond(z)\mathtt{B}$, where $\mathtt{B}$ is a constant $2 \times 2$ matrix of rank one. The scalar factor $b_\diamond$ is an even entire function, and convergence is uniform on compact subsets of $\mathbb{C}$.*

A more restrictive version of this theorem was given in [29]. The main objects under investigation in [29] were parametrized factors $A$ at the boundary between zero Lyapunov exponent (subcritical or critical factors) and positive Lyapunov exponent (supercritical factors). In this context, products of supercritical factors were computed numerically, and limit functions $b_\diamond$ were found with surprisingly regular zeros. Our goal here is to describe this type of limits.

Our main focus is on Schrödinger factors

$$A(x) = \begin{bmatrix} \lambda v(t+x) - E & -1 \\ 1 & 0 \end{bmatrix}, \tag{1.5}$$

and perturbations of such factors. Here, $\lambda$, $t$, and $E$ are real parameters, and $v$ is a 1-periodic continuous function on $\mathbb{R}$. The choice $v(x) = -2\cos(2\pi x)$ defines the almost Mathieu (AM) family of factors [15,22,25]. Schrödinger factors arise naturally in the study of Schrödinger operators on $\ell^2(\mathbb{Z})$ of the form [2]

$$(\mathcal{H}_\lambda^t u)_q = \lambda v(t + q\alpha) u_q - u_{q+1} - u_{q-1}\,, \qquad q \in \mathbb{Z}\,. \tag{1.6}$$

To be more specific, $A^{*q}(0)$ is the transfer matrix that maps $\left[\begin{smallmatrix} u_0 \\ u_{-1}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} u_q \\ u_{q-1}\end{smallmatrix}\right]$, for a formal solution $u$ of the equation $\mathcal{H}_\lambda^t u = Eu$. Many properties of $\mathcal{H}_\lambda^t$ are reflected by properties of the corresponding products (1.1), and vice versa. The Lyapunov exponent of the AM factors with $\lambda > 0$ is known to be $L(G) = \max\{0, \log\lambda\}$ for all energies in the spectrum [19].

The family of AM operators $t \mapsto \mathcal{H}_\lambda^t$ describe the motion of an electron on $\mathbb{Z}^2$ under the influence of a magnetic flux $2\pi\alpha$ per unit cell, after restricting to wave functions $\phi(q,p) = e^{-2\pi i p t} u_q$. The full Hamiltonian for this system is known as the Hofstadter Hamiltonian [1,6].

Another quantity associated with $\mathcal{H}_\lambda^t$ that is relevant in the present context is the integrated density of states $E \mapsto \mathfrak{k}(E)$. Notice that $\mathcal{H}_\lambda^t$ is self-adjoint and bounded. Denote by $P_L$ the canonical inclusion map from $\ell^2\bigl(\mathbb{Z} \cap [-L, L]\bigr)$ into $\ell^2(\mathbb{Z})$. Given $E \in \mathbb{R}$, the integrated density of states $\mathfrak{k}_L(\mathcal{H}_\lambda^t, E)$ is defined as the fraction of eigenvalues of $P_L^* \mathcal{H}_\lambda^t P_L$ that belong to $(-\infty, E]$. The integrated density of states for $\mathcal{H}_\lambda^t$ can be obtained as the limit $\mathfrak{k}(E) = \lim_{L\to\infty} \mathfrak{k}_L(\mathcal{H}_\lambda^t, E)$. Clearly, $\mathfrak{k}$ is an increasing function, taking the value $\mathfrak{k}(E) = 0$ for $E$ below the spectrum of $\mathcal{H}_\lambda^t$ and the value $\mathfrak{k}(E) = 1$ for $E$ above the spectrum. And

[2] We adopt here the "physical" choice of signs, which makes the kinetic term positive.

it is constant on any spectral gaps (connected components of the resolvent set). For the AM operator, it is known that there are no isolated eigenvalues [21], so $\mathfrak{k}$ is continuous. For $\lambda > 1$, the spectrum is pure point, with exponentially decreasing eigenfunctions [18]. (This holds for arbitrary Diophantine $\alpha$, and for "nonresonant" values of $t$ that include our choice $\alpha/2$ below.)

Among the intriguing features of the AM operator is the following resonance phenomenon. $\mathcal{H}_\lambda^t$ has an infinite number of spectral gaps. Each gap can be labeled canonically by an integer $k$, known as the Hall conductance, and $\mathfrak{k}(E) \equiv k\alpha \pmod 1$ holds for all energies $E$ in the spectral gap labeled by $k$. For details we refer to [7,8,11,14,23].

A related quantity for Schrödinger pairs $G = (\alpha, A)$ is the rotation number $\operatorname{rot}(G)$. In cases where $A$ is of the form (1.5), with a potential $v$ that is continuous, we can define the rotation number by the equation

$$\operatorname{rot}(G) = \lim_{q\to\infty} \frac{\operatorname{Rot}_N(G)}{N}\,, \qquad \text{(for Schrödinger } G) \tag{1.7}$$

where $\operatorname{Rot}_N(G)$ is half the number of sign changes on $\{1, 2, \ldots, N\}$ of a nonzero solution $u$ of the equation $\mathcal{H}_\lambda^t u = Eu$, counting 0 as "positive". Notice that $0 \le \operatorname{rot}(G) \le 1/2$. The rotation number for more general pairs $G$, and its properties, will be described in Section 3. Assuming that $\alpha$ is irrational, the value of $\operatorname{rot}(G)$ is independent of the solution $u$. For an AM pair $G_E$ with spectral energy $E$, it can be shown [11] that the rotation number is related to the integrated density of states via $\mathfrak{k}(E) = 2\operatorname{rot}(G_E)$. Roughly speaking, this relation expresses the fact that the wave number (inverse wave length) of eigenfunctions increases with the energy $E$.

**Convention 1.** In what follows, we fix the phase $t$ in the definition of the Schrödinger factor (1.5) and the Schrödinger operator (1.6) to the value $t = \alpha/2$, unless specified otherwise. This makes the symmetric AM factors $A_\circ$ reversible in the sense of Definition 1.1.

The above-mentioned gap-labeling property associates spectral gaps with rotation numbers that belong to $\frac{1}{2}\mathbb{Z}[\alpha]$. We will now describe a phenomenon that is associated with a different set of rotation numbers in $\mathbb{Q}[\alpha]$.

**Definition 1.3.** *A number $\rho \in [0, 1/2]$ will be called positive periodic (as a rotation number) if $\rho = \frac{w}{v} + \frac{u}{v}\alpha$ for some integers $u, v, w$ with $v > 2$ and $\gcd(u, v, w) = 1$, and if $\left|\frac{u}{v} - \frac{w}{v}\alpha\right| \le \frac{1}{2}$.*

We will call $\rho \in [-1/2, 0]$ negative periodic if $\rho + 1/2$ is positive periodic. Notice that, for any given integer $v > 2$, the (positive or negative) periodic rotation numbers in $\frac{1}{v}\mathbb{Z}[\alpha]$ constitute a discrete subset of $\mathbb{R}$.

In what follows, $\alpha$ denotes the inverse golden mean, unless specified otherwise. Let $p_n$ be the $n$-th Fibonacci number, and define $q_n = p_{n+1}$. Our main result in this paper is the following.

**Theorem 1.4.** *Let $\rho$ be positive periodic. Then there exists a positive integer $n$ and two nonzero even entire functions $b_\diamond$ and $a_\diamond$, such that the following holds. Consider the family of AM maps $G_{\lambda,E}$ parametrized by the coupling constant $\lambda > 0$ and the energy $E$. Then there exists a real analytic function $\epsilon$, defined on an open neighborhood of zero,* *such that if $\delta = \lambda^{-1}$ belongs to the domain of $\epsilon$, and if $E = \lambda\epsilon(\delta)$, then $G$ has rotation number* $\mathrm{rot}(G) = \rho$. *In this case, there exist sequences of positive real numbers $k \mapsto M_k$ and $k \mapsto W_k$, such that*

$$\begin{aligned} \lim_{k\to\infty} M_k\, A_\circ^{*p_{3nk}}\bigl(\alpha^{3nk}x\bigr) &= b_\diamond(x)\mathtt{A}^\dagger\,, \\ \lim_{k\to\infty} W_k\, A_\circ^{*q_{3nk}}\bigl(\alpha^{3nk}x\bigr) &= a_\diamond(x)\mathtt{A}\,, \end{aligned} \tag{1.8}$$

*for some constant $2\times 2$ matrix* $\mathtt{A}$ *of rank one. Convergence in (1.8) is uniform on compact subsets of $\mathbb{C}$. An analogous result holds for any two-parameter family that is sufficiently close to the* AM *family (in a sense described later).*

Both $b_\diamond$ and $a_\diamond$ have infinitely many zeros, all simple and on the real axis. The zeros can be determined rather explicitly, and this is used to construct the functions $b_\diamond$ and $a_\diamond$. We note that the function $\epsilon = \epsilon(\delta)$ and the matrix $\mathtt{A}$ in the above theorem can depend on the family. But the scaling limits $b_\diamond$ and $a_\diamond$ are universal.

We expect that an analogous result holds for all $\lambda > 1$. Numerically, this is observed at energy $E = 0$, which corresponds to $\rho = {}^1\!/_4$. For this particular value of $\rho$, a proof seems feasible, since the emergence of the limit zeros is quite transparent in this case [29]. We also expect that Theorem 1.4 generalizes to arbitrary quadratic irrationals $\alpha$.

**Remark 2.** For non-Schrödinger maps $G = (\alpha, A)$, the rotation number $\mathrm{rot}(G)$ in Theorem 1.4 has to be replaced by the rotation number $\varrho(G)$ defined in Section 3. Using $\varrho(G)$ in place of $\mathrm{rot}(G)$, an analogous theorem holds for negative periodic rotation numbers as well. This corresponds to replacing $A$ by $-A$. For a Schrödinger factor $A$, we can get $-A$ via a conjugacy by $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ and then replacing $v, E$ by their negatives. The conjugacy is equivalent to replacing $u$ by $q \mapsto (-1)^q u_q$ in the equation (1.6).

Theorem 1.4 is best understood in a dynamics context. The pairs $G = (\alpha, A)$ with $A(x) \in \mathrm{SL}(2,\mathbb{R})$ are naturally associated with skew-product maps

$$G(x,y) = \bigl(x+\alpha, A(x)y\bigr)\,, \qquad x \in X\,, \quad y \in Y\,, \tag{1.9}$$

on $X \times Y$, where $X = \mathbb{T}$ and $Y = \mathbb{R}^2$. The $q$-th iterate of $G$ is $G^q = \bigl(q\alpha, A^{*q}\bigr)$, with $A^{*q}$ as defined in (1.1). So Theorem 1.4 can be viewed as describing the asymptotic behavior of long orbits for skew-product maps. Due to the scaling involved in (1.8), we will also need to consider the case $X = \mathbb{R}$. The condition $\det(A) = 1$ will be weakened as well, and $Y = \mathbb{R}$ will be used for limit cases.

