RG methods in Hamiltonian dynamics

- invariant tori, KAM theorem
- Lindstedt series,
  perturbative renormalization
- nonperturbative regime
- RG (renormalization group) approach
- examples of RG transformations in ...
- self-similar ω's and H's
- the RG transformation $\mathcal{R}$
- action of $\mathcal{R}$ near a trivial fixed point
- numerical results:
  nontrivial fixed point, critical torus
- related open problems

Consider (only) analytic H's

$$H(q,p)\ ,\quad q\in\mathbb{T}^d\ ,\quad p\in B$$

$$\dot{F}=\{F,H\}=\nabla_1 F\cdot\nabla_2 H-\nabla_1 H\cdot\nabla_2 F$$

$$\dot{q}=\nabla_2 H$$

$$\dot{p}=-\nabla_1 H$$

Integrable example:

$$H^0(q,p)=\omega\cdot p+\tfrac{1}{2}p\cdot Mp$$

has invariant ω-torus, where

$$q=q_0+t\omega$$

$$p=0$$

Consider only 1-1 canonical transf.

$$\{F\circ U,G\circ U\}=\{F,G\}\circ U$$

with

$$U=I+u\ ,\qquad \bar{u}=0$$

A KAM theorem

(persistence of certain invariant tori under small perturbations)

Consider $H = H^0 + \varepsilon h$ and assume

(a) non-degeneracy : $\det(M) \neq 0$

(b) Diophantine cond. $(c > 0,\ r > d-1)$

$$|\omega \cdot \nu| > \frac{c}{\|\nu\|^r} \qquad \forall \nu \in \mathbb{Z}^d \setminus \{0\}$$

Then for $\varepsilon < \ldots$ there exists $\beta \in \mathbb{R}^d$ and

$$\Gamma : \mathbb{T}^d \times \{0\} \longrightarrow \mathbb{T}^d \times B$$

with

(1) $\beta = \mathcal{O}(\varepsilon)$

(2) $\Gamma = I + \gamma^*$, $\bar{\gamma} = 0$, $\gamma = \mathcal{O}(\varepsilon)$

(3) analyticity, bounds ...

(4) $t \mapsto \Gamma(q_0 + t\omega, 0)$

is an orbit for $H_\beta = H(\cdot, \cdot - \beta)$

KAM construction :

$$\Gamma_n : \mathbb{T}^d \times B_n \to \mathbb{T}^d \times B \qquad \text{canonical}$$

$$\Gamma_n \to \Gamma \qquad \left(\bigcap_n B_n = \{0\} \right)$$

$$H_{\beta_n} \circ \Gamma_n \longrightarrow H^0 \qquad (\beta_n \to \beta)$$

$$\left(\begin{array}{c} H^0 \sim H_{\beta_n} \circ \Gamma_n \\ \{\Gamma_n, H^0\} \sim \{\Gamma_n, H_{\beta_n} \circ \Gamma_n\} = \{I, H_{\beta_n}\} \circ \Gamma_n \end{array} \right)$$

$$\{\Gamma, H^0\} = \{I, H_\beta\} \circ \Gamma$$

Alternative : solve directly

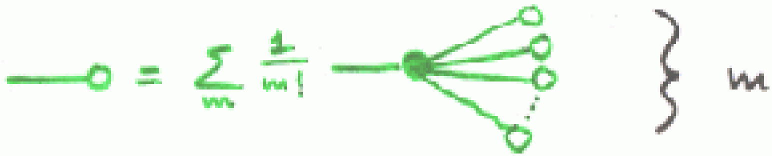
Equation for $\gamma = \Gamma - I$ (case $\beta = 0$)

$$\gamma = \varepsilon \, \mathcal{D}^{-1} \underbrace