On trigonometric skew-products over irrational circle-rotations

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Abstract. We describe some asymptotic properties of trigonometric skew-product maps over irrational rotations of the circle. The limits are controlled using renormalization. The maps considered here arise in connection with the self-dual Hofstadter Hamiltonian at energy zero. They are analogous to the almost Mathieu maps, but the factors commute. This allows us to construct periodic orbits under renormalization, for every quadratic irrational, and to prove that the map-pairs arising from the Hofstadter model are attracted to these periodic orbits. Analogous results are believed to be true for the self-dual almost Mathieu maps, but they seem presently beyond reach.

1. Introduction and main results

The work presented here was motivated in part by the difficulty in controlling limits of products

$$A^{*q}(x) \stackrel{\text{def}}{=} A(x+(q-1)\alpha)\cdots A(x+2\alpha)A(x+\alpha)A(x)\,, \tag{1.1}$$

when A is a self-dual almost Mathieu (AM) factor. To be more precise, an AM factor is a matrix-valued function $A = \left[\begin{smallmatrix} E-V & -1 \\ 1 & 0\end{smallmatrix}\right]$ with $V(x) = 2\lambda\cos(2\pi(x+\xi))$ and $E,\xi \in \mathbb{R}$. The associated products (1.1) describe generalized eigenfunction of the Hofstadter Hamiltonian (defined below) if E belongs to its spectrum. Interestingly, a simpler description is possible in the self-dual case $\lambda = 1$ and for energy $E = 0$ [11,12]. As we will explain below, this leads to products (1.1) where A is one of the two scalar functions

$$A^s(x) = \frac{\sin(\pi(x+{}^{\alpha}\!/\!_4))}{\sin(\pi(-x-{}^{3\alpha}\!/\!_4))}\,, \qquad A^c(x) = \frac{\cos(\pi(x+{}^{\alpha}\!/\!_4))}{\cos(\pi(-x-{}^{3\alpha}\!/\!_4))}\,. \tag{1.2}$$

We restrict our analysis to quadratic irrationals α. Taking $q \to \infty$ in (1.1) along a sequence of continued fractions denominators for α, and scaling x appropriately, we prove the existence of a nontrivial limiting function A_*.

Analogous results are believed to be true for the self-dual AM factors as well, but they seem presently beyond reach. Thus, it is useful to first study the simpler factors (1.2). But the methods and results described here should be of independent interest.

The large q behavior of products of the type (1.1), but with trigonometric factors like $A(x) = 2\sin(\pi x)$, has been studied in a variety of different contexts [7,9,13,19,20,21,22]. The work in [19,20,22] was motivated by questions in mathematical physics and dynamical systems as well. Just like the work presented here, it focuses on quadratic irrationals α and continued fractions denominators q. Limits are considered for specific and/or individual values of x; while here, we consider convergence as meromorphic functions.

We start by describing how these products are obtained from the Hofstadter model. The Hofstadter Hamiltonian on $\ell^2(\mathbb{Z}^2)$ describes electrons moving on $\mathbb{Z}^2$, under the influence of a flux $2\pi\alpha$ per unit cell [3,8]. It is given by

$$H^\alpha = U + U^* + \lambda(V+V^*)\,, \qquad UVU^{-1}V^{-1} = e^{-2\pi i\alpha}\,, \tag{1.3}$$

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where λ is a positive constant and U, V are magnetic translations. We consider the Landau gauge, where $(U\phi)(n,m) = \phi(n-1,m)$ and $(V\phi)(n,m) = e^{2\pi i n\alpha}\phi(n,m-1)$.

We are interested mainly in the self-dual case $\lambda = 1$. In this case, and for irrational values of α, the spectrum of H^α is a Cantor set [18] of measure zero [10]. The metric structure of this spectrum and related quantities result from a rich interplay between arithmetic and analysis. The non-commuting property of the pair (U, V) enters the arithmetic part via the parameter α. In addition, it complicates the analysis. The following shows that this can be avoided for zero energy. After describing the steps that lead to this simplification, we will proceed by analyzing the resulting products (1.1).

The operators U, V, and H^α commute with the two dual magnetic translations $(\mathcal{U}\phi)(n,m) = \phi(n,m-1)$ and $(\mathcal{V}\phi)(n,m) = e^{2\pi i m\alpha}\phi(n-1,m)$. Restricting the Hamiltonian H^α to generalized eigenvectors $\phi_\xi(n,m) = e^{-2\pi i n\xi}u_n$ of the translation $\mathcal{U}$, one ends up with the AM operator, which leads to the above-mentioned AM matrices.

Following [12], we consider instead generalized eigenvectors of the diagonal translation $\mathcal{U}^{-1}\mathcal{V}$. They are of the form $\psi_\xi = \Theta_\xi w$ for some sequence $w : \mathbb{Z} \to \mathbb{C}$, where

$$(\Theta_\xi w)(n,m) = \Theta_{\xi,n,m} w_{n+m}\,, \qquad \Theta_{\xi,n,m} = e^{-\pi i m(m+1)\alpha} e^{\pi i(m-n)\xi}\theta_{n+m}\,, \tag{1.4}$$

and where each θ_k can be an arbitrary phase factor. The corresponding eigenvalue is $e^{2\pi i\xi}$.

Let us restrict now to the self-dual case $\lambda = 1$. Choosing $\theta_k = e^{\frac{\pi}{2}ik(k+1)\alpha}$, a tedious but straightforward computation shows that $H^\alpha\Theta_\xi = \Theta_\xi\mathcal{H}^\alpha$, where

$$(\mathcal{H}^\alpha w)_k = 2\cos(\pi((k+1)\alpha - \xi))w_{k+1} + 2\cos(\pi(k\alpha - \xi))w_{k-1}\,. \tag{1.5}$$

This defines a self-adjoint operator $\mathcal{H}^\alpha$ on $\ell^2(\mathbb{Z})$. Clearly, the spectrum of $\mathcal{H}^\alpha$ agrees with the spectrum of the self-dual Hofstadter Hamiltonian H^α. What is particularly convenient about this operator is that, for energy $E = 0$, the equation $\mathcal{H}^\alpha w = Ew$ represents a one-step recursion instead of a two-step recursion:

$$w_{k+1} = -\frac{a_k}{a_{k+1}}\,w_{k-1}\,, \qquad a_n = \cos(\pi(n\alpha - \xi))\,. \tag{1.6}$$

(A one-step recursion is obtained for $\lambda \neq 1$ as well, but its structure is less useful.) In other words, the product setting is now commutative.

A variant of this equation has been studied in [12] for rational values of α. In order to summarize some of the features, let $\alpha = p/q$ with q odd and $\gcd(p,q) = 1$. Define $\omega_k = a_1 \cdots a_{k-3}a_{k-1}a_{k+2}a_{k+4}\cdots a_{2q}$ for $k = 0, 2, \ldots, 2q$. In particular, $\omega_0 = a_2a_4\cdots a_{2q}$ and $\omega_{2q} = a_1a_3\cdots a_{2q-1}$. Notice that $\omega_0 = \omega_{2d}$. Thus, ω extends to a q-periodic sequence defined on $2\mathbb{Z}$. Setting $\omega_k = \omega_{k-q}$ for all odd k yields a q-periodic sequence on $\mathbb{Z}$. The sequence w defined by $w_k = i^k\omega_k$ is a solution of (1.6). Notice that w is a holomorphic function of the variable ξ. An associated meromorphic solution is given by $k \mapsto 1/w_{-k}$.

Consider now irrational values of α. In the absence of periodicity considerations, the equation (1.6) can be solved independently on the even and odd sublattice of $\mathbb{Z}$. Thus, we restrict now to the even sublattice and write $w_{2j} = y_j$. Let $x = {}^{\alpha}\!/_{4} - {}^{\xi}\!/_{2}$. Setting $k = 2j+1$ in (1.6), we obtain the recursion $y_{j+1} = \mathcal{A}(x + j\alpha)y_j$, where $\mathcal{A} = A^sA^c$ is the product of the two functions defined in (1.2).

Notice that A^s and A^c are periodic with period 1. By contrast, $\mathcal{A}$ is periodic with periodic with $1/2$. A change of variables $x = x'/2$ can convert this into a period 1, but then $x + j\alpha$ becomes $x' + j\alpha'$, with $\alpha' = 2\alpha$. Neither a period $1/2$ nor a new angle α' seems desirable. So we propose solving (1.6) via two separate recursions,

$$w_{2j} = y_j^s y_j^c\,, \qquad y_{j+1}^s = A^s(x + j\alpha)y_j^s\,, \quad y_{j+1}^c = A^c(x + j\alpha)y_j^c\,. \tag{1.7}$$

The sequences $j \mapsto y_j^s$ and $j \mapsto y_j^c$ can be studied independently, with the two problems being very similar. Notice that the iteration leads to products of the form (1.1). Our goal here is to determine the behavior of these products, as $q \to \infty$ along a sequence of continued fractions denominators for α, while the argument x is being rescaled appropriately.