Our proof of Theorem 1.4 involves the use of a renormalization transformation $\mathcal{R}$ that acts on pairs $P = (F, G)$ of skew-product maps $F = (1, B)$ and $G = (\alpha, A)$. We consider families of pairs $P_{\delta,\epsilon}$ that admit a scaling limit as $\delta \to 0$. For Schrödinger factors (1.5), we use $\delta = \lambda^{-1}$, multiply $A$ by $\delta$, and set $E = \lambda\epsilon$. If $\varrho(G_{\delta,\epsilon}) = \rho$ with $\rho$ positive periodic, then the asymptotic behavior (1.8) is governed by a fixed point $P_\diamond$ of (a modified version of) the transformation $\mathcal{R}^{3n}$. The local stable manifold $\mathcal{W}^s$ of this transformation at $P_\diamond$ is of codimension 1 and agrees with the level set $\varrho(G) = \rho$. So the real analytic curve $\delta \mapsto \epsilon(\delta)$ described in Theorem 1.4 characterizes the intersection of the given family with this manifold $\mathcal{W}^s$. Similar manifolds were constructed in [24] for subcritical skew-flows with Diophantine rotation vectors.

Given the relation $2\,\mathrm{rot}(G_E) = \mathfrak{k}(E)$ for the AM family, the above suggests that the energies $E = \lambda\epsilon\bigl(\lambda^{-1}\bigr)$ are eigenvalues of the operator $\mathcal{H}_\lambda^{\alpha/2}$. Analytic curves of eigenvalues are a standard feature in real analytic families of compact self-adjoint operators, even in the presence of level crossings [5]. This applies e.g. to the operators $P_L^*\mathcal{H}_\lambda^t P_L$ mentioned earlier. So it seems possible that $E = \lambda\epsilon\bigl(\lambda^{-1}\bigr)$ is a limit as $L \to \infty$ of such eigenvalue curves.

From a dynamics perspective, the zeros at $\pm\rho$ of the limit function $a_\diamond$ seem associated with factors $A_\circ(x)$ in large products that map the expanding direction (from the product to the right of this factor) to the contracting direction of the product to the left of this factor. This suggest that $x$ is near $\pm\rho$, which is where $A_\circ(x)$ can cover a wide range of rotation angles. The location of such factors may be associated with peaks of an eigenvector.

We note that the Hofstadter model has a large number of symmetries [13,16,17] and other important arithmetic features. It seems fair to say that their interplay with analysis is only poorly understood. This is our motivation for focusing on the inverse golden mean. Periodicity may be regarded as exceptional, but periodic orbits play an important role in dynamical systems, since they heavily influence the motion nearby. The AM family can be regarded as exceptional as well, but, as Theorem 1.4 shows, some of its “rigid” behavior is shared by nearby families.

The remaining part of this paper is organized as follows. In Section 2 we introduce the renormalization transformation $\mathcal{R}$ and prove a result that implies Theorem 1.2. Section 3 is devoted to limits of Schrödinger factors and the rotation number. In Section 4 we construct the periodic orbits of $\mathcal{R}$ that are associated with positive periodic rotation numbers. A proof of Theorem 1.4 is given in Section 5.

**Remarks 3.**

- To leading order, the scaling constants in Theorem 1.4 are $M_k \sim e^{-p_{3nk}L}$ and $W_k \sim e^{-q_{3nk}L}$, where $L$ is the Lyapunov exponent of $G$. For the AM factors, we suspect that a choice $M_k = Me^{-q_{3nk}L}$ and $W_k = We^{-p_{3nk}L}$ will work. But trying to verify this would go beyond the scope of this paper.
- If we allowed $v = 1$ and $v = 2$ in Definition 1.3, then $\{0, {}^{\alpha}\!/_2, {}^1\!/_2\}$ would count as positive periodic as well. The associated energies for the AM operators correspond to spectral gaps. We believe that Theorem 1.4 can be proved for these rotation numbers as well, but they would require special treatment.
- Numerically, critical fixed points $\bigl($for the appropriate power of $\mathcal{R}^3\bigr)$ have been found for rotation numbers in $\{0, {}^1\!/_8, {}^1\!/_6, {}^1\!/_2 - {}^{\alpha}\!/_2, {}^1\!/_4, {}^{\alpha}\!/_2, {}^2\!/_6, {}^3\!/_8, {}^1\!/_2\}$. Existence proofs have been given in [27,29] in the case of rotation numbers $\{0, {}^1\!/_4, {}^1\!/_2\}$.
- Conjecture 1.1 in [29] on critical periods only covers rational rotation numbers. A reasonable amendment would be to include all (positive or negative) periodic rotation numbers.
- Numerically, the unstable manifold associated with the critical fixed point of $\mathcal{R}^3$ described in Theorem 2.3 of [29] leads into a supercritical fixed point $P_\diamond$ of the type described here. This concerns the case $\rho = {}^1\!/_4$, where the curve $E = \lambda\epsilon\bigl(\lambda^{-1}\bigr)$ is

trivial, namely $E=0$ by symmetry. Still, a possible renormalization picture would be that something similar occurs for any periodic rotation number $\rho$.

## 2. Renormalization

We use renormalization as a combinatorial tool for generating factors of the type that appear in (1.8). From an arithmetic point of view, the renormalization (RG) transformation $\mathcal{R}$ defined below lifts the Gauss map for real numbers to pairs of maps $(F,G)$. From an analysis point of view, $\mathcal{R}$ constitutes a dynamical system on a space of such pairs.

Consider a pair $(F,G)$ of skew-product maps $F=(1,B)$ and $G=(\alpha,A)$ on $\mathbb{R}\times\mathbb{R}^2$, with factors $B$ and $A$ that take values in $\mathrm{SL}(2,\mathbb{R})$. If $G$ commutes with $F$, then $G$ can be viewed as a map on a cylinder in $\mathbb{R}\times\mathbb{R}^2$ whose points are the orbits of $F$. Assume that $\alpha<1$ is a positive irrational number, and denote by $c$ the integer part of $\alpha^{-1}$. Then the basic renormalized pair is defined by

$$\mathcal{R}(F,G)=\bigl(\check{F},\check{G}\bigr)\,,\qquad \check{F}=\Lambda^{-1}G\Lambda\,,\quad \check{G}=\Lambda^{-1}FG^{-c}\Lambda\,,\tag{2.1}$$

where $\Lambda(x,y)=(\alpha x,y)$. By construction, the first component of $\check{F}$ is again 1, while the first component of $\check{G}$ is $\check{\alpha}=\alpha^{-1}-c$. Notice that $\alpha\mapsto\check{\alpha}$ is the Gauss map that appears in the continued fraction expansion of $\alpha$.

In what follows, we restrict to the inverse golden mean, which satisfies $\alpha^{-1}-1=\alpha$. So we can fix $c=1$ in (2.1). The transformation $\mathcal{R}$ extends readily to factors in $\mathrm{GL}(2,\mathbb{R})$. In this case, $G^{-1}$ is replaced by the quasi-inverse $G^\dagger$ defined below. We note that a map $G=(\alpha,A)$ is reversible in the sense of Definition 1.1 precisely if

$$\mathcal{S}^{-1}G\mathcal{S}=G^\dagger\overset{\text{def}}{=}\bigl(-\alpha,A(\,\boldsymbol{.}-\alpha)^\dagger\bigr)\,,\qquad \mathcal{S}(x,y)=(-x,Sy)\,.\tag{2.2}$$

A pair $P=(F,G)$ is said to be reversible if both $F$ and $G$ are reversible.

The conjugacy by $\Lambda$ only renormalizes the translational parts of our skew-products. We will often have to renormalize the matrix part as well. When considering periodic orbits of length $k$, it is best to do this only once per period. So for $\mathcal{R}^k$ we use a scaling

$$\Lambda_k=\mathcal{L}_k\Lambda^k\,,\qquad \mathcal{L}_k(x,y)=(x,L_k y)\,,\tag{2.3}$$

where $L_k$ a suitable matrix in $\pm\mathrm{SL}(2,\mathbb{R})$ that commutes with $S$. This matrix can depend on the pair $(F,G)$. It will be specified as needed, when the choice matters.

The definition (2.1) of $\mathcal{R}$ involves the basic composition operator $\mathcal{C}(F,G)=(G,FG^\dagger)$. Due to the symmetry $A\mapsto -A$ mentioned in Remark 2, we restrict to powers of $\mathcal{C}$ that are multiples of 3. Then it is convenient to replace $\mathcal{C}^3$ by the transformation $\mathcal{C}_3$ defined by

$$\mathcal{C}_3(F,G)=\bigl(GF^\dagger G\,\boldsymbol{,}\,G^\dagger FG^\dagger FG^\dagger\bigr)\,.\tag{2.4}$$

The only difference between $\mathcal{C}^3$ and $\mathcal{C}_3$ is the order of the factors. This is irrelevant for commuting pairs; but for non-commuting pairs, which need to be included in our analysis, the transformation $\mathcal{C}^3$ does not in general preserve reversibility, while $\mathcal{C}_3$ does.

Given any $n \geq 1$, we define the RG transformation $\mathcal{R}_{3n}$ by setting

$$\mathcal{R}_{3n}(F,G) = \bigl(\Lambda_{3n}^{-1}\hat{F}\Lambda_{3n}\,,\Lambda_{3n}^{-1}\hat{F}\Lambda_{3n}\bigr)\,, \qquad \bigl(\hat{F},\hat{G}\bigr) = \mathcal{C}_3^n(F,G)\,. \tag{2.5}$$

Consider first the case $n=1$. Let $\bigl(\tilde{F},\tilde{G}\bigr) = \mathcal{R}_3(F,G)$. A straightforward computation shows that the symmetric factor $\tilde{B}_\circ$ for $\tilde{F}$ is

$$\tilde{B}_\circ(x) = L_3^{-1}A_\circ\bigl(\alpha^3x - \tfrac{1-\alpha}{2}\bigr)B_\circ\bigl(\alpha^3x\bigr)^\dagger A_\circ\bigl(\alpha^3x + \tfrac{1-\alpha}{2}\bigr)L_3\,, \tag{2.6}$$

and that the symmetric factor $\tilde{A}_\circ$ of $\tilde{G}$ is

$$\begin{aligned}\tilde{A}_\circ(x) = L_3^{-1}A_\circ\bigl(\alpha^3x + (1-\alpha)\bigr)^\dagger B_\circ\bigl(\alpha^3x + \tfrac{1-\alpha}{2}\bigr)A_\circ\bigl(\alpha^3x\bigr)^\dagger \times\\ \times B_\circ\bigl(\alpha^3x - \tfrac{1-\alpha}{2}\bigr)A_\circ\bigl(\alpha^3x - (1-\alpha)\bigr)^\dagger L_3\,.\end{aligned} \tag{2.7}$$

Notice that, if $B_\circ$ and $A_\circ$ are analytic in a strip $|\operatorname{Im} x| < \delta$, then $\tilde{B}_\circ$ and $\tilde{A}_\circ$ are analytic in the strip $|\operatorname{Im} x| < \alpha^{-3}\delta$. The following is equally straightforward to verify.

Let $r_{\scriptscriptstyle B}$ and $r_{\scriptscriptstyle A}$ be positive real numbers.

**Proposition 2.1.** *Assume that $\alpha^3 r_{\scriptscriptstyle A} + \frac{1}{2}\alpha^2 \leq r_{\scriptscriptstyle B} \leq \alpha^{-3}r_{\scriptscriptstyle A} - \frac{1}{2}\alpha^{-1}$. Notice that $r_{\scriptscriptstyle B} = \alpha/2$ and $r_{\scriptscriptstyle A} = 1/2$ satisfy this condition. If the domains of $B_\circ$ and $A_\circ$ include the (real or complex) disks $|x| \leq r_{\scriptscriptstyle B}$ and $|x| \leq r_{\scriptscriptstyle B}$, respectively, then the domains of $\tilde{B}_\circ$ and $\tilde{A}_\circ$ include the same disks. Assume now that*

$$r_{\scriptscriptstyle B} > \tfrac{\alpha}{2}\,, \qquad r_{\scriptscriptstyle A} > \tfrac{1}{2}\,, \qquad \alpha^3 r_{\scriptscriptstyle A} + \tfrac{1}{2}\alpha^2 < r_{\scriptscriptstyle B} < \alpha^{-3}r_{\scriptscriptstyle A} - \tfrac{1}{2}\alpha^{-1}\,. \tag{2.8}$$

*Suppose that the domains of $B_\circ$ and $A_\circ$ include the disks $|x| < r_{\scriptscriptstyle B}$ and $|x| < r_{\scriptscriptstyle B}$, respectively. Then the domains of the symmetric factors associated with $\mathcal{R}_3^n(F,G)$ include the same disks. Furthermore, these domains grow asymptotically like $\alpha^{-3n}$, as $n \to \infty$. Moreover, the values of these factors on any given compact set are determined (for sufficiently large $n$) by the values of $B_\circ$ on the disk $|x| < r_{\scriptscriptstyle B}$ and the values of $A_\circ$ on the disk $|x| < r_{\scriptscriptstyle A}$.*

This motivates the following choice of function spaces. Given $r > 0$, denote by $\mathcal{F}_r$ the space of all real analytic functions $f$ on $(-r,r)$ that have a finite norm

$$\|f\|_r = \sum_{n=0}^{\infty}\bigl|f^{(n)}(0)\bigr|\,\frac{r^n}{n!}\,. \tag{2.9}$$

Notice that every function $f \in \mathcal{F}_r$ extends analytically to the complex disk $|x| < r$. Furthermore, $\mathcal{F}_r$ is a Banach algebra under pointwise multiplication of functions. This was crucial in [27,29] but is less important here.

The space of matrix functions

$$x \mapsto A_\circ(x) = \begin{bmatrix} a_\circ(x) & b_\circ(x)\\ c_\circ(x) & d_\circ(x)\end{bmatrix} \tag{2.10}$$

with entries in $\mathcal{F}_r$ will be denoted by $\mathcal{F}_r^4$. The norm of $A_\circ \in \mathcal{F}_r^4$ is defined as $\|A_\circ\|_r = \|a_\circ\|_r + \|b_\circ\|_r + \|c_\circ\|_r + \|d_\circ\|_r$. Given a pair $r = (r_{_B}, r_{_A})$ of positive real numbers, we define $\mathcal{H}_r$ to be the vector space of all pairs $\mathcal{P} = (B_\circ, A_\circ)$ in $\mathcal{F}_{r_B}^4 \times \mathcal{F}_{r_A}^4$, equipped with the norm $\|\mathcal{P}\|_r = \|B_\circ\|_{r_B} + \|A_\circ\|_{r_A}$.

Given that every skew-product map that appears in our analysis has a pre-determined first component, we will identify a skew-product map $G = (\alpha, A)$ with its symmetric factor $A_\circ$. Referring to (2.10), we note that $G$ is reversible if and only if $a_\circ$ and $d_\circ$ are even, while $b_\circ(x) = -c_\circ(-x)$. The subspace of reversible pairs in $\mathcal{H}_r$ will be denoted by $\mathcal{H}_r'$.

**Convention 4.** We always assume that domain parameters $r = (r_{_B}, r_{_A})$ satisfy the domain condition (2.8). And whenever $L_k : \mathcal{H}_r \to \pm\mathrm{SL}(2, \mathbb{R})$ remains unspecified, it is assumed to be continuous.

**Lemma 2.2.** *The transformation $\mathcal{R}_3 : \mathcal{H}_r \to \mathcal{H}_r$ is compact and preserves reversibility.*

The reversibility-preserving property is a consequence of the fact that the products $GF^\dagger G$ and $G^\dagger FG^\dagger FG^\dagger$ that appear in (2.4) are palindromic. Compactness is a consequence of the fact that $\mathcal{R}_3$ is analyticity-improving, in the sense that $\mathcal{R}_3$ maps bounded sets in $\mathcal{H}_r$ to bounded sets in $\mathcal{H}_{r'}$, for some $r_{_B}' > r_{_B}$ and $r_{_A}' > r_{_A}$. Now one uses that the inclusion maps $\mathcal{F}_{r_B'} \to \mathcal{F}_{r_B}$ and $\mathcal{F}_{r_A'} \to \mathcal{F}_{r_A}$ are compact.

The same holds of course for the transformations $\mathcal{R}_{3n}$ with $n > 1$.

When the exact growth of products is not needed, we also consider a modified version of our RG transformation, defined by

$$\mathfrak{R}_{3n} = \mathfrak{N} \circ \mathcal{R}_{3n}\,, \tag{2.11}$$

where $\mathfrak{N}$ is a suitable normalization. Unless specified otherwise, $\mathfrak{N}$ consists in normalizing $B_\circ \mapsto \|B_\circ\|_{r_B}^{-1} B_\circ$ and $A_\circ \mapsto \|A_\circ\|_{r_A}^{-1} A_\circ$.

The following implies Theorem 1.2.

**Theorem 2.3.** *Choose $L_3 = S$. Let $P' = (F', G')$ be a pair in $\mathcal{H}_r$, of the form $F' = (1, \mathbf{1})$ and $G' = (\alpha, A')$. Assume hat $G'$ has a positive Lyapunov exponent and is reversible. Denote by $K$ the set of all accumulation points of the sequence $n \mapsto \mathfrak{R}_3^n(P')$. Then every pair $P = (F, G)$ in $K$ has the following property. $F = (1, B)$ and $G = (\alpha, A)$, with $B_\circ(x) = b_\circ(x)\mathtt{A}^\dagger$ and $A_\circ(x) = a_\circ(x)\mathtt{A}$ for all $x$, where $b_\circ$ and $a_\circ$ are nonzero even entire functions and $\mathtt{A}$ is a constant $2 \times 2$ matrix of rank 1.*

**Proof.** First, notice that $SC_\circ(x)S = C_\circ(-x)^\dagger$ for any reversible map $H = (\gamma, C)$. So the conjugacy by $L_3 = S$ in (2.6) and (2.7) just replaces each factor $C(x)$ by $C(-x)^\dagger$. In particular, the leftmost factor in (2.7) becomes $A_\circ\bigl(-\alpha^3 x - (1-\alpha)\bigr)$.

Since the transformation $\mathfrak{R}_3$ is compact, the set $K$ is compact and non-empty. And by Proposition 2.1, the symmetric factors $B_\circ$ and $A_\circ$ of any pair $P \in K$ are entire analytic. They have norm 1 and thus cannot be zero. Without the normalization $\mathfrak{N}$, the norm of $\mathcal{R}_3^n(P_0)$ tends to infinity, since $G$ has a positive Lyapunov exponent. So the factors $B_\circ$ and $A_\circ$ have determinant zero and rank 1.

Consider the representation (2.10) for $A_\circ$. By reversibility, the functions $a_\circ$ and $d_\circ$ are even, while $b_\circ(x) = -c_\circ(-x)$. Consider first the case where one of $a_\circ, b_\circ, c_\circ, d_\circ$ is identically zero. Then $a_\circ d_\circ = b_\circ c_\circ = 0$, implying that $A_\circ = a_\circ\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ or $A_\circ = d_\circ\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]$. Using now (2.6) and (2.7), we see that $\tilde{P} = \mathfrak{R}_3(P)$ has symmetric factors with the properties described in Theorem 2.3.

Assume now that none of $a_\circ, b_\circ, c_\circ, d_\circ$ is identically zero. Consider the ratio $r_0 = c_\circ/a_\circ$. This is a meromorphic function describing the angle between the range of $A_\circ$ and one of the coordinate axis. Denote by $r_n$ the corresponding ratio for the pair $P_n = \mathfrak{R}_3^n(P)$. By (2.7) we have

$$r_1(x) = r_0\bigl(-\alpha^3 x - (1-\alpha)\bigr)\,. \tag{2.12}$$

Notice that the function $x \mapsto -\alpha^3 x - (1-\alpha)$ has $x_0 = -{}^{\alpha}\!/_{2}$ as a fixed point. Setting $R_n(t) = r_n\bigl(x_0 + t\bigr)$, the identity (2.12) becomes

$$R_n(t) = R_0\bigl((-\alpha)^{3n} t\bigr)\,, \tag{2.13}$$

with $n = 1$. The same holds for any $n > 0$ by iteration. Unless $R$ has a pole at the origin, we see that $R_n \to R_0(0)$ on some open neighborhood of the origin. If $R$ has a pole at the origin, we repeat the above with $r_0 = a_\circ/c_\circ$ in place of $r_0 = c_\circ/a_\circ$. Using compactness again, this shows that $K$ includes a pair $P$ with constant ratio $r_0$ in some open neighborhood of $x_0$. In the case where $r_0 = a_\circ/c_\circ$, this implies that $a_\circ = r_0 c_\circ$ and $b_\circ = r_0 d_\circ$ on all of $\mathbb{C}$. If $r_0 = 0$, then we are in the case discussed at the beginning. Otherwise, the reversibility property $b_\circ = -c_\circ$ shows that $a_\circ, b_\circ, c_\circ, d_\circ$ are all constant multiples of $c_\circ$. A similar argument applies in the case $r_0 = c_\circ/a_\circ$. In either case, since $a_\circ$ and $d_\circ$ are even and not both identically zero, $K$ includes a pair $P$ that admits the representation described in Theorem 2.3. **QED**

We are interested in describing the points $x$ where the limit factors $B_\circ$ and $A_\circ$ in Theorem 2.3 are zero. For a limit pair $P = (F, G)$ with $F = (1, B)$ and $G = (\alpha, A)$, the equations (2.6) and (2.7) can be used to describe how zeros propagate under renormalization. Let $\mathcal{B}$ be the set of zeros of $B_\circ$ and $\mathcal{A}$ the set of zeros of $A_\circ$. Let $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ be the set of zeros of the symmetric factors for $\tilde{P} = \mathcal{R}_3(P)$. From (2.6) and (2.7) we see that

$$\tilde{\mathcal{B}} = \alpha^{-3}\hat{\mathcal{B}}\,, \qquad \hat{\mathcal{B}} = \mathcal{B} \bigcup_{m=\pm 1} \Bigl(\mathcal{A} + \tfrac{m}{2}\alpha^2\Bigr)\,, \tag{2.14}$$

and

$$\tilde{\mathcal{A}} = \alpha^{-3}\hat{\mathcal{A}}\,, \qquad \hat{\mathcal{A}} = \bigcup_{m=\pm 1} \Bigl(\mathcal{B} + \tfrac{m}{2}\alpha^2\Bigr) \bigcup_{m=0,\pm 1} \Bigl(\mathcal{A} + m\alpha^2\Bigr)\,. \tag{2.15}$$

The map $(\mathcal{B}, \mathcal{A}) \mapsto \bigl(\tilde{\mathcal{B}}, \tilde{\mathcal{A}}\bigr)$ will be described in more detail in Section 4.