Consider $A \in \{A^s, A^c\}$. Given that A is periodic with period 1, we may assume that α lies between 0 and 1. Let $\alpha_0 = \alpha$. The continued fractions expansion of α is defined inductively by setting $c_k = \lfloor \alpha_k^{-1} \rfloor$ and $\alpha_{k+1} = \alpha_k^{-1} - c_k$ for $k = 0, 1, 2, \ldots$. Here $\lfloor s \rfloor$ denotes the integer part of a real number s. The best rational approximants of α are the rationals p_k/q_k defined recursively via $p_{k+1} = c_k p_k + p_{k-1}$ and $q_{k+1} = c_k q_k + q_{k-1}$, starting with $p_0 = 0$, $p_1 = q_0 = 1$, and $q_1 = c_0$. The difference $\alpha - p_k/q_k$ can be estimated by using the standard identity

$$(-1)^k (q_k\alpha - p_k) = \bar{\alpha}_{k+1} \stackrel{\text{def}}{=} \alpha_0\alpha_1 \cdots \alpha_k\,. \tag{1.8}$$

A non-rational α is said to be a quadratic irrational, if α is the root of a quadratic polynomial with integer coefficients. A well-know fact about quadratic irrationals is that their continued fractions expansion $\alpha = 1/(c_0 + 1/(c_1 + 1/(c_2 + \ldots)))$ is eventually periodic. That is, there exist $l > 0$ such that $c_{k+l} = c_k$ for sufficiently large k. For this and other basic facts about continued fractions we refer to [17]. If $c_{k+l} = c_k$ holds for all $k \geq 0$, then we will call α a periodic irrational with period l.

Convention 1. The function A^s and A^c defined by (1.2) will be referred to as the sine factor and cosine factor, respectively. Both will also be called trigonometric factors. Unless stated otherwise, α is now assumed to be a quadratic irrational between 0 and 1. Its shortest eventual period will be denoted by l.

Given a pair (α, A) with A meromorphic, and a positive integer q, define A^{*q} as in equation (1.1). We formulate our main result in two parts. The first part is the following.

Theorem 1.1. *Given a quadratic irrational α, there exists an even integer $n \in \{l, 2l, 3l, \ldots\}$ and a nonnegative integer μ, such that the following holds. α_μ is periodic, and $\mu = 0$ if α is periodic. Let A be one of the trigonometric factors defined in (1.2). Then the limit*

$$A_* = \lim_{t\to\infty} A^{*q_{\mu+tn}}(\bar{\alpha}_{\mu+tn}\,\boldsymbol{\cdot}) \tag{1.9}$$

exists as an analytic function from $\mathbb{C}$ to the Riemann sphere $\mathbb{C} \cup \{\infty\}$, with the convergence being uniform on compact subsets of $\mathbb{C}$. The zeros and poles of A_* are all simple and lie in $\mathbb{R} \setminus \mathbb{Z}[\alpha]$.

Other features of the limit (1.9) will be described later, after we have prepared the proper context.

One of the consequences of Theorem 1.1 is the existence of recurrent orbits for the recursion (1.7). These orbits do not belong to $\ell^2(\mathbb{Z})$. But after a suitable truncation, they yield approximate eigenfunction of $\mathcal{H}^\alpha$, and thus of H^α via (1.4). To be more precise, consider the one-sided sequence y^s defined by setting $y^s_{j+1} = A^s(j\alpha)y^s_j$ for $j \ge 0$, starting with $y^s_0 = 1$. By (1.9), this sequence is recurrent in the sense that $y^s_{q_{\mu+tn}} \to A^s_*(0)$ as $t \to \infty$. The same holds for the sequence y^c defined via the factor A^c. As an immediate consequence we have the following. Let $y = y^s y^c$.

Corollary 1.2. *With y as defined above, and for $t \ge 0$, define $\phi_t \in \ell^2\bigl(\mathbb{Z}^2\bigr)$ by setting $\phi_t(n,m) = \Theta_{\alpha,n,m} y_{(n+m)/2}$ whenever $n \pm m = 0, 2, 4, \ldots, 2q_{\mu+tn}$, and $\phi_t(n,m) = 0$ otherwise. Then $\|H^\alpha \phi_t\|_{\ell^2} \le \epsilon_t \|\phi_t\|_{\ell^2}$ as $t \to \infty$, with $0 < \epsilon_t = \mathcal{O}(1/t)$.*

The point here is not that zero belongs to the spectrum of H^α; this fact is well-known. The interesting part of this corollary is the information about approximate eigenvectors. It should be noted, however, that this only involves the convergence of (1.9) at the origin. The observed behavior of the sequence $j \mapsto y^s_j$ for the inverse golden mean $\alpha = \bigl(\sqrt{5}-1\bigr)/2$ is shown in Figure 1 and described in Section 2.

A result analogous to Corollary 1.2 was given in [23] for the self-dual AM factors; but it required an assumption on the sequence $q \mapsto A^{*q}$, as a substitute for (1.9).

Consider the circles $\mathbb{S} = \mathbb{R} \cup \{\infty\}$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Given a continuous function $A : \mathbb{R} \mapsto \mathbb{S}$, the recursion $y_{j+1} = A(x + j\alpha)y_j$ can be combined with a translation $x \mapsto x + \alpha$ of the line $X = \mathbb{R}$ to define a skew-product map G,

$$G(x,y) = (x + \alpha, A(x)y)\,, \qquad x \in X\,, \quad y \in \mathbb{R}\,. \tag{1.10}$$

The function A will be referred to as the factor of G. We also use the notation $G = (\alpha, A)$. If A is periodic with period 1, then we will also consider the circle $X = \mathbb{T}$ in (1.10). Notice that the q-th iterate of G is given by

$$G^q = \bigl(q\alpha, A^{*q}\bigr)\,, \tag{1.11}$$

with A^{*q} as defined in (1.1). So Theorem 1.1 can be interpreted as saying that the trigonometric map G converges under proper iteration and rescaling to a map (α_μ, A_*). This motivates the following renormalization approach.

As is common in the renormalization of maps on spaces $\mathbb{T} \times Y$, we first generalize the notion of periodicity by considering pairs of maps. If A is periodic with period 1, then we pair $G = (\alpha, A)$ with the map $F = (1, 1)$ whose second component is the constant function 1. Then the periodicity of A is expressed by the property that G commutes with F. More generally, let us now consider pairs (F, G) of maps $F = (1, B)$ and $G = (\alpha, A)$ that commute. Then the renormalized pair is defined by the equation

$$\mathfrak{R}(F,G) = \bigl(\check{F}, \check{G}\bigr)\,, \qquad \check{F} = \Lambda^{-1} G \Lambda\,, \quad \check{G} = \Lambda^{-1} F G^{-c} \Lambda\,, \tag{1.12}$$

where $c = \lfloor \alpha^{-1} \rfloor$ and $\Lambda(x,y) = \bigl(\alpha x, y\bigr)$. Here, G^{-c} denotes the c-th iterate of the inverse map $G^{-1} = \bigl(-\alpha, A(\,.\, - \alpha)^{-1}\bigr)$. The first component of $\check{F}$ is 1, and the first component of

$\check{G}$ is $\check{\alpha} = \alpha^{-1} - c$. Notice that $\alpha \mapsto \check{\alpha}$ is the Gauss map that appears in the continued fractions expansion of α. That is, if $\alpha = 1/(c_0 + 1/(c_1 + 1/(c_2 + \ldots)))$, then $c = c_0$ and $\check{\alpha} = 1/(c_1 + 1/(c_2 + 1/(c_3 + \ldots)))$. If α is periodic, then the possibility arises that some pair $((1, B), (\alpha, A))$ is periodic under the iteration of $\mathfrak{R}$. Such periodic orbits indeed exist, as the following theorem implies.

Theorem 1.3. *With μ, n, and A as in Theorem 1.1, the limit*

$$B_* = \lim_{t\to\infty} A^{*q_{\mu+tn-1}}(\bar{\alpha}_{\mu+tn}\,\boldsymbol{\cdot}) \tag{1.13}$$

exists and shares the properties of the limit A_* that are described in Theorem 1.1. The pair (F_*, G_*) with $F_* = (1, B_*)$ and $G_* = (\alpha_\mu, A_*)$ commutes and is a fixed point of $\mathfrak{R}^n$.