## 3. Limit skew-products and the rotation number

In order to describe the behavior of factors (1.5) with large values of $\lambda$, we divide these matrices by $\lambda$ and consider the resulting factors

$$A(x)=\begin{bmatrix} v(t+x)-\epsilon & -\delta \\ \delta & 0 \end{bmatrix}, \tag{3.1}$$

where $\delta=\lambda^{-1}$ and $\epsilon=\lambda^{-1}E$. Here, and in what follows, we assume that $\lambda>1$. Orbits for the associated skew-product map $G=(\alpha,A)$ on $\mathbb{T}\times\mathbb{R}^2$ are in one-to-one correspondence with solutions of the equation $H_\delta^t u=\epsilon u$, where

$$(H_\delta^t u)_q=v(t+q\alpha)u_q-\delta u_{q+1}-\delta u_{q-1}\,,\qquad q\in\mathbb{Z}\,. \tag{3.2}$$

Clearly, the operator $H_\delta^t$ on $\ell^2(\mathbb{Z})$ has a well-defined limit as $\delta\to 0$, given by setting $\delta=0$ in the above equation. A sequence $u\in\ell^2(\mathbb{Z})$ is an eigenvector of $H_0^t$ with energy $\epsilon$ if and only if $[v(t+q\alpha)-\epsilon]u_q=0$ for all $q$. So for every integer $q$, the function $u=\delta_q$ supported at $q$ is an eigenvector of $H_0^t$ with eigenvalue $v(t+q\alpha)$. Given that these eigenvectors span a dense subspace of $\ell^2(\mathbb{Z})$, the spectrum of $H_0^t$ agrees with the range of $v$, and it consists purely of eigenvalues and their accumulation points.

The integrated density of states $\mathfrak{k}(\epsilon)$ is the fraction (in the usual limit sense) of integers $q$ for which $v(t+q\alpha)\le\epsilon$. By ergodicity, this is simply the measure of the set of all points $x\in\mathbb{T}$ for which $v(x)\le\epsilon$.

Let us compute $\mathfrak{k}(\epsilon)$ for the AM potential $v(x)=-2\cos(2\pi x)$. For simplicity consider $t={}^{\alpha}\!/_{2}$. Then the symmetric limit factor associated with (3.1) is

$$A_\circ(x)=a_\circ(x)\begin{bmatrix}1&0\\0&0\end{bmatrix},\qquad a_\circ(x)=-\epsilon-2\cos(2\pi x)\,. \tag{3.3}$$

Given $\epsilon$ in the spectrum $[-2,2]$ of $H_0^t$, we define $\rho=\rho(\epsilon)$ in $[0,{}^1\!/_2]$ by the equation $\epsilon=-2\cos(2\pi\rho)$. Then

$$a_\circ(x)=2\cos(2\pi\rho)-2\cos(2\pi x)=4\sin(\pi(x-\rho))\sin(\pi(x+\rho))\,. \tag{3.4}$$

If we represent $\mathbb{T}$ as the interval $[-{}^1\!/_2,{}^1\!/_2]$ with the endpoints identified, then $a_\circ$ is negative on $(-\rho,\rho)$ and positive on the complement of $[-\rho,\rho]$. This shows that $\mathfrak{k}(\epsilon)=2\rho$.

In order to determine the rotation number of the skew-product map $G_\epsilon=(\alpha,A)$ with symmetric factor (3.3), let us first reformulate the definition of the number $\mathrm{Rot}_N(G)$ that appears in (1.7). It counts half the number of times that the first component of the vector $A^{*k}(t)y$ changes sign as $k$ is increased from 1 to $N$. The limit (1.7) is independent of $t$ and of the initial condition $y\in\mathbb{R}^2\setminus\{0\}$. We choose the same definition for degenerate skew-products with $A(x)=a_\circ(x)\mathtt{A}$, but we assume that $\mathtt{A}y\ne 0$.

Notice that $A^{*k}(t)y$ changes sign as $k$ is increased to $k+1$ precisely when $a_\circ(t+k\alpha)$ is negative. So by ergodicity, the average number of sign changes is the measure of $[-\rho,\rho]$. The limit (1.7) is half this number, which is $\rho$. So in summary, we have

$$\mathfrak{k}(\epsilon)=2\rho\,,\qquad \mathrm{rot}(G_\epsilon)=\rho\,,\qquad \epsilon=-2\cos(2\pi\rho)\,. \tag{3.5}$$

In particular, $\mathfrak{k}(\epsilon) = 2\,\mathrm{rot}(G_\epsilon)$, which corresponds to the known relation for the AM factors. Notice that $H_0^t + 2\delta\mathrm{I} \geq H_\delta^t \geq H_0^t - 2\delta\mathrm{I}$, where $H \geq K$ means that $H - K$ is a positive operator. Using the definition of $\mathfrak{k}_L$ given in the introduction, we have $\mathfrak{k}_L\bigl(H_0^t + 2\delta\mathrm{I}, \epsilon\bigr) \leq \mathfrak{k}_L\bigl(H_\delta^t, \epsilon\bigr) \leq \mathfrak{k}_L\bigl(H_0^t - 2\delta\mathrm{I}, \epsilon\bigr)$. Taking $L \to \infty$ and then $\delta \to 0$, and using that $\mathfrak{k}(\epsilon) = 2\,\mathrm{rot}(G_\epsilon)$ for $0 \leq \delta < 1$, we obtain the following.

**Proposition 3.1.** *Let $G_\delta = (\alpha, A)$, with $A$ given by (3.1) and $v = -2\cos(2\pi\,\boldsymbol{.})$. Then $\mathrm{rot}(G_\delta) \to \mathrm{rot}(G_0)$ as $\delta \to 0$, uniformly in $\epsilon$.*

The same holds for real analytic potentials close to $v = -2\cos(2\pi\,\boldsymbol{.})$.

Notice that a skew-product map $G = (\alpha, A)$ with $A(x) = a(x)\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$ is for practical purposes a skew-product map $g = (\alpha, a)$ on $\mathbb{T} \times \mathbb{R}$, where $a(x)$ acts on $\mathbb{R}$ by multiplication. The rotation number of $g$, determined via sign changes, is clearly the same as the rotation number of $G$.

Recall that our definition (1.7) of the rotation number $\mathrm{rot}(G)$ is restricted to Schrödinger factors as defined by (1.5). The formula (1.7) can be applied to other pairs $G = (\alpha, A)$ as well. But it does not yield the desired result, in general, not even modulo $1/2$.

For a more general definition of a rotation number for skew-product maps $G = (\alpha, A)$ on $\mathbb{T} \times \mathbb{R}^2$, assume that $A : \mathbb{T} \to \mathrm{SL}(2, \mathbb{R})$ is continuous. Consider the map $y \mapsto A(x)y$ on $\mathbb{R}\setminus\{0\}$. Writing $y$ as $\|y\|\left[\begin{smallmatrix}\cos t\\ \sin t\end{smallmatrix}\right]$ and $\eta = A(x)y$ as $\|\eta\|\left[\begin{smallmatrix}\cos \tau\\ \sin \tau\end{smallmatrix}\right]$, we have a continuous function $\tau = \tau(x,t)$ from $\mathbb{T} \times \mathbb{S}$ to $\mathbb{S}$, where $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$. Each of the maps $t \mapsto \tau(x,t)$ on $\mathbb{S}$ admits a continuous lift $t \mapsto t + g(x,t)$ to $\mathbb{R}$. This lift is unique up to an additive constant $2\pi m_x$ with $m_x \in \mathbb{Z}$. Assuming now that $A$ is homotopic to $x \mapsto \mathbf{1}$, $g$ can be chosen continuous, with $m_x = m$ for all $x$. Using the map $\mathcal{G}$ defined by $\mathcal{G}(x,t) = (x + \alpha, t + g(x,t))$, one now defines

$$\varrho(G) = \lim_{N\to\infty} \frac{\Sigma_N(G)}{N}\,, \qquad \Sigma_N(G) \stackrel{\mathrm{def}}{=} \frac{1}{2\pi}\sum_{n=1}^{N} g\bigl(\mathcal{G}^n(x,t)\bigr)\,. \tag{3.6}$$

Here, $\alpha$ can be any irrational number. Using the unique ergodicity of $x \mapsto x + \alpha$ on $\mathbb{T}$, it is possible to prove [8,9] that the limit (3.6) exists, is independent of $(x,t)$, and that convergence is uniform in $(x,t)$. Due to the freedom of choosing $m \in \mathbb{Z}$, the rotation number $\varrho(G)$ is unique only modulo 1.

$\Sigma_N(G)$ measures the amount of rotation on the interval $\{1, 2, \ldots, N\}$, in units of 1 per revolution. For Schrödinger maps $G = (\alpha, A)$, we assume that the lift $g$ has been chosen in such a way that $g(0,0) \in [0, 2\pi)$. Then $\Sigma_N(G)$ agrees within an error $\leq 1$ with half the number of sign changes $\mathrm{Rot}_N(G)$. This follows from the fact that the second row of $A(x)$ is $[1\ 0]$, implying that $A(x)$ maps the right (left) half-plane to the upper (lower) half-plane. So every full revolution is associated with a single sign change positive $\to$ negative and a single sign change negative $\to$ positive; see also [10,12,20]. This implies in particular that $\mathrm{rot}(G) = \varrho(G)$ for Schrödinger factors (1.5) with continuous potentials $v$. (Replacing $A$ by $-A$ in this case yields $\varrho \mapsto \varrho - 1/2$ modulo 1, while sign counting would yield $\mathrm{rot} \mapsto 1/2 - \mathrm{rot}$.)

# 4. Periodic orbits

In this section we consider pairs of skew-product maps $p = (f, g)$ with $f = (1, b)$ and $g = (\alpha, a)$, where $b$ and $a$ are even real analytic functions whose values $a(x)$ and $b(x)$ act on $\mathbb{R}$ by multiplication. The RG transformations $\mathcal{R}_{3n}$ extend naturally to such pairs, if we define $b^\dagger = b$ and $a^\dagger = a$. Conjugacy has no effect for scalar factors, so $\Lambda_k = \Lambda^k$ for all $k$. The condition that $a$ and $b$ be even corresponds to reversibility.

Our goal here is to construct periodic orbits under renormalization and to describe some of their properties.

**Convention 5.** Throughout this section, the second component $A$ in a pair $(\alpha, A)$ representing a skew-product map denotes the symmetric factor. So the composition rule here is $(\beta, B)(\alpha, A) = (\alpha + \beta, C)$ with $C(x) = B(x + {}^{\alpha}\!/\!{}_{2})A(x - {}^{\beta}\!/\!{}_{2})$.

First we need to establish some properties of the map $(\mathcal{B}, \mathcal{A}) \mapsto \bigl(\tilde{\mathcal{B}}, \tilde{\mathcal{A}}\bigr)$ defined by the equations (2.14) and (2.15). A crucial ingredient in the proof below is the well-known identity

$$p_k - q_k\alpha = (-\alpha)^{k+1}\,, \qquad q_k = p_{k+1}\,, \tag{4.1}$$

where $p_k$ denotes the $k$-th Fibonacci number.