We note that (1.9) and (1.13) are stating that $\mathfrak{R}^{\mu+tn}(F, G) \to (F_*, G_*)$ as $t \to \infty$.

A result analogous to Theorem 1.3 is believed to hold for skew-product maps with self-dual AM factors. The renormalization operator $\mathfrak{R}_{\rm AM}$ for such maps is defined the same way as $\mathfrak{R}$, but the scaling Λ acts on the variable y as well. Periodic orbits for $\mathfrak{R}_{\rm AM}$ that attract self-dual AM pairs are expected to exist for infinitely many energies in the spectrum; see e.g. the discussion in [25]. But existing proofs are restricted to the inverse golden mean $\alpha = \bigl(\sqrt{5} - 1\bigr)/2$ and cover only two periods: a period $n = 3$ [23] that appears to attract the self-dual AM pair for the largest (smallest) energy in the spectrum, and and a period $n = 6$ [26] that appears to attract the self-dual AM pair for energy zero. Both proofs are based on local methods (computer-assisted perturbation theory about an approximate solution). Convergence result like (1.9) and (1.13) are considered global in renormalization, and they are notoriously difficult to prove. This was one of our main motivations for considering skew-product maps associated with the operator (1.5).

For further information on skew-product maps with factors in $\mathrm{SL}(2, \mathbb{R})$, as they relate to the work presented here, we refer to [25] and references therein. Renormalization methods for such maps have been used also in connection with the problem of reducibility [16].

Our proof of (1.9) and (1.13) is based on estimates on the zeros of the functions on the right hand sides of these two equations. The limit $t \to \infty$ is performed first for the zeros; then the functions A_* and B_* are constructed from the limiting set of zeros. The zero-sets are discussed in Section 3, while their regularity (as sequences) and convergence is considered in Section 4.

2. Additional observations and remarks

A property that greatly simplifies the renormalization of AM maps [23] is reversibility. The same is true for the maps considered here.

Definition 2.1. *Define $\mathcal{S}(x, y) = (-x, y)$. We say that the map $G = (\alpha, A)$ is reversible (with respect to $\mathcal{S}$) if $G^{-1} = \mathcal{S}G\mathcal{S}$. The function A° defined by $A^\circ(z) = A(z - {}^{\alpha}\!/_{2})$ will be referred to as the symmetric factor of G.*

Notice that reversibility is preserved under composition of commuting maps. For the factor A, reversibility means that $A(x - \alpha) = A(-x)^{-1}$. The corresponding property for

A° is simply $A^\circ(z)^{-1} = A^\circ(-z)$. It is not hard to see that any meromorphic function A° with this property admits a representation $A^\circ(z) = a(z)/a(-z)$, with a entire analytic. In particular, $a(z) = \sin(\pi(z - {}^{\alpha}\!/_4))$ is a possible choice for the sine factor A^s, and $a(z) = \cos(\pi(z - {}^{\alpha}\!/_4))$ for the cosine factor A^c. We note that the reversibility of the maps considered here is due to our choice of the variable $x = {}^{\alpha}\!/_4 - {}^{\xi}\!/_2$.

Figure 1 shows the sequence $j \mapsto y_j^s$ described before Corollary 1.2, for the inverse golden mean $\alpha = \bigl(\sqrt{5} - 1\bigr)/2$. The graph on the right fits a behavior $y_j^s \sim Y(\log j) j^\tau$ for large j, with Y periodic and $\tau \simeq 0.86$. The period of Y appears to be $\log\bigl(\alpha^{-6}\bigr)$, indicating that $\mathfrak{R}$ has a period 6 in this case. The same values are found for the sequence $j \mapsto y_j^c$.

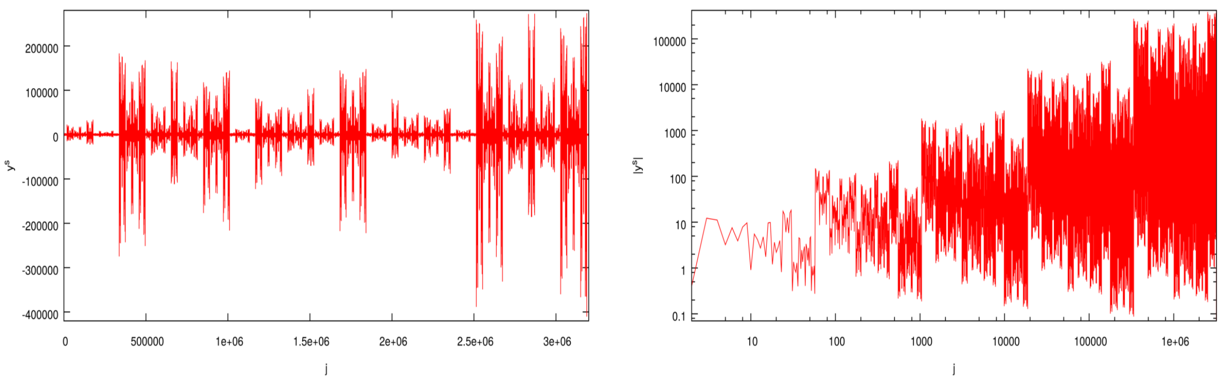

Figure 1. The orbit $j \mapsto y_j^s$ for the inverse golden mean (left) and its absolute value (log scale, right).

Similar behavior is observed for the self-dual AM model at energy zero. The main difference is a smaller growth rate, $\tau_{\rm AM} \simeq 0.39$. This may be related to the fact that the AM operator uses (generalized eigenvectors of) the dual vertical translation $\mathcal{U}$, while the operator $\mathcal{H}^\beta$ uses the dual diagonal translation $\mathcal{U}^{-1}\mathcal{V}$.

Further remarks.

- It should be possible to determine (maybe explicitly) minimal values for the integers μ and n that appear in Theorems 1.1 and 1.3, say as functions of the pre-period and period of α. Here, we prove little more than existence.
- The symmetric factor $A^\circ = a/a(-\,\boldsymbol{.}\,)$ of a reversible map is determined by the logarithmic derivative a'/a of its numerator a. So there is a natural additive formulation of the methods and results described here. See also Remark 6 in Section 4.
- The renormalization operator $\mathfrak{R}$ could be defined as a dynamical system on a suitable space of map-pairs, but our analysis did not require this. Our main goal here is to control the limits (1.9) and (1.13).
- Consider the fixed point of $\mathfrak{R}^n$ described in Theorem 1.3, and assume that $\mathfrak{R}$ has been defined properly as a dynamical system. Based on Remark 7 in Section 4, we expect that the derivative of $\mathfrak{R}^n$ at this fixed point has an eigenvalue $\bar{\sigma}_n^{-1}$, where $\sigma = \alpha_\mu$. This supports an observation made for the self-dual AM maps:
- The second largest eigenvalue that was observed numerically in [25,26] for the inverse golden mean (with $n = 6$ at energy zero) agrees with the above-mentioned value $\bar{\sigma}_n^{-1}$.

- The largest eigenvalue in the self-dual AM case is believed to be associated with variations of the energy; so we are not expecting to see it here, where the value of the energy is fixed to zero.
- For self-dual AM factors, limits like (1.9) can exist only if the energy E is chosen in the spectrum of H^α. For irrational α, finding a point in the spectrum is nontrivial, except for $E = 0$. Given that $E = 0$ also leads to a simpler equation (1.6), one may wonder whether the behavior of the model near energy zero is in some sense more trivial than near other energies. The numerical results in [25], which cover both zero and nonzero energies, suggest that this is not the case.

3. Sets of zeros and renormalization

The goal in this section is to show that the zeros of a pair "stabilize" under the iterating of $\mathfrak{R}^n$, for a suitable choice of n. By the zeros of a pair $P = (F, G)$ we mean the zeros of the factors of F and G. Due to the scaling Λ, the zeros of P in a large interval $|x| < r$ determine the zeros of $\mathfrak{R}^n(P)$ in an interval $|x| < cr$ that is larger by a factor $c > 1$. So it suffices to control the zeros in some fixed interval. This is done by controlling the spacing of zeros (via the three-gap theorem), and by determining a pair of invariant zeros.