**Lemma 4.1.** *Let $\mathcal{B}_0 = \{\}$. Given $\mathcal{A}_0 \subset \mathbb{R}$, define $\mathcal{B}_n = \tilde{\mathcal{B}}_{n-1}$ and $\mathcal{A}_n = \tilde{\mathcal{A}}_{n-1}$ for $n = 1, 2, 3, \ldots$ Consider now $\mathcal{A}_0 = \{\rho\}$ with $0 \le \rho \le {}^1\!/\!{}_2$. Then $\mathcal{B}_n \cap [-{}^1\!/\!{}_2, {}^1\!/\!{}_2]$ and $\mathcal{A}_n \cap (-{}^1\!/\!{}_2, {}^1\!/\!{}_2)$ contain at most one point, for each $n$. The sequence $k \mapsto \mathcal{A}_k \cap [-{}^1\!/\!{}_2, {}^1\!/\!{}_2]$ is periodic if and only if $\rho$ is positive periodic or belongs to $\{0, {}^{\alpha}\!/\!{}_2, {}^1\!/\!{}_2\}$. Furthermore, some set $\mathcal{A}_n \cap [-{}^1\!/\!{}_2, {}^1\!/\!{}_2]$ includes a point $\tilde{\rho} \in \frac{1}{2}\mathbb{Z}[\alpha]$ if and only if $\tilde{\rho} - \rho \in \mathbb{Z}[\alpha]$.*

**Proof.** Let $B_0 = \{\}$ and $A_0 = \{\rho\}$, considered as sets on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $B_n = \hat{B}_{n-1}$ and $A_n = \hat{A}_{n-1}$ for $n = 1, 2, 3, \ldots$, using the map $(B, A) \mapsto \bigl(\hat{B}, \hat{A}\bigr)$ defined by (2.14) and (2.15). Consider the functions $b_0, a_0 : \mathbb{T} \to \mathbb{R}$ defined by $b_0(x) = 1$ and $a_0(x) = \sin(\pi(x - \rho))$. If we set $b_n = \hat{b}_{n-1}$ and $a_n = \hat{a}_{n-1}$ for $n = 1, 2, 3, \ldots$, where

$$\begin{aligned} \hat{b}(x) &= a\bigl(x - \tfrac{1-\alpha}{2}\bigr) b(x) a\bigl(x + \tfrac{1-\alpha}{2}\bigr)\,, \\ \hat{a}(x) &= a\bigl(x + (1-\alpha)\bigr) b\bigl(x + \tfrac{1-\alpha}{2}\bigr) a\bigl(x\bigr) b\bigl(x - \tfrac{1-\alpha}{2}\bigr) a\bigl(\alpha^3 x - (1-\alpha)\bigr)\,, \end{aligned} \tag{4.2}$$

then the zeros of $b_n$ and $a_n$ are precisely the sets $B_n$ and $A_n$, respectively. Notice that

$$\begin{aligned} b_n(x) &= a_0\bigl(x + \tfrac{q_{3n-1}}{2}\alpha\bigr) \cdots a_0\bigl(x + \tfrac{1}{2}\alpha\bigr) a_0\bigl(x - \tfrac{1}{2}\alpha\bigr) \cdots a_0\bigl(x - \tfrac{q_{3n-1}}{2}\alpha\bigr)\,, \\ a_n(x) &= a_0\bigl(x + \tfrac{q_{3n}-1}{2}\alpha\bigr) \cdots a_0\bigl(x + \alpha\bigr) a_0(x) a_0\bigl(x - \alpha\bigr) \cdots a_0\bigl(x - \tfrac{q_{3n}-1}{2}\alpha\bigr)\,. \end{aligned} \tag{4.3}$$

So the zeros of $a_n$ constitute an orbit on $\mathbb{T}$ under translation $x \mapsto x + \alpha \pmod 1$. This orbit has length $q_{3n}$ and its center point is at $\rho$. By the three-gap theorem [2,3,4], the gaps between adjacent points in $A_n$ can only take the values $\alpha^{3n-2}$, $\alpha^{3n-1}$, and $\alpha^{3n}$. In fact, the gap $\alpha^{3n-2}$ cannot occur, since it gets closed in the last step of the orbit. So the gaps in the scaled orbit $\mathcal{A}_n = \alpha^{-3n} A_n$ are either $\alpha^{-1}$ or 1. Thus, $\mathcal{A}_n$ contains at most one point in $(-{}^1\!/\!{}_2, {}^1\!/\!{}_2)$. Similarly, the gaps in $B_n$ are no shorter $\alpha^{3n-1}$, so the scaled orbit $\mathcal{B}_n = \alpha^{-3n} B_n$ has at most one point in $[-{}^1\!/\!{}_2, {}^1\!/\!{}_2]$.

Consider now the condition that $\mathcal{A}_n$ contains a given real number $\tilde{\rho}$. This condition is equivalent to $\tilde{a}_n(\tilde{\rho}) = 0$, where $\tilde{a}_n(x) = a_n\bigl(\alpha^{3n}x\bigr)$. From (4.3) we see that $\tilde{a}_n(\tilde{\rho}) = 0$ exactly when

$$\bigl[(-\alpha)^{3n} - 1\bigr]\tilde{\rho} - \nu_2 + \nu_1\alpha = \rho - \tilde{\rho}\,, \qquad |\nu_1| \le \frac{q_{3n}-1}{2}\,, \tag{4.4}$$

for some $\nu \in \mathbb{Z}^2$. Using that $(p_{3n-1}-1) - p_{3n}\alpha = (-\alpha)^{3n} - 1$, this can be written as

$$\bigl[(p_{3n-1}-1) - p_{3n}\alpha\bigr]\tilde{\rho} - \nu_2 + \nu_1\alpha = \rho - \tilde{\rho}\,, \qquad |\nu_1| \le \frac{q_{3n}-1}{2}\,. \tag{4.5}$$

If $\tilde{\rho} = 0$, then (4.5) can be satisfied from some $n$ if and only if $\rho \in \mathbb{Z}[\alpha]$. And a straightforward computation shows that $\rho = 0$ yields $\mathcal{A}_n \cap [-1/2, 1/2] = \{0\}$ for all $n$. Consider now an arbitrary $\tilde{\rho} \in \frac{1}{2}\mathbb{Z}[\alpha]$. In this case, the equation (4.5) can be written as

$$\left[\frac{p_{3n-1}-1}{2} - \frac{p_{3n}}{2}\alpha\right] 2\tilde{\rho} - \nu_2 + \nu_1\alpha = \rho - \tilde{\rho}\,, \qquad |\nu_1| \le \frac{q_{3n}-1}{2}\,. \tag{4.6}$$

Notice that $p_{3n-1}-1$ and $p_{3n}$ are even. So we must have $\rho \in \tilde{\rho} + \mathbb{Z}[\alpha]$. For $\rho = 1/2$ and $\rho = \alpha/2$, an explicit computation shows that $\mathcal{A}_n \cap [-1/2, 1/2] = \{\pm\rho\}$ for all $n$. We note also that (4.6) has no solution for $\tilde{\rho} = 1/2 - \alpha/2$.

Consider now the periodicity condition $\tilde{a}_n(\rho) = 0$, which corresponds to setting $\tilde{\rho} = \rho$ in (4.5). Clearly, we need $\rho \in \mathbb{Q}[\alpha]$ for (4.5) to have a solution in this case. So write $\rho = \frac{w}{v} + \frac{u}{v}\alpha$ for some $v > 0$. Without loss of generality, we can restrict restrict to values of $n$ for which $p_{3n-1} \equiv 1$ and $p_{3n} \equiv 0$ modulo $v$. This condition is satisfied e.g. if $n$ is a multiple of the Pisano period $\ell(v)$. By definition, the Pisano period $\ell(v)$ is the period of the Fibonacci sequence $k \mapsto p_k$ modulo $v$. Then we have $\tilde{a}_n(\rho) = 0$, if and only if

$$\left[\frac{p_{3n-1}-1}{v} - \frac{p_{3n}}{v}\alpha\right](w + u\alpha) - \nu_2 + \nu_1\alpha = 0\,, \qquad |\nu_1| \le \frac{q_{3n}-1}{2}\,, \tag{4.7}$$

for some $\nu \in \mathbb{Z}^2$. Multiplying out the product $[\ldots](w+u\alpha)$ in the above equation and using that $\alpha^2 = 1 - \alpha$, one finds that

$$\nu_1 = \frac{p_{3n}}{v}w - \frac{q_{3n}-1}{v}u\,, \qquad \nu_2 = \frac{p_{3n-1}-1}{v}w - \frac{p_{3n}}{v}u\,. \tag{4.8}$$

And the condition $|\nu_1| \le \frac{q_{3n}-1}{2}$ becomes

$$\left|\frac{p_{3n}}{q_{3n}-1}\frac{w}{v} - \frac{u}{v}\right| \le \frac{1}{2}\,. \tag{4.9}$$

Assume now that $\rho$ is neither 0 nor $1/2$, since these values have been covered already. Then equality in (4.9) can hold only for finitely many values of $n$. Given that $\frac{p_{3n}}{q_{3n}-1} \to \alpha$, we have $\tilde{a}_n(\rho) = 0$ as $n \to \infty$ along multiples of $\ell(v)$ if and only if $\left|\frac{w}{v}\alpha - \frac{u}{v}\right| < \frac{1}{2}$, meaning that $\rho$ is positive periodic or equal to ${}^{\alpha}\!/_2$. This show that the sequence $k \mapsto \mathcal{A}_k \cap [-{}^1\!/_2, {}^1\!/_2]$ is periodic if and only if $\rho$ is positive periodic or belongs to $\{0, {}^{\alpha}\!/_2, {}^1\!/_2\}$. **QED**

We note that points evolve independently under $(\mathcal{B}, \mathcal{A}) \mapsto (\tilde{\mathcal{B}}, \tilde{\mathcal{A}})$. Having $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \{x_1, x_2, \ldots, x_m\} \subset (-{}^1\!/_2, {}^1\!/_2)$ results in $\mathcal{B}_n \cap [-{}^1\!/_2 {}^1\!/_2]$ and $\mathcal{A}_n \cap (-{}^1\!/_2, {}^1\!/_2)$ containing $m$ points or none, for all $n$. Furthermore, if $\mathcal{A}_0$ is invariant under $x \mapsto -x$, then so are all the sets $\mathcal{A}_n$ and $\mathcal{B}_n$.

**Convention 6.** In the remaining part of this paper, $\rho$ is a fixed but arbitrary positive periodic rotation number. Referring to the periodicity statement in Lemma 4.1, the fundamental period of the sequence $k \mapsto \mathcal{A}_k$, generated from $\mathcal{B}_0 = \{\}$ and $\mathcal{A}_0 = \{\rho\}$, will from now on be denoted by $n$.

Recall from (2.11) that $\mathfrak{R}_{3n} = \mathfrak{N} \circ \mathcal{R}_{3n}$, where $\mathfrak{N}$ is a suitable multiplicative normalization. The following theorem deals only with (zeros of) specific functions, so we "normalize" here simply by identifying nonzero functions that are constant multiples of each other.

**Theorem 4.2.** *$\mathfrak{R}_{3n}$ has a fixed point $p_\diamond = ((1, b_\diamond), (\alpha, a_\diamond))$ with the following properties. The functions $b_\diamond$ and $a_\diamond$ are entire and even, and as functions of $\sqrt{x}$, they are of order ${}^1\!/_2$. The zeros of $b_\diamond$ and $a_\diamond$ are all real and simple. $a_\diamond$ has zeros at $\pm\rho$ and no other zeros in the interval $[-{}^1\!/_2, {}^1\!/_2]$.*

Following an approach used in [28], we construct the functions $b_\diamond$ and $a_\diamond$ by first determining their zeros. As mentioned above, we identify nonzero functions $u$ and $v$ that are constant multiples of each other; in symbols, $u \equiv v$. The set of zeros of a function $u$ will be denoted by $\mathcal{Z}(u)$.

By reversibility, the factors of our skew-product maps $f = (1, b)$ and $g = (\alpha, a)$ admit a factorization $b(x) \equiv \mathfrak{b}(x)\mathfrak{b}(-x)$ and $a(x) \equiv \mathfrak{a}(x)\mathfrak{a}(-x)$. This factorization is canonical (via orbits) and propagates under renormalization. The functions $\mathfrak{b}$ and $\mathfrak{a}$ will be referred to as the semi-factors of $f$ and $g$, respectively. Consider the sequence of zeros $j \mapsto b_j$ of $\mathfrak{b}$ and the sequence of zeros $j \mapsto a_j$ of $\mathfrak{a}$. If the sums $\sum_j |b_j|^{-2}$ and $\sum_j |a_j|^{-2}$ are finite, then

$$b(x) \equiv \prod_j \left(1 - \frac{x^2}{b_j^2}\right), \qquad a(x) \equiv \prod_j \left(1 - \frac{x^2}{b_j^2}\right), \tag{4.10}$$

with uniform convergence on compact subsets of $\mathbb{C}$. In the cases considered below, all zeros are simple, so we may work with the sets $\mathcal{B} = \mathcal{Z}(\mathfrak{b})$ and $\mathcal{A} = \mathcal{Z}(\mathfrak{a})$ instead of the sequences $j \mapsto b_j$ and $j \mapsto a_j$, respectively.

If $\mathcal{P} = (\mathcal{B}, \mathcal{A})$ is a pair of subsets of $\mathbb{R}$, define the pair $\tilde{\mathcal{P}} = (\tilde{\mathcal{B}}, \tilde{\mathcal{A}})$ by the equations (2.14) and (2.15). Denoting the map $\mathcal{P} \mapsto \tilde{\mathcal{P}}$ by $R$, we define $\mathcal{P}_t = R^{tn}(\mathcal{P})$ for $t = 0, 1, 2, \ldots$. To recall the connection of $R$ with the transformation $\mathcal{R}_3$, consider two reversible maps $f = (1, b)$ and $g = (\alpha, a)$. If $\mathcal{B} = \mathcal{Z}(\mathfrak{b})$ and $\mathcal{A} = \mathcal{Z}(\mathfrak{a})$, then we have $\mathcal{B}_t = \mathcal{Z}(\mathfrak{b}^t)$ and $\mathcal{A}_t = \mathcal{Z}(\mathfrak{a}^t)$. Here $\mathfrak{b}^t$ and $\mathfrak{a}^t$ are the semi-factors associated with the pair $p^t = \mathcal{R}_3^{tn}(p)$, where $p = (f, g)$.

**Proof of Theorem 4.2.** In order to simplify the description, consider the explicit pair $p = ((1, 1), (\alpha, a))$ with $a(x) = 4\sin(\pi(x - \rho))\sin(\pi(x + \rho))$. Its semi-factors are given by

$$\mathfrak{b}(x) = 1, \qquad \mathfrak{a}(x) \equiv \sin(\pi(x - \rho)). \tag{4.11}$$

Notice that $\mathfrak{a}$ is periodic with period 2. But its zeros are periodic with period 1, that is, $\mathcal{B}=\{\}$ and $\mathcal{A}=\{\rho\}+\mathbb{Z}$. Since the transformation $R$ involves a scaling by $\alpha^{-3}$, the sets $\mathcal{B}_t$ and $\mathcal{A}_t$ are periodic with period $r_t=\alpha^{-3nt}$. So we can regard them as subsets of the circle $\mathbb{T}_t=\mathbb{R}/(r_t\mathbb{Z})$. Representing this circle by the interval $[-r_t/2,r_t/2]$ whose endpoints are being identified, the zero sets $\mathcal{B}'_t\subset\mathbb{T}_t$ and $\mathcal{A}'_t\subset\mathbb{T}_t$ associated with $p^t$ are given by $\mathcal{B}'_t=\mathcal{B}_t\cap\mathbb{T}_t$ and $\mathcal{A}'_t=\mathcal{A}_t\cap\mathbb{T}_t$, respectively.

As described in the proof of Lemma 4.1, the set $\mathcal{A}'_t$ is a scaled orbit of length $q_{3nk}$ under $x\mapsto x+\alpha$, centered at $x=\rho$. As a result, the gaps between two adjacent zeros in $\mathcal{A}'_t$ are either $\alpha^{-1}$ or 1. Similarly, the gaps between two adjacent zeros in $\mathcal{B}'_t$ are no shorter than $\alpha^{-1}$. Recall also that $\mathcal{A}'_0\subset\mathcal{A}'_1$ by Lemma 4.1. Thus, given that $\mathcal{B}'_0\subset\mathcal{B}'_1$, the inclusions $\mathcal{B}'_t\subset\mathcal{B}'_{t+1}$ and $\mathcal{A}'_t\subset\mathcal{A}'_{t+1}$ hold for all $t$.

Now we can define $\mathcal{B}_\infty=\lim_t\mathcal{B}_t$ and $\mathcal{A}_\infty=\lim_t\mathcal{A}_t$, using either the lim inf or lim sup, with the same result. Due to the above-mentioned gap properties, the zeros $b^t_j$ of $\mathfrak{b}^t$ and the zeros $a^t_j$ of $\mathfrak{a}^t$ satisfy $\sum_j(b^t_j)^{-2}<\infty$ and $\sum_j(a^t_j)^{-2}<\infty$, respectively. This includes the case $t=\infty$ as well. We note that all zeros are real. And they are simple: we have $\rho\notin\frac{1}{2}\mathbb{Z}[\alpha]$, so $-\rho$ is not on the orbit of $\rho$ under translation $x\mapsto x+\alpha$ on $\mathbb{T}$.

Clearly, $\mathcal{P}_\infty=(\mathcal{B}_\infty,\mathcal{A}_\infty)$ is a fixed point of $R^{3n}$. Now define $b_\diamond$ and $a_\diamond$ as products (4.10), for the zeros $b_j=b^\infty_j$ and $a_j=a^\infty_j$, respectively. Since we are dealing with a class of entire functions $\big($even, and of order $1/2$ in the variable $\sqrt{x}\,\big)$ that are determined up to a constant factor by their zeros, it is not hard to see that the pair $p_\diamond$ with symmetric factors $b_\diamond$ and $a_\diamond$ is a fixed point of $\mathcal{R}_{3n}$, up to a normalization by constant factors. **QED**

We note that $b^t\to b_\diamond$ and $a^t\to a_\diamond$, uniformly on compact subsets of $\mathbb{C}$. This is a straightforward consequence of our uniform gap-bounds, together with the property that $\mathcal{B}'_t\subset\mathcal{B}'_{t+1}$ and $\mathcal{A}'_t\subset\mathcal{A}'_{t+1}$ for all $t$. But we need not prove this here, since it follows from Theorem 4.3 below.

For a more detailed description of the convergence argument we refer to [28], where a similar analysis has been carried out (related to the critical case) for meromorphic factors $a(x)=\boldsymbol{a}(x)/\boldsymbol{a}(-x)$ and quadratic irrational $\alpha$.

We note that the product (4.10) for the function $a_\diamond$ is in effect an infinite product of sine functions, albeit rescaled. The classical product of sines is the Sudler product $P_m(\alpha)=\prod_{j=1}^m|2\sin\pi j\alpha|$. Perturbed versions of this product, for $\alpha$ the golden mean, have been investigated recently in [26].

Let us return to the issue of normalization. As can be seen from (2.6) and (2.7), we may (and thus will) replace $b_\diamond\mapsto -b_\diamond$ and/or $a_\diamond\mapsto -a_\diamond$, in such a way that $b_\diamond(0)>0$ and $a_\diamond(0)<0$. This corresponds to the signs of the limit AM pair, with $b_\circ=1$ and $a_\circ$ given by (3.3).

The symmetric factors of $\tilde{p}_\diamond=\mathcal{R}_{3n}(p_\diamond)$ are of the form $\tilde{b}_\diamond=b_\diamond e^{\tilde{v}_\diamond}$ and $\tilde{a}_\diamond=a_\diamond e^{\tilde{u}_\diamond}$ for some real constants $\tilde{v}_\diamond$ and $\tilde{u}_\diamond$. Consider a pair $p=((1,a),(\alpha,b))$ with symmetric factors $b=b_\diamond e^v$ and $a=a_\diamond e^u$. If $u$ and $v$ are constants, then the symmetric factors for $\tilde{p}=\mathcal{R}_{3n}(p)$ are $\tilde{b}=\tilde{b}_\diamond e^{\tilde{v}}$ and $\tilde{a}=\tilde{a}_\diamond e^{\tilde{u}}$, where

$$\begin{bmatrix}\tilde{v}\\ \tilde{u}\end{bmatrix}=\begin{bmatrix}\tilde{v}_\diamond\\ \tilde{u}_\diamond\end{bmatrix}+U^{3n}\begin{bmatrix}v\\ u\end{bmatrix},\qquad U=\begin{bmatrix}0&1\\1&1\end{bmatrix}. \tag{4.12}$$

Since $U$ is hyperbolic, it is clear that we can determine $v$ and $u$ in such a way that $\tilde{v} = v$ and $\tilde{u} = u$. Thus, we will assume from now on that $b_\diamond$ and $a_\diamond$ have been normalized in such a way that $p_\diamond$ is a fixed point of $\mathcal{R}_{3n}$.

For the normalization $\mathfrak{N}$ that enters the definition (2.11) of $\mathfrak{R}_{3n}$, we now choose $(b, a) \mapsto \bigl(Mb, a\bigr)$, with $M$ defined by the condition $Mb(0) = b_\diamond(0)$, assuming $b(0) \neq 0$. Clearly, $p_\diamond$ is a fixed point of $\mathfrak{R}_{3n}$.

Denote by $\kappa$ the positive zero of $b_\diamond$ that is closest to the origin. Presumably $\kappa > {}^{\alpha}\!/_2$, but we are not sure. In any case, all zeros of $p_\diamond$ are determined (via orbits) from the two zeros of $a_\diamond$ at $\pm\rho$. Under renormalization, zeros of $b_\diamond$ in $[-{}^{\alpha}\!/_2, {}^{\alpha}\!/_2]$ could only generate zeros of $b_\diamond$ outside $[-{}^{\alpha}\!/_2, {}^{\alpha}\!/_2]$ and/or zeros of $a$ outside $[{}^1\!/_2, {}^1\!/_2]$.

The following holds for some open neighborhood $Z$ of $\kappa$.