Consider the continued fractions approximants p_k/q_k for a given irrational number $\alpha = 1/(c_0 + 1/(c_1 + 1/(c_2 + \ldots)))$ between 0 and 1. The recursion relation described before (1.8) can be written as

$$\begin{bmatrix} p_{k-1} & q_{k-1} \\ p_k & q_k \end{bmatrix} = \mathcal{C}_k(\alpha) \overset{\text{def}}{=} \begin{bmatrix} 0 & 1 \\ 1 & c_{k-1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & c_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & c_0 \end{bmatrix} . \tag{3.1}$$

Here $k \geq 1$. $\mathcal{C}_0(\alpha)$ is defined to be the identity matrix $\mathbf{1}$. In what follows, we omit the argument α whenever it is clear what its values is. Notice that $\det(\mathcal{C}_k) = \pm 1$.

Proposition 3.1. *Assume that α is periodic with period l. Then there exist a positive integer t such that $\mathcal{C}_{tl} \equiv \mathbf{1} \pmod 4$.*

Proof. Given that there are only finitely many 2×2 integer matrices modulo 4, we can find integers $s > r \geq 0$ such that $\mathcal{C}_l^s \equiv \mathcal{C}_l^r \pmod 4$. So the claim holds for $t = s - r$. **QED**

Definition 3.2. *Let $\ell(\alpha)$ be the smallest even value of $\ell \in \{l, 2l, 3l, \ldots\}$ with the property that $\mathcal{C}_\ell \equiv \mathbf{1} \pmod 4$. Let $k(\alpha)$ be the smallest value of $k \geq 0$ such that α_k is periodic. Define $\sigma = \alpha_k$ for $k = k(\alpha)$.*

In what follows, we always consider pairs $P = (F, G)$ that commute, with F of the form $F = (1, B)$. If m is even, then the m-th iterate of $\mathfrak{R}$ can be written as

$$\mathfrak{R}^m(F, G) = \bigl(\tilde{F}, \tilde{G}\bigr) , \qquad \begin{aligned} \tilde{F} &= \Lambda_m^{-1} F^{p_{m-1}} G^{-q_{m-1}} \Lambda_m , \\ \tilde{G} &= \Lambda_m^{-1} F^{-p_m} G^{q_m} \Lambda_m , \end{aligned} \tag{3.2}$$

where $\Lambda_m(x, y) = (\bar{\alpha}_m x, y)$. The same holds if m is odd, except that the exponents in the expressions for $\tilde{F}$ and $\tilde{G}$ have the opposite signs.

Notation 2. The map $G=(\alpha,A)$ with the sine (cosine, or trigonometric) factor A will be referred to as the sine (cosine, or trigonometric) map.

For reference later on, consider the case $B=1$. Assume that m is even. Then, up to a scaling by $\bar\alpha_m$, the factor of $\tilde G$ is A^{*q_m}. Assume that G is the sine map. Regard A and A^{*q_m} as functions on the circle $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Then A has a single zero at $-{}^{\alpha}\!/_{4}$. So A^{*q_m} has q_m zeros. These zeros are the first q_m points on the orbit of $-{}^{\alpha}\!/_{4}$ under repeated rotation by α. Thus, they are all simple. Similarly for the poles of A^{*q_m}, which lie on the orbit of $-{}^{3\alpha}\!/_{4}$. Since the difference between $-{}^{3\alpha}\!/_{4}$ and $-{}^{\alpha}\!/_{4}$ does not belong to $\mathbb{Z}[\alpha]$, the set of poles is disjoint from the set of zero. Thus, no cancellations occur between zeros and poles. The same holds for odd values of m and for the cosine map.

Remark 3. Modulo 1, the symmetric factor A° of the sine map has its zero at $a={}^{\alpha}\!/_{4}$ and its pole at $-a$. For the symmetric factor of the cosine map, the zero is at $a={}^{\alpha}\!/_{4}-{}^{1}\!/_{2}$ and the pole at $-a$.

Lemma 3.3. *Let $\ell=\ell(\alpha)$ and $k=k(\alpha)$. Then there exists a nonnegative integer $\kappa=\kappa(\alpha)$ of the from $\kappa=k+s\ell$ with $s\ge 0$, and real numbers a_*, b_*, such that the following holds. Let $G=(\alpha,A)$ be the sine map and let $F=(1,1)$. If $m=\kappa+t\ell$ with $t\ge 1$, then the symmetric factors for the renormalized maps (3.2) satisfy $\tilde A^\circ(a_*)=0$ and $\tilde B^\circ(b_*)=0$. If α is periodic, then $\kappa=0$, $a_*={}^{\alpha}\!/_{4}$, and $b_*=-a_*$.*

Proof. Let $\kappa=k+s\ell$ with $s\ge 0$ to be determined. Set $m=\kappa+t\ell$, where t can be any positive integer. Consider (3.2) with this value of m.

We start by determining a zero of $\tilde A^\circ$. Notice that $\mathcal{C}_m\equiv\mathcal{C}_k \pmod 4$ by Proposition 3.1 and by the definition of ℓ. Consider first the case where k is even and q_k odd. Then m is even and q_m odd, so

$$\tilde A^\circ(z)=\prod_{|j|<q_m/2} A^\circ(\bar\alpha_m z-j\alpha)\qquad (m\text{ even},\ q_m\text{ odd})\,. \tag{3.3}$$

Recall that A° has a zero at $a={}^{\alpha}\!/_{4}$. In order to find a suitable zero of $\tilde A^\circ$, imagine that $\bar\alpha_m$ is small. Then we need to find a value of j for which $j\alpha$ is close to ${}^{\alpha}\!/_{4}$ modulo 1.

For every positive integer i, denote by p_i' (q_i') the remainder in the division of p_i (q_i) by 4. Let $n=k+t\ell$. By (1.8) we have $q_n\alpha-p_n=\bar\alpha_n\alpha_n$, and thus

$$\frac{q_n-q_n'}{4}\alpha+\frac{q_n'}{4}\alpha-\frac{p_n'}{4}\equiv\frac14\bar\alpha_n\alpha_n\qquad \pmod 1\,. \tag{3.4}$$

Similarly, $q_{n-1}\alpha-p_{n-1}=-\bar\alpha_n$, and thus

$$\frac{q_{n-1}-q_{n-1}'}{4}\alpha+\frac{q_{n-1}'}{4}\alpha-\frac{p_{n-1}'}{4}\equiv-\frac14\bar\alpha_n\qquad \pmod 1\,. \tag{3.5}$$

Now multiply (3.4) by $u=p_{n-1}'$ and (3.5) by $v=-p_n'$. Adding the resulting congruences yields

$$j_n\alpha+\frac{d}{4}\alpha\equiv\frac14\bar\alpha_n(u\alpha_n-v)\qquad \pmod 1\,, \tag{3.6}$$

where $d=\det(\mathcal{C}_n)=\pm 1$ and

$$j_n=u\frac{q_n-q_n'}{4}+v\frac{q_{n-1}-q_{n-1}'}{4}+\frac{d'-d}{4}\,,\qquad d'=p_{n-1}'q_n'-p_n'q_{n-1}'\,. \tag{3.7}$$

Clearly j_n is an integer. So we have

$$A^\circ(\bar\alpha_n z-j_n d\alpha)=A^\circ(\bar\alpha_n(z-a)+{}^{\alpha}\!/_4)\,,\qquad a=\frac{d}{4}(u\alpha_n-v)\,. \tag{3.8}$$

Notice that this factor vanishes for $z=a$, since A° has a zero at ${}^{\alpha}\!/_4$.

Assume first that α is periodic. In this case we choose $s=0$. Then the assumption that m is even and q_m odd is satisfied automatically. In fact, $\mathcal{C}_n\equiv\mathbf{1}\,(\mathrm{mod}\ 4)$, and thus $u=d=d'=1$ and $v=0$. So the factor (3.8) vanishes at $a_*=a={}^{\alpha}\!/_4$. And the same holds for $\tilde A^\circ(z)$. Next, assume that α is nonperiodic. Notice that, modulo 4, the matrix $\mathcal{C}_m$ does not depend on the value of t. In particular, u, v, d, and d' are independent of t. Since $m=n+s$, we can choose $s\ge 0$ in such a way that $|j_n|<q_m/2$. This can be done independently of t. Then (3.8) implies that $\tilde A^\circ$ vanishes at $a_*=a\bar\alpha_n/\bar\alpha_m$. Notice that $\bar\alpha_n/\bar\alpha_m=\sigma_\ell^{-s}$.

This settles the case where k is even and q_k odd. If k and q_k are both odd, then (3.3) holds with z replaced by $-z$ on the right hand side. So we can repeat the above, with a replaced by $-a$.

Next consider the case where both k and q_k are even. Then so are m and q_m, and $\tilde A^\circ$ is given by

$$\tilde A^\circ(z)=\prod_{j=-q_m/2}^{q_m/2-1}A^\circ(\bar\alpha_m z+{}^{\alpha}\!/_2+j\alpha)\qquad(m\ \text{even},\ q_m\ \text{even})\,. \tag{3.9}$$

Now we need $j\alpha$ close to $-{}^{\alpha}\!/_4$ modulo 1, instead of ${}^{\alpha}\!/_4$. Adapting the above arguments to this situation is trivial.

Finally, consider the case where k is odd and q_k even. Then (3.9) holds with z replaced by $-z$ on the right hand side. Now j_n is the same as in the first case, and the argument is essentially the same.

Next, consider the zeros of $\tilde B_0$. The expression for $\tilde B_0(z)$ is the same as that for $\tilde A_0(\pm z)$, except that there are fewer factors in the corresponding products (3.3) and (3.9). But, as in one of the cases above, we can choose $s>0$, if necessary, to guarantee that the factor with $j=j_n$ appears in the given product. In the case where α is periodic, one finds that $\tilde B_0(z)$ vanishes at $-{}^{\alpha}\!/_4$. This concludes the proof of Lemma 3.3. **QED**

Remark 4. An analogous result holds for the cosine map. It is clear that κ can be chosen to have the same value for both trigonometric maps. Similarly for the integers μ and n in Lemma 3.5 below.

Given integers $\mu\ge 0$ and $n\ge 1$, to be specified later, define

$$(F_t,G_t)=\mathfrak{R}^{\mu+tn}(F,G)\,,\qquad t=0,1,2,\ldots \tag{3.10}$$

The factors of F_t and G_t will be denoted by B_t and A_t, respectively; and the symmetric factors by B_t° and A_t°, respectively. Denote by $\mathcal{B}_t$ and $\mathcal{A}_t$ the set of zeros of B_t° and A_t°, respectively. To be more precise, these are subsets of $\mathbb{R}$. Notice however that B_t° and A_t° are periodic with period $r_m = \bar{\alpha}_m^{-1}$, where $m = \mu + tn$. So they define functions on the circle $\mathbb{T}_m = \mathbb{R}/(r_m\mathbb{Z})$. The corresponding zero sets on $\mathbb{T}_m$ will be denoted by $\mathcal{B}'_t$ and $\mathcal{A}'_t$, respectively. In what follows, we identify a circle $\mathbb{R}/(r\mathbb{Z})$ with the interval $I_r = [-{}^r\!/_2, {}^r\!/_2)$. In this sense, we have $\mathcal{B}'_t = \mathcal{B}_t \cap I_{r_m}$ and $\mathcal{A}'_t = \mathcal{A}_t \cap I_{r_m}$.

For simplicity we restrict now to the sine model and mention the cosine model only when there is a noteworthy difference. In what follows, we use the abbreviation $\ell = \ell(\alpha)$ and $\kappa = \kappa(\alpha)$.

Proposition 3.4. *Consider $\mu = \kappa$ and $n = \ell$. Given any positive real number r, there exists infinitely many pairs of positive integers $t_0 < t_1$, such that $\mathcal{B}_{t_1} \cap I_r = \mathcal{B}_{t_0} \cap I_r$ and $\mathcal{A}_{t_1} \cap I_r = \mathcal{A}_{t_0} \cap I_r$.*

Proof. Here we use a known relationship between the continued fractions expansion of α and first-return maps under a rotation by α [14,15,24]. Instead of starting with a circle $(-1, 0]$ modulo 1, it is convenient to use $(-1, \alpha]$ modulo $1 + \alpha$, where α is "identified" with 0 by being the first image of 0 under the rotation by α.

To simplify notation, assume that α is periodic, so that $\mu = 0$. Let $t > 0$ and $m = tn$. Consider first the scaled set $\mathfrak{A}_t \subset \mathbb{T}$, obtained from $\mathcal{A}'_t$ by scaling with a factor $\bar{\alpha}_m = \bar{\alpha}_n^t$. The points in $\mathfrak{A}_t$ constitute an orbit of length q_m under a translation by α on $\mathbb{T}$. By the three-gap theorem [4,5,6], this orbit divides $\mathbb{T}$ into q_m arcs of at most three distinct lengths. (If there are three, then one is the sum of the two others.)

Since α is periodic with period n, these lengths are of the form $\delta_1\bar{\alpha}_m$, $\delta_2\bar{\alpha}_m$, and possibly $(\delta_1 + \delta_2)\bar{\alpha}_m$. This follows from the fact that (up to a reflection) the first-return map of a rotation by $\alpha_0 = \alpha$ on a circle $(-1, \alpha_0]$ is a rotation by $\alpha_0\alpha_1$ on the circle $(-\alpha_0, \alpha_0\alpha_1]$; see e.g. Lemma 2.12 in [24]. After repeating this n times, the rotation is by $\bar{\alpha}_n\alpha_n$ and the return map is to $(-\bar{\alpha}_n, \bar{\alpha}_n\alpha_n]$. Thus, up to a scaling factor $\pm\bar{\alpha}_n$, we end up with the original rotation. (If $\mu > 0$, then the same argument can be applied to the μ-th first-return map.)

This shows that neighboring points in $\mathcal{A}_t$ are separated by distances δ_1, δ_2, and possibly $\delta_1 + \delta_2$, that are independent of t. In addition, we know from Lemma 3.3 that each set $\mathcal{A}_t$ contains a given point a_*. So for any fixed $r > 0$, there are only finitely many possibilities for the set $\mathcal{A}_t \cap I_r$ with $t \geq 1$. By an analogous argument, there are only finitely many possibilities for $\mathcal{B}_t \cap I_r$. **QED**

Define $\mathcal{B}_t(r) = \mathcal{B}_t \cap I_r$ and $\mathcal{A}_t(r) = \mathcal{A}_t \cap I_r$.

Lemma 3.5. *There exist integers $s \geq 0$ and $\tau \geq 1$, as well as a real number $R > 0$, such that the following holds. Let $\mu = \kappa + s\ell$ and $n = \tau\ell$. Consider the definition (3.10) with these values of μ and n. Define $R_0 = 2R$ and $R_t = R + \bar{\sigma}_n^{-1}(R_{t-1} - R)$ for $t = 1, 2, 3, \ldots$. Then for every $t \geq 0$, the pair of sets $\mathcal{A}_{t+1}(R_{t+1})$ and $\mathcal{B}_{t+1}(R_{t+1})$ is determined by the pair of sets $\mathcal{A}_t(R_t)$ and $\mathcal{B}_t(R_t)$. Furthermore, $\mathcal{A}_{t+1}(R_t) = \mathcal{A}_t(R_t)$ and $\mathcal{B}_{t+1}(R_t) = \mathcal{B}_t(R_t)$.*

Proof. With $s \geq 0$ and $\tau \geq 1$ to be determined, let $\mu = \kappa + s\ell$ and $n = \tau\ell$. Consider the pairs (F_t, G_t) as defined by (3.10). Notice that the first components of F_t and G_t are 1 and σ, respectively. Here, σ is the periodic part of α described in Definition 3.2. In what follows, q_k denotes the k-th continued fractions denominator for σ, and p_k denotes the corresponding numerator.

The pair (F_t, G_t) can be obtained by starting with (F_0, G_0) and iterating $\mathfrak{R}^n$ t times. The procedure is the same at each step, so consider just $t = 1$. Since p_n is even and q_n odd, the product $F_0^{-p_n} G_0^{q_n}$ can be computed inductively by starting with $H = G_0$ and joining factors $H \mapsto KHK$ with $K \in \{F_0^{-1}, G_0\}$. Notice that, if $H = (\gamma, C)$ and $K = (\delta, D)$ are reversible, then so is the composed map $\hat{H} = KHK$, and its symmetric factor is given by

$$\hat{C}^\circ(z) = D^\circ\bigl(z + \tfrac{\gamma+\delta}{2}\bigr) C^\circ(z) D^\circ\bigl(z - \tfrac{\gamma+\delta}{2}\bigr)\,. \tag{3.11}$$

Define $J_k = \frac{p_k + q_k - 1}{2}$ for $k > 0$. The above shows that the symmetric factor of G_1 is of the form

$$A_1^\circ(z) = A_0^\circ(\bar{\sigma}_n z) \prod_{j=1}^{J_n} U_j^\circ(\bar{\sigma}_n z + u_j) U_j^\circ(\bar{\sigma}_n z - u_j)\,, \tag{3.12}$$

with $U_j^\circ \in \{A_0^\circ, B_0^\circ(-\,\boldsymbol{.})\}$ and $u_j \in \mathbb{R}$. An analogous expression is obtained for the symmetric factor of F_1. That is,

$$\tilde{B}_1^\circ(z) = B_0^\circ(\bar{\sigma}_n z) \prod_{j=1}^{J_{n-1}} V_j^\circ(\bar{\sigma}_n z + v_j) V_j^\circ(\bar{\sigma}_n z - v_j)\,, \tag{3.13}$$

with $V_j^\circ \in \{B_0^\circ, A_0^\circ(-\,\boldsymbol{.})\}$ and $v_j \in \mathbb{R}$.