**Theorem 4.3.** *Let $b$ be even and real analytic on $[-{}^{\alpha}\!/_2, {}^{\alpha}\!/_2]$, satisfying $b(0) > 0$. Let $a$ be even and real analytic on $[-{}^1\!/_2, {}^1\!/_2]$, satisfying $a(0) < 0$. Assume that $b$ has no zero in $(0, {}^{\alpha}\!/_2] \setminus Z$. Assume that $a$ has a simple zero at $\rho$ and no other zeros in $(0, {}^1\!/_2]$. If $p = ((1, b), (\alpha, a))$, then $\mathfrak{R}_{3n}^k(p) \to p_\diamond$ as $k \to \infty$, uniformly on compact subsets of $\mathbb{C}$.*

Given any real numbers $r_{\scriptscriptstyle A}$ and $r_{\scriptscriptstyle B}$ satisfying (2.8), define $\mathcal{G}_r = \mathcal{F}_{r_B} \times \mathcal{F}_{r_A}$ and denote by $\mathcal{G}_r'$ the subspace of reversible pairs.

**Proof of Theorem 4.3.** Consider first the sequence of pairs $p_m = \mathcal{R}_{3n}^m(p)$. For sufficiently large $m$, the zero sets of $b_m$ and $a_m$ agree with the zero sets of $b_\diamond$ and $a_\diamond$, respectively, on any given compact subset of $\mathbb{C}$. This includes the absence of non-real zeros, since their imaginary parts expand by a factor $\alpha^{-3}$ under $\mathcal{R}_3$, as can be seen from (2.14) and (2.15).

Now pick $r_{\scriptscriptstyle A}$ and $r_{\scriptscriptstyle B}$ satisfying (2.8). If $m_0$ has been chosen sufficiently large, then for $m \geq m_0$, the pair $p_m$ belongs to $\mathcal{G}_r'$, and its zeros agree with those of $p_\diamond$ on the disks $|x| \leq r_{\scriptscriptstyle B}$ for $b$ and $|x| \leq r_{\scriptscriptstyle A}$ for $a$. To simplify notation, we denote $p_{m_0}$ again by $p$.

Then we have $b = b_\diamond e^v$ and $a = a_\diamond e^u$ for a pair of functions $(u, v) \in \mathcal{G}_r$. And the symmetric factors of $\tilde{p} = \mathfrak{R}_{3n}(p)$ are again of the form $\tilde{b} = b_\diamond e^{\tilde{v}}$ and $\tilde{a} = a_\diamond e^{\tilde{u}}$, with $(\tilde{u}, \tilde{v}) \in \mathcal{G}_r$. The map $L : (v, u) \mapsto (\tilde{v}, \tilde{u})$ on $\mathcal{G}_r'$ is linear, compact, and rather simple: its eigenvectors are pairs of polynomials of degree $2k$ with eigenvalues $\alpha^{3n(2k\pm1)}$; except at degree 0, where our normalization condition $\mathfrak{N}$ acts. Without $\mathfrak{N}$, the constant functions transform via the matrix $U^{3n}$ in (4.12). Our normalization condition $\mathfrak{N}$ eliminates the expanding eigenvalue $\alpha^{-3n}$ and makes the constant terms contract as well. Thus, $u$ and $v$ tend to zero under iteration of $L$, implying that $\mathfrak{R}_{3n}^k(p) \to p_\diamond$ in $\mathcal{G}_r'$, as $k \to \infty$. **QED**

**Remark 7.** Theorem 1.2 and Lemma 4.1 indicate that the periodic orbits $p_\diamond$ associated with periodic rotation numbers $\rho$ are merely periodic points in a supercritical attractor for $\mathfrak{R}_3$. It should be possible to determine non-periodic orbits associated with many other values of $\rho$. In such an analysis, there would be no reason to restrict $\alpha$ to the inverse golden mean or another quadratic irrational. But we have not looked into these questions.

## 5. Hyperbolicity and consequences

Let $p_\diamond$ be the fixed point of $\mathfrak{R}_{3n}$ that was described in the last section.

Theorem 4.3 implies that the stable subspace of $D\mathfrak{R}_{3n}(p_\diamond)$ has codimension 1 and includes all reversible pairs $p=((1,b),(\alpha,a))$ that satisfy $a(\rho)=0$. Perturbations with $a(\rho)\neq 0$ are unstable: (2.14) and (2.15) show that small perturbations of zeros grow at a rate $\alpha^{-3}$ under $\mathcal{R}_3$. This implies the following.

**Corollary 5.1.** *The derivative of $D\mathfrak{R}_{3n}(p_\diamond)$ on $\mathcal{G}'_r$ has a simple eigenvalue $\alpha^{-3n}$ and no other spectrum outside the open unit disk. The stable subspace is characterized by the condition $a_\circ(\rho)=0$.*

Consider now the transformation $\mathcal{R}_3$ for skew-products with factors in $\mathrm{GL}(2,\mathbb{R})$. Given the form of our matrices (3.1), we associate with the fixed point $p_\diamond$ the pair $P_\diamond=(F_\diamond,G_\diamond)$ with symmetric factors

$$B_\diamond(x)=b_\diamond(x)\mathtt{A}^\dagger\,,\qquad A_\diamond(x)=a_\diamond(x)\mathtt{A}\,,\qquad \mathtt{A}=\begin{bmatrix}1&0\\0&0\end{bmatrix}. \tag{5.1}$$

Recall from (2.5) that the transformation $\mathcal{R}_{3n}$ includes a scaling $\Lambda_{3n}(x,y)=\bigl(\alpha^{3n}x,L_{3n}y\bigr)$. Consider first the choice $L_{3n}=S^n$. Then it is straightforward to check that $P_\diamond$ is a fixed point of $\mathcal{R}_{3n}$. If we restrict $\mathcal{R}_{3n}$ to pairs $P=(F,G)$ with symmetric factors $B_\circ(x)=b_\circ(x)\mathtt{A}^\dagger$ and $A_\circ(x)=a_\circ(x)\mathtt{A}$, where $\mathtt{A}$ is fixed as in (5.1), then $D\mathcal{R}_{3n}(P_\diamond)$ is clearly equivalent to $D\mathcal{R}_{3n}(p_\diamond)$. But in the full space $\mathcal{H}'_r$, the derivative of $D\mathcal{R}_{3n}(P_\diamond)$ has an eigenvalue 1 associated with conjugacies by constant matrices that commute with $S$. This eigenvalue can be eliminated as follows.

Let $\bigl(\check F,\check G\bigr)=\mathcal{R}_{3n}(F,G)$, still with the choice $L_{3n}=S^n$. As an extra normalization step, we include a conjugacy $\check A_\circ\mapsto\tilde A_\circ=e^{-\sigma_{3n}S}\check A_\circ e^{\sigma_{3n}S}$ and $\check B_\circ\mapsto\tilde B_\circ=e^{-\sigma_{3n}S}\check B_\circ e^{\sigma_{3n}S}$, in order to normalize the off-diagonal terms $\tilde b_\circ$ and $\tilde c_\circ$ of the matrix $\tilde A_\circ$. To be more precise, we choose $\sigma_{3n}=\sigma_{3n}(P)$ in such way that

$$\tilde c_\circ(x_0)=0\,,\qquad x_0=-\tfrac{\alpha}{2}\,. \tag{5.2}$$

This is possible if $\check A_\circ$ is sufficiently close to $A_\diamond$, since $a_\diamond(x_0)\neq 0$. Here we use that $\rho\neq x_0$. Our reason for normalizing at $x_0$ is that this is the point about which we have the scaling (2.13). This normalization step can now be incorporated into the scaling $\Lambda_{3n}$ by choosing

$$L_{3n}=S^n e^{\sigma_{3n}S}=S^n\bigl[\cosh(\sigma_{3n})\mathbf{1}+\sinh(\sigma_{3n})S\bigr]\,,\qquad \sigma_{3n}=\sigma_{3n}(P)\,. \tag{5.3}$$

In what follows, the transformation $\mathcal{R}_{3n}$ is defined with the above choice of $L_{3n}$.

We also have to define a proper version of $\mathfrak{R}_{3n}=\mathfrak{N}\circ\mathcal{R}_{3n}$. This can be done the same way as for scalar factors. Let $\tilde P=\mathfrak{R}_{3n}(P)$. If $\tilde B_\circ$ denotes the symmetric factor of $\tilde F$, then $\mathfrak{N}$ consists of the scaling $\tilde B_\circ\mapsto M\tilde B_\circ$, where $M$ is determined by the condition that

$$M\,\mathrm{tr}\bigl(\tilde B_\circ(0)\bigr)=b_\diamond(0)\,. \tag{5.4}$$

Clearly, $P_\diamond$ is a fixed point of $\mathfrak{R}_{3n}$.

Consider now the derivative of $\mathfrak{R}_{3n}(P_\diamond)$, applied to a pair $P=(F,G)$ with symmetric factors

$$B_\circ = \begin{bmatrix} a_B & b_B \\ c_B & d_B \end{bmatrix}, \qquad A_\circ = \begin{bmatrix} a_A & b_A \\ c_A & d_A \end{bmatrix}. \tag{5.5}$$

**Theorem 5.2.** *$D\mathfrak{R}_{3n}(P_\diamond)$ is hyperbolic, with a simple eigenvalue $\alpha^{-3n}$ and no other spectrum outside the open unit disk. The stable subspace of $D\mathfrak{R}_{3n}(P_\diamond)$ is characterized by the condition $a_A(\rho)=0$ in the representation (5.5).*

**Proof.** Consider a representation analogous to (5.5) for the pair $\tilde P = D\mathfrak{R}_{3n}(P_\diamond)P$, with a tilde over the matrices and their elements.

The factors $\tilde B_\circ$ and $\tilde A_\circ$ naturally decompose into sums of products of matrices from $\{B_\circ, A_\circ, B_\diamond, A_\diamond\}$. But only a few products are nonzero, due to the fact that each product can only have one factor $B_\circ$ or $A_\circ$. The other factors all are $B_\diamond$ or $A_\diamond$, which are "sparse" in the sense that they only have one nonzero matrix element.

In order to describe some nonzero terms, let us split $B_\circ$ into a "1d" part that has $a_B=b_B=c_B=0$, and into a "2d" part that has $d_B=0$. Similarly, we split $A_\circ$ into a 1d part that has $b_A=c_A=d_A=0$, and into a 2d part that has $a_A=0$. If we split $\tilde B_\circ$ and $\tilde A_\circ$ analogously, then $D\mathfrak{R}_{3n}(P_\diamond)$ becomes a $2\times 2$ "matrix". A useful feature of this matrix is that it is upper triangular, meaning that the entry "1d $\mapsto$ 2d" is zero. So the eigenvalues of $D\mathfrak{R}_{3n}(P_\diamond)P$ are those of the operators "1d $\mapsto$ 1d" and "2d $\mapsto$ 2d". The former is hyperbolic, with a single expanding direction, as described in Corollary 5.1.