Consider first the case $n = \ell$, meaning $\tau = 1$. We note that the translations u_j in the product (3.12) depend on the order in which the factors A_0° and $B_0^\circ(-\,\boldsymbol{.})$ are joined to that product. Similarly for the translations v_j in the product (3.13). We fix this order once and for all. (But we are not trying to minimize the sizes of u_j and v_j.)

In the case $n = \tau\ell$ with $\tau > 1$, we write $\mathfrak{R}^n$ as the τ-th iterate of $\mathfrak{R}^\ell$ and use the same translations in each step. The resulting translations u_j and v_j depend on τ, but only via compositions. What is important is that there is no dependence on the factors A_0° and B_0°.

Notice that a point a belongs to the zero set $\mathcal{A}_1$ of A_1° precisely if $\bar{\sigma}_n a$ belongs to $\mathcal{A}_0$, or if $\bar{\sigma}_n a \pm u_j$ belongs to $\mathcal{A}_0$ or $\mathcal{B}_0$, depending on whether $U_j^\circ = A_0^\circ$ or $U_j^\circ = B_0^\circ(-\,\boldsymbol{.})$, respectively. Thus, the set $\mathcal{A}_1$ is determined from $(\mathcal{B}_0, \mathcal{A}_0)$ via $q_n + p_n$ affine maps $g_i(z) = g_i(0) + \bar{\sigma}_n^{-1} z$. Similarly, $\mathcal{B}_1$ is determined from $(\mathcal{B}_0, \mathcal{A}_0)$ via $q_{n-1} + p_{n-1}$ affine maps $f_i(z) = f_i(0) + \bar{\sigma}_n^{-1} z$. These maps g_i and f_i are independent of the functions A_0° and B_0°. Furthermore, each expands by a factor $\bar{\sigma}_n^{-1} > 1$.

Pick $R > 0$ such that $|f_i(z)| > R$ and $|g_i(z)| > R$ whenever $|z| \geq R$, for all values of i. This can be done independently of the choice of τ that defines $n = \tau\ell$, since the maps f_i and g_i for $\tau > 1$ are compositions of the maps f_i and g_i for $\tau = 1$. Nor does it depend on the choice of $s \geq 0$ that defines $\mu = \kappa + s\ell$.

Let $R_0 = 2R$. Consider temporarily $s = 0$, and set $r = R_0$. Then, using one of the pairs (t_0, t_1) from Proposition 3.4, define $\tau = t_1 - t_0$.

From now on we fix $s = t_0$. Then $\mathcal{B}_1(R_0) = \mathcal{B}_0(R_0)$ and $\mathcal{A}_1(R_0) = \mathcal{A}_0(R_0)$.

Define a function h by setting $h(r) = R + \bar{\sigma}_n^{-1}(r - R)$. Then we have $|f_i(z)| > h(r)$ and $|g_i(z)| > h(r)$ whenever $|z| \geq r \geq R$. Thus, for any given $t \geq 0$, the pair of sets $\mathcal{B}_{t+1}(h(r))$ and $\mathcal{A}_{t+1}(h(r))$ is determined by the pair of sets $\mathcal{B}_t(r)$ and $\mathcal{A}_t(r)$, whenever $r > R$. Applying this with $r = R_{t-1}$ and setting $R_t = h(R_{t-1})$, for $t = 1, 2, 3, \ldots$, we obtain the fist claim in Lemma 3.5.

Recall that $\mathcal{B}_1(R_0) = \mathcal{B}_0(R_0)$ and $\mathcal{A}_1(R_0) = \mathcal{A}_0(R_0)$. Since $(\mathcal{B}_0(R_0), \mathcal{A}_0(R_0))$ determines $(\mathcal{B}_1(R_1), \mathcal{A}_1(R_1))$, and $(\mathcal{B}_1(R_0), \mathcal{A}_1(R_0))$ determines $(\mathcal{B}_2(R_1), \mathcal{A}_2(R_1))$, we must have $\mathcal{B}_2(R_1) = \mathcal{B}_1(R_1)$ and $\mathcal{A}_2(R_1) = \mathcal{A}_1(R_1)$. Iterating this argument proves the second claim in Lemma 3.5. **QED**

4. Zero sequences and logarithmic derivatives

The main goal here is to construct the limit functions (1.9) and (1.13) via their sequences of zeros.

Consider the zero-sets $\mathcal{A}_t$ and $\mathcal{B}_t$ for the functions A_t° and B_t°, respectively. In order to obtain estimates, we will use that these sets are regular in a suitable sense. In particular, they have well-defined average densities (the number of points lying between $\pm{}^r\!/_2$, divided by r, in the limit $r \to \infty$). This follows from the fact that these sets are periodic with period $\bar{\alpha}_m^{-1}$, where $m = \mu + tn$. Given that the number of zeros per period is q_m and q_{m-1}, respectively, the average densities of $\mathcal{A}_t$ and $\mathcal{B}_t$ are given by

$$\rho(\mathcal{A}_t) = \bar{\alpha}_m q_m = \varrho_a + \mathcal{O}\bigl(\bar{\sigma}_n^{2t}\bigr)\,, \qquad \rho(\mathcal{B}_t) = \bar{\alpha}_m q_{m-1} = \varrho_b + \mathcal{O}\bigl(\bar{\sigma}_n^{2t}\bigr)\,, \tag{4.1}$$

for some positive constants ϱ_a and ϱ_b. For the estimates of $\bar{\alpha}_m q_m$ and $\bar{\alpha}_m q_{m-1}$, we have used that the eigenvalues of $\mathcal{C}_n(\sigma)$ are $\bar{\sigma}_n$ and $\bar{\sigma}_n^{-1}$ in modulus, which is well-known and follows essentially from (1.8).

If $\mathcal{S}$ is a discrete subset of $\mathbb{R}$, unbounded below and above, we associate with $\mathcal{S}$ the increasing sequence $j \mapsto s_j$ from $\mathbb{Z}$ onto $\mathcal{S}$ with the property that s_0 is the nonnegative point in $\mathcal{S}$ that is closest to zero. The sequences associated with the zero-sets $\mathcal{A}_t$ and $\mathcal{B}_t$ will be denoted by $j \mapsto a_{t,j}$ and $j \mapsto b_{t,j}$, respectively.

Proposition 4.1. *There exists $c > 0$ such that the following holds. Let $m = \mu + tn$ with $t \geq 0$. Then the sequences $j \mapsto a_{t,j}$ and $j \mapsto b_{t,j}$ satisfy the bounds*

$$|j - \rho(\mathcal{A}_t) a_{t,j}| \leq c \log q_m\,, \qquad |j - \rho(\mathcal{B}_t) b_{t,j}| \leq c \log q_{m-1}\,, \qquad j \in \mathbb{Z}\,. \tag{4.2}$$

Proof. For $q > 1$ consider the set $D_q = \{j\alpha : 1 \leq j \leq q\}$ modulo 1. The discrepancy of α for a subinterval I of $[0, 1]$ is defined as

$$\mathcal{D}_q(I) = \bigl||I \cap D_q| - q|I|\bigr|\,. \tag{4.3}$$

A classic result [1,2] in discrepancy theory, concerning arbitrary irrationals $\alpha > 0$ of bounded type, asserts that there exist $c' > 0$ such that

$$\mathcal{D}_q([0, x]) \leq c' \log q\,, \qquad 0 \leq x \leq 1\,. \tag{4.4}$$

An analogous bound (with c' increased by a factor of 2) holds for intervals $I = [a, b]$, since $\mathcal{D}_q([a, b]) \le \mathcal{D}_q([0, a)) + \mathcal{D}_q([0, b])$. This implies e.g. that the bound (4.4) generalizes to sets $D_q = \{\vartheta + j\alpha : 1 \le j \le q\}$ modulo 1, for arbitrary ϑ.

Writing $D_q = \{\delta_1, \delta_2, \ldots, \delta_q\}$ with $0 < \delta_1 < \delta_2 < \ldots < \delta_q \le 1$, we find that

$$\bigl|k - q\delta_k\bigr| = \bigl|[0, \delta_k] \cap D_q\bigr| - q\delta_k\bigr| = \mathcal{D}_q([0, \delta_k]) \le c \log q\,. \tag{4.5}$$

Let $\rho = \rho(\mathcal{A}_t)$. Setting $q = q_m = \bar{\alpha}_m^{-1}\rho$ and $\delta_k = \bar{\alpha}_m a'_k$, the above yields

$$\bigl|k - \rho\, a'_k\bigr| \le c \log q_m\,. \tag{4.6}$$

This is a bound on the zeros $a'_1, a'_2, \ldots .a'_{q_m}$ of A_t in the interval $[0, r_m]$, where $r_m = \bar{\alpha}_m^{-1}$. By choosing ϑ appropriately and changing the index to $j = k - \frac{q_m+1}{2}$, we obtain the first bound (4.2) for $|j| \le \frac{q_m-1}{2}$. By periodicity, this bound extends to $j \in \mathbb{Z}$. The second bound in (4.2) is proved analogously. **QED**

Next, we consider the limit as $t \to \infty$. By Lemma 3.5, the limit sets $\liminf_t \mathcal{A}_t$ and $\limsup_t \mathcal{A}_t$ agree, so we can choose either one to define $\mathcal{A}_* = \lim_t \mathcal{A}_t$. Similarly define $\mathcal{B}_* = \lim_t \mathcal{B}_t$. Denote by $j \mapsto a_{*,j}$ and $j \mapsto b_{*,j}$ the sequences associated with the sets $\mathcal{A}_*$ and $\mathcal{B}_*$, respectively. As a consequence of the last statement in Lemma 3.5, we have the following.

Corollary 4.2. *There exists a positive real number $\theta < 1$ such that the following holds for $t_0 > 0$ sufficiently large. Let $t \ge t_0$ and $m = \mu + tn$. Then $a_{*,j} = a_{t,j}$ whenever $|j| \le \theta q_m$.*

This allows us to bound the zeros $a_{*,j}$ and $b_{*,j}$ by using Proposition 4.1.

Proposition 4.3. *There exists $C > 0$, such that for all $j \in \mathbb{Z}$ with $|j|$ sufficiently large,*

$$|j - \rho(\mathcal{A}_*)a_{*,j}| \le C \log |j|\,, \qquad |j - \rho(\mathcal{B}_*)b_{*,j}| \le C \log |j|\,. \tag{4.7}$$

where $\rho(\mathcal{A}_*) = \varrho_a$ and $\rho(\mathcal{B}_*) = \varrho_b$.

Proof. Define $q(t) = q_m$ with $m = \mu + tn$. The first bound in (4.2) implies that there exists $C > 0$, such that

$$|j - \varrho_a a_{t,j}| \le \tfrac{1}{2} C \log q(t)\,, \qquad |j| \le q(t)\,, \tag{4.8}$$

for all $t \ge 0$. Here we have also used (4.1), together with the fact that the sequence $k \mapsto \bar{\alpha}_k q_k$ is bounded.

Let $\theta, t_0 > 0$ be as described in Corollary 4.2. Consider $t_1 \ge t_0$ to be specified later. Let $j \in \mathbb{Z}$ such that $|j| > \theta q(t_0 - 1)$. Pick the smallest value of $t > t_1$ such that $|j| \le \theta q(t)$. Then $|j| > \theta q(t-1)$. Using that the ratio $q(t)/q(t-1)$ is bounded by some constant $c > 0$ that does not depend on t, it follows that $q(t) \le \theta^{-1} c|j|$. Substituting this bound on $q(t)$ into (4.8), and using that $a_{*,j} = a_{t,j}$, we obtain the first bound in (4.7), provided that t_1 has been chosen sufficiently large. The second bound in (4.7) is proved analogously. **QED**

Proposition 4.3 shows that the sequences $j \mapsto a_{*,j}$ and $j \mapsto b_{*,j}$ are asymptotically regular, in the following sense.

Definition 4.4. *We say that a sequence $s : \mathbb{Z} \to \mathbb{C}$ is asymptotically regular if there exist constants $\rho, C > 0$ and $J_0 > 1$ such that*

$$|\rho s_j - j| \le C \log |j| \qquad \text{whenever } |j| \ge J_0 \,. \tag{4.9}$$

A consequence of asymptotic regularity is the following.

Proposition 4.5. *Let s be asymptotically regular, with constant ρ, C, J_0 in (4.9). Let $J \ge J_0$ such that $2C \log J \le J$. Then*

$$\sum_{j>J} \left| \frac{1}{s_j} + \frac{1}{s_{-j}} \right| \le 8C\rho \frac{1 + \log J}{J} \,. \tag{4.10}$$

Proof. Write $s_j = \rho^{-1}\bigl(j + c_j \log |j|\bigr)$ with $|c_j| \le C$. Then for $j > J$,

$$\left| \frac{1}{s_j} + \frac{1}{s_{-j}} \right| = \frac{\rho}{j} \left| \frac{1}{1 + c_j (\log |j|)/j} - \frac{1}{1 - c_{-j} (\log |j|)/j} \right| \le 8C\rho \frac{\log j}{j^2} \,. \tag{4.11}$$

The bound (4.10) now follows from the fact that

$$\sum_{j>J} \frac{\log j}{j^2} \le \int_J^\infty \frac{\log t}{t^2} dt = \frac{1 + \log J}{J} \,. \tag{4.12}$$

QED

The above will be used to estimate logarithmic derivatives of symmetric factors such as A_t°. Given a sequence of complex numbers $j \mapsto z_j$ with $z_j^{-1} = \mathcal{O}(j^{-1})$ as $j \to \pm\infty$, we define

$$\sum_j{}' \frac{1}{z - z_j} \stackrel{\text{def}}{=} \lim_{J \to \infty} \sum_{j=-J}^{J} \frac{1}{z - z_j} \,, \tag{4.13}$$

provided that the limit exists. To be more precise, convergence for any given $z \in \mathbb{C}$ is considered a statement about the tail of the sum, where finitely many terms may be omitted. Thus, if the sum (4.13) converges for $z = 0$, then it converges uniformly on compact subsets of $\mathbb{C}$. The value of (4.13) at a pole is defined to be ∞.

Consider now pairs of functions (ϕ, ψ) of the form

$$\psi(z) = \sum_j{}' \frac{1}{z - a_j} \,, \qquad \phi(z) = \sum_j{}' \frac{1}{z - b_j} \,, \tag{4.14}$$

that have residue 1 at each pole. Setting $a_j = a_{*,j}$ and $b_j = b_{*,j}$ defines the functions $\psi_* = \psi$ and $\phi_* = \phi$, respectively. Notice that the above sums Σ'_j converge in this case, uniformly on compact subsets of $\mathbb{C}$, as a result of Proposition 4.3 and Proposition 4.5.

Similarly, setting $a_j = a_{t,j}$ and $b_j = b_{t,j}$ in (4.14) defines $\psi_t = \psi$ and $\phi_t = \phi$, respectively. Convergence in this case is guaranteed by Proposition 4.1. As will become clear later, the functions ψ_t and ϕ_t yield the logarithmic derivatives of A_t° and B_t° via $(A_t^\circ)'(z)/A_t^\circ(z) = \psi_t(z) + \psi_t(-z)$ and $(B_t^\circ)'(z)/B_t^\circ(z) = \phi_t(z) + \phi_t(-z)$, respectively.

Remark 5. It is possible to generate these functions by using an RG transformation $\mathcal{R}$ for pairs (f, g) of additive skew-product maps $f = (-1, \phi)$ and $g = (\alpha, \psi)$, where $g(x, y) = (x + \alpha, \psi(x + {}^{\alpha}\!/_{2}) + y)$ and $f(x, y) = (x - 1, \phi(x - {}^{1}\!/_{2}) + y)$. To be more specific, $\mathcal{R}(f, g) = (\lambda^{-1} g\lambda, \lambda^{-1} f g^c \lambda)$, with a scaling $\lambda(x, y) = (-\alpha x, y)$, and with c as described after (1.12). But we will not need this formulation here.

Lemma 4.6. *$\psi_t \to \psi_*$ and $\phi_t \to \phi_*$ as $t \to \infty$, uniformly on compact subsets of $\mathbb{C}$.*

Proof. Let $R > 0$ and consider the restriction of $\psi_t - \psi_*$ to the disk $D = \{z \in \mathbb{C} : |z| \le R\}$. With θ and t_0 as described in Corollary 4.2, restrict to $t \ge t_0$ and define $J_t = \theta q(t)$, where $q(t) = q_m$ with $m = \mu + tn$. We also assume that $t \ge t_0$ is sufficiently large, such that $|a_{*,j}| > 2R$ and $|a_{t,j}| > 2R$ whenever $|j| > J_t$. Then by Corollary 4.2 we have

$$\psi_*(z) - \psi_t(z) = \sum_{|j|>J_t}{}' \frac{1}{z - a_{*,j}} - \sum_{|j|>J_t}{}' \frac{1}{z - a_{t,j}}\,, \tag{4.15}$$

up to removable singularities.