Consider now the "2d $\mapsto$ 2d" part. The first observation is that $\tilde a_B$ and $\tilde d_A$ are zero when $\sigma_{3n}=0$. So these functions are determined by $c_A(x_0)$ only. As a result, $D\mathfrak{R}_{3n}(P_\diamond)$ has an an eigenvalue 0 with eigenvectors whose only nonzero components are $a_B$ and $d_A$. Furthermore, the terms $a_B$ and $d_A$ do not contribute to the off-diagonal entries of $\tilde B_\circ$ or $\tilde A_\circ$. So it suffices to consider the operator "off-diagonal $\mapsto$ off-diagonal". Here, only the rightmost or leftmost factors contribute, and this is always a factor $A_\circ$ or $A_\circ^\dagger$. So $\tilde c_B$ and $\tilde b_B$ are determined by $c_A$ and $b_A$. And $D\mathfrak{R}_{3n}(P_\diamond)$ has an an eigenvalue 0 with eigenvectors whose only nonzero components are $c_B$ and $b_B$.

Consider now the case $\sigma_{3n}=0$, where the conjugacy by $e^{\sigma_{3n}S}$ is trivial. Then $\tilde c_A$ satisfies

$$\frac{\tilde c_A}{a_\diamond}(x_0+t) = \frac{c_A}{a_\diamond}\bigl(x_0-\alpha^{3n}t\bigr)\,, \tag{5.6}.$$

near $t=0$, where $x_0=-{}^{\alpha}\!/_2$. This corresponds of course to the relation (2.13). So $D\mathfrak{R}_{3n}(P_\diamond)$ has eigenvalues $(-\alpha)^{3nm}$ with eigenvectors whose only nonzero components are

$$c_A(x) = a_\diamond(x)(x-x_0)^m\,, \qquad c_B(x) = -b_\diamond(x)(x-x_0)^m\,, \tag{5.7}$$

as well as $b_A(x)=-c_A(-x)$ and $b_B(x)=-c_B(-x)$. These eigenfunctions are contracted for $m>0$. But for $m=0$ we have an eigenvalue 1. This is due to the fact that we considered $\sigma_{3n}=0$. With $\sigma_{3n}=\sigma_{3n}(P)$ as defined by the condition (5.2), the eigenvalue becomes 0.

This shows that the only unstable direction of $D\mathfrak{R}_{3n}(P_\diamond)$ is the one inherited from $D\mathfrak{R}_{3n}(p_\diamond)$, which is associated with a nonzero value of $a_A(\rho)$. So the stable subspace $W^s$ of $D\mathfrak{R}_{3n}(P_\diamond)$ consists of all pairs $P$ with the property that $a_A(\rho)=0$. **QED**

Our proof of Theorem 1.4 below involves the pre-limit versions

$$\mathrm{rot}_{N}(G)=\frac{\mathrm{Rot}_N(G)}{N}\,,\qquad \varrho_{N}(G)=\frac{\Sigma_N(G)}{N}\,, \tag{5.8}$$

of the rotation numbers defined in (1.7) and (3.6), respectively. Let $P=(F,G)$ with $F=(1,\mathbf{1})$. Then the rotation numbers (5.8) for $N=q_{3nk}$ can be computed via iterates $P_k=\mathfrak{R}_{3n}^k(P)$, using that for even $k$,

$$P_k=(F_k,G_k)\,,\qquad G_k=\Lambda_{3nk}^{-1}\bigl(F^\dagger\bigr)^{p_{3nk}}G^{q_{3nk}}\Lambda_{3nk}\,. \tag{5.9}$$

Here, $\Lambda_{3nk}(x,y)=\bigl(\alpha^{3nk}x,e^{\sigma_{3nk}S}y\bigr)$, and $\sigma_{3nk}$ is the sum $\sigma_{3nk}(P)=\sum_{j<k}\sigma_{3n}(P_j)$.

Assuming that $G=(\alpha,A)$ commutes with $F$, meaning that $A$ is periodic with period 1, the factors $F^\dagger$ in (5.9) contribute nothing to the matrix part of $G_k$. We will be using (5.9) in the case where $P_k\to P_\diamond$, with the convergence being asymptotically geometric. Then the sequence of normalization constant $k\mapsto\sigma_{3nk}(P)$ converges, since $\sigma_{3n}(P_\diamond)=0$.

**Proof of Theorem 1.4.** Consider first the claim that (1.8) holds with $k\mapsto e^{-p_{3nk}L}$ and $k\mapsto e^{-q_{3nk}L}$ replaced by suitable sequences $k\mapsto M_k$ and $k\mapsto M_k'$, respectively.

Let $R_d$ be a small rectangle in $\mathcal{H}_r'$, centered at $P_\diamond$. To be more precise, let us call the stable (codimension 1) subspace $W^s$ of $D\mathfrak{R}_{3n}(P_\diamond)$ vertical and the unstable (dimension 1) subspace $W^u$ horizontal. Then $R_d$ is a rectangle centered at $P_\diamond$, of width and height $2d$, whose left/right sides are vertical and top/bottom sides are horizontal. Denote by $\mathcal{W}^s$ ($\mathcal{W}^u$) the local (un)stable manifold of $\mathfrak{R}_{3n}$ at $P_\diamond$. It is tangent to $P_\diamond+W^s$ ($P_\diamond+W^u$) at $P_\diamond$. If $d>0$ is chosen sufficiently small, then $\mathcal{W}^s$ ($\mathcal{W}^u$) leaves the rectangle through the top/bottom (left/right) sides within $\mathcal{O}\bigl(d^2\bigr)$ from their centers.

Consider first the AM factor, multiplied by $\delta=\lambda^{-1}$, with energy $E=\lambda\epsilon$. So we have a pair $P=(F,G)$ with $F=(1,\mathbf{1})$ and $G=(\alpha,A)$, and the symmetric factor $A_\circ$ is

$$A_\circ(x)=\begin{bmatrix}a_\circ(x)&-\delta\\ \delta&0\end{bmatrix}\,,\qquad a_\circ(x)=-\epsilon-2\cos(2\pi x)\,. \tag{5.10}$$

To indicate the dependence on $\delta$ and $\epsilon$, we will use the notation $P^\delta$ or $P^\delta(\epsilon)$.

Consider first the case $\delta=0$. Let $\epsilon(0)=-2\cos(2\pi\rho)$. If we fix $\epsilon=\epsilon(0)$, then the sequence $k\mapsto\mathfrak{R}_{3n}^k\bigl(P^0\bigr)$ converges to $P_\diamond$ by Theorem 4.3. If we consider the family $\epsilon\mapsto P^0(\epsilon)$, with $\epsilon$ close to $\epsilon(0)$, then for sufficiently large $k$, the family $\epsilon\mapsto\mathfrak{R}_{3n}^k(P^0(\epsilon))$ enters and leaves $R_d$ through the vertical sides and it is transversal to the stable manifold $\mathcal{W}^s$. This follows from the way the zeros of $a_\circ$ transform under renormalization; and we assume that $d>0$ has been chosen sufficiently small.

The same holds for the family $\epsilon\mapsto P^\delta(\epsilon)$, for $\delta\neq 0$ sufficiently close to 0. So if $d>0$ has been chosen sufficiently small, then the renormalized family $\epsilon\mapsto\mathfrak{R}_{3n}^k(P^\delta(\epsilon))$ crosses $\mathcal{W}^s$ transversally at some value $\epsilon=\epsilon(\delta)$. Clearly, the curve $\delta\mapsto\epsilon(\delta)$ is real analytic near the origin, and it does not depend on $k$.

What remains to be shown is that $G^\delta(\epsilon(\delta))$ has rotation number $\rho$. To this end, let $P^\delta=P^\delta(\epsilon(\delta))$ and $P_k^\delta=\mathfrak{R}_{3n}^k\bigl(P^\delta\bigr)$. To simplify notation, let us restrict to $\delta>0$.

We note that, if $k$ is sufficiently large, then the symmetric factor $A^\delta_{k\circ}(0)$ of $G^\delta_k$ is close to $A_\diamond(0) = a_\diamond(0)\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Thus, given that $a_\diamond(0) \neq 0$, the first component of $A^\delta_{k\circ}(0)\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$ stays bounded away from 0 and thus does not change sign, as $\delta$ is varied. This holds uniformly in $k$ and $\delta$, if $k \geq k_0$ and $\delta \leq \delta_0$ for some $k_0 > 0$ and some $\delta_0 > 0$.

We need a slight variation of this property. Recall from (5.9) that $A^\delta_{k\circ}(0)$ is the product $\bigl(A^\delta\bigr)^{*q_{3nk}}_\circ(0)$, conjugated by a matrix $e^{\sigma_{3nk}S}$ with $\sigma_{3nk} = \sigma_{3nk}\bigl(P^\delta\bigr)$. Here, and in what follows, we assume that $k$ is even. Notice that $\sigma_{3nk}\bigl(P^\delta\bigr) \to 0$ as $\delta \to 0$, uniformly in $k$. Thus, by decreasing the value $\delta_0 > 0$, if necessary, the absence of sign changes described above still holds if $A^\delta_{k\circ}(0)$ is replaced by $\bigl(A^\delta\bigr)^{*q_{3nk}}_\circ(0)$.

Now choose a fixed but arbitrary positive $\delta \leq \delta_0$. Then the above implies that

$$\operatorname{rot}_{N_k}\bigl(G^\delta\bigr) = \operatorname{rot}_{N_k}\bigl(G^0\bigr)\,, \qquad N_k = q_{3nk}\,, \quad k \geq k_0\,. \tag{5.11}$$

Here, and in what follows, we use as starting point $x = \frac{1-N}{2}\alpha$ in the definition of $\operatorname{Rot}_N(G)$. This corresponds to the argument of the rightmost factor in our symmetric products $A^{*N}_\circ(0)$. Due to the uniform convergence in (3.6), we still have $\operatorname{Rot}_N(G) \to \operatorname{rot}(G)$ as $N \to \infty$.

Let now $\varepsilon > 0$. If $k \geq k_0$ is sufficiently large, then $\bigl|\operatorname{rot}_{N_k}\bigl(G^\delta\bigr) - \operatorname{rot}\bigl(G^\delta\bigr)\bigr| < \varepsilon/2$ and $\bigl|\operatorname{rot}_{N_k}\bigl(G^0\bigr) - \rho\bigr| < \varepsilon/2$. Combining this with (5.11), we have

$$\begin{aligned}\bigl|\operatorname{rot}\bigl(G^\delta\bigr) - \rho\bigr| &\leq \bigl|\operatorname{rot}\bigl(G^\delta\bigr) - \operatorname{rot}_{N_k}\bigl(G^\delta\bigr)\bigr|\\ &\quad + \bigl|\operatorname{rot}_{N_k}\bigl(G^\delta\bigr) - \operatorname{rot}_{N_k}\bigl(G^0\bigr)\bigr| + \bigl|\operatorname{rot}_{N_k}\bigl(G^0\bigr) - \rho\bigr| < \varepsilon\,.\end{aligned} \tag{5.12}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\operatorname{rot}\bigl(G^\delta\bigr) = \rho$.

Given a positive value $\delta_1 \leq \delta_0$, the above can be extended to more general real analytic families of maps. We have to assume that, after a suitable rescaling and possible reparametrization, the resulting family of factors $A^{\delta,\epsilon}_\circ$ is sufficiently close in $\mathcal{F}^4_r$ to the rescaled AM family (5.10), uniformly in $\delta$ (close to $\delta_1$) and $\epsilon$ (close to $\epsilon(\delta_1)$). Then we can repeat the above argument for this family (using $\Sigma_N$ in place of $\operatorname{Rot}_N$) and get the same conclusion. The main point here is that transversality to $\mathcal{W}^s$ is stable under small perturbations. **QED**

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