Consider first $s_j = 1/(z - a_{*,j})$ with $|j| > J_t$ and $z \in D$. By Proposition 4.3, the sequence $j \mapsto s_j$ is asymptotically regular, with constants that are independent of $z \in D$. Applying Proposition 4.5 to this sequence, we see that the first sum in (4.15) tends to zero as $t \to \infty$, uniformly on the disk $|z| \le R$.

Consider now the second sum in (4.15). Due to the first bound in (4.7), the restriction $|j| > J_t$ implies that $|j - \rho(\mathcal{A}_t) a_{t,j}| \le c \log |j|$, for some constant $c > 0$ that is independent of t. Let $s_j = 1/(z - a_{t,j})$ with $|j| > J_t$ and $z \in D$. Clearly, the sequence $j \mapsto s_j$ is asymptotically regular, with constants that are independent of $z \in D$ and of t, for sufficiently large t. Applying Proposition 4.5 to this sequence, we see that the second sum in (4.15) tends to zero as $t \to \infty$, uniformly on the disk $|z| \le R$.

This shows that the left hand side of (4.15) tends to zero as $t \to \infty$, uniformly on D. An analogous argument shows that $\phi_* - \phi_t \to 0$ uniformly on D. Since $R > 0$ was arbitrary, the assertion follows. **QED**

Now we are ready for a

Proof of Theorems 1.1 and 1.3. Consider the set $\mathcal{F}$ of meromorphic functions whose partial fractions expansion admits a representation (4.13), with $z_j^{-1} = \mathcal{O}(j^{-1})$ as $j \to \pm\infty$. Recall that $\psi_*, \phi_* \in \mathcal{F}$ and $\psi_t, \phi_t \in \mathcal{F}$ for all t. In addition, these function have residue 1 at each pole. Clearly, if f belongs to $\mathcal{F}$, then so does $f(\,\boldsymbol{.} - c)$, as well as $f(c\,\boldsymbol{.})$ if $c \ne 0$. Furthermore, the sum of two function in $\mathcal{F}$ is again a function in $\mathcal{F}$.

Using that $\cot \in \mathcal{F}$ by symmetry, we have

$$\frac{\pi \sin'(\pi(z - {}^{\alpha}\!/_{4}))}{\sin(\pi(z - {}^{\alpha}\!/_{4}))} = \sum_j{}' \frac{1}{z - a_j}\,, \qquad \frac{\sin(\pi(z - {}^{\alpha}\!/_{4}))}{\sin(\pi(- {}^{\alpha}\!/_{4}))} = \prod_j{}' \left(1 - \frac{z}{a_j}\right), \tag{4.16}$$

where $a_j = {}^{\alpha}\!/_4 + j$. The product Π'_j is defined as a limit analogous to the limit (4.13) defining Σ'_j. Notice the absence of an exponential factor in this product representation. Identities analogous to (4.16) hold with the cosine in place of the sine; but to simplify the description, we consider just the sine case.

By construction, $A_t^\circ(z) = a_t(z)/a_t(-z)$, where a_t is a product of translated and scaled factors $\sin(\pi(\,\bullet\, - {}^{\alpha}\!/_4))$. As mentioned before Remark 3, the zeros of a_t are all simple, and there are no cancellations between zeros and poles. Thus, we have $a_t'/a_t = \psi_t$, and the logarithmic derivative of A_t° admits a representation

$$\frac{(A_t^\circ)'(z)}{A_t^\circ(z)} = \psi_t(z) + \psi_t(-z) = \sum_j{}' \left(\frac{1}{z - a_{t,j}} + \frac{1}{z + a_{t,j}}\right). \tag{4.17}$$

Using that $A_t^\circ(0) = 1$, this implies that

$$A_t^\circ(z) = \prod_j{}' \bigl(1 - z/a_{t,j}\bigr)\bigl(1 + z/a_{t,j}\bigr)^{-1}. \tag{4.18}$$

Now define

$$A_*^\circ(z) = \prod_j{}' \bigl(1 - z/a_{*,j}\bigr)\bigl(1 + z/a_{*,j}\bigr)^{-1}. \tag{4.19}$$

Then the analogue of (4.17) holds, where "t" is replaced by "$*$". Notice that $A_*^\circ(0) = 1$. Using Lemma 4.6, together with the fact that $A_t^\circ(0) = 1$ for all t, we find that $A_t^\circ \to A_*^\circ$, uniformly on compact subsets of $\mathbb{C}$.

With a definition of B_*° analogous to (4.19), the same type of argument shows that $B_t^\circ \to B_*^\circ$, uniformly on compact subsets of $\mathbb{C}$. This in turn defines the skew-product maps F_* and G_* described in Theorem 1.3. Clearly F_* and G_* commute, since F_t and G_t commute for each t.

The last claim in Theorem 1.3 is that $P_* = (F_*, G_*)$ is a fixed point of $\mathfrak{R}^n$. To see why this holds, consider how $\mathfrak{R}^n$ acts on a pair of maps $P = (F, G)$ in terms of the associated pair of functions (ϕ, ψ). Denote by $(\tilde\phi, \tilde\psi)$ the pair of functions associated with the renormalized pair of maps $\tilde P = \mathfrak{R}^n(P)$. Consider first the case where $F = F_t$ and $G = G_t$ for some $t \ge 0$. As seen in the proof of Lemma 3.5, there exist $q_n + p_n$ affine functions υ_i, with q_n of them being of the form $\upsilon_i(a,b) = \upsilon_i(0,0) + a$ and the other p_n of the form $\upsilon_i(a,b) = \upsilon_i(0,0) + b$, such that

$$\tilde\psi(z) = \bar\sigma_n \sum_j{}' \sum_i \frac{1}{\bar\sigma_n z - \upsilon_i(a_j, b_j)} = \sum_j{}' \sum_i \frac{1}{z - \bar\sigma_n^{-1}\upsilon_i(a_j, b_j)}. \tag{4.20}$$

Since the sums Σ_i range over a fixed finite set, the sums Σ'_j in this equation converge, uniformly on compact subsets of $\mathbb{C}$. Similarly, there are $q_{n-1} + p_{n-1}$ affine functions ν_i such that

$$\tilde\phi(z) = \sum_j{}' \sum_i \frac{1}{z - \bar\sigma_n^{-1}\nu_i(a_j, b_j)}. \tag{4.21}$$

The functions υ_i and ν_i only depend on σ and n, so the identities (4.20) and (4.21) carry over to the limit $t \to \infty$. Consider now the limit functions $\psi = \psi_*$ and $\phi = \phi_*$. By Lemma 3.5, the set of poles $\bar{\sigma}_n^{-1}\upsilon_i(a_j, b_j)$ of $\tilde{\psi}$ agrees with $\mathcal{A}_*$, and the set of poles $\bar{\sigma}_n^{-1}\nu_i(a_j, b_j)$ of $\tilde{\phi}$ agrees with $\mathcal{B}_*$. As a result, we have $\tilde{\psi} = \psi$ and $\tilde{\phi} = \phi$. This in turn implies that $\tilde{A}^\circ = A^\circ$ and $\tilde{B}^\circ = B^\circ$, since all these symmetric factors take the value 1 at the origin. Thus, for $P = P_*$ we have $\tilde{P} = P$, as claimed. **QED**

Remark 6. For the inverse golden mean (and possibly all quadratic irrationals) it is possible to choose the translations u_i in the proof of Lemma 3.5 in such a way that one of the zeros $a_{*,j}$ is a fixed point for one of the functions $a \mapsto \bar{\sigma}_n^{-1}(\upsilon_i(0,0) + a)$. Given that this function has derivative $\bar{\sigma}_n^{-1}$, this suggests that $\mathfrak{R}^n$ expands in (at least) one direction by a factor $\bar{\sigma}_n^{-1}$ at (F_*, G_*). Here we use the fact that $A_*^\circ \simeq A_t^\circ$ and $B_*^\circ \simeq B_t^\circ$ for large t, so the zeros of A_* and B_* can be perturbed in a way that is consistent with commutativity and reversibility, by perturbing the zeros of the sine factor.

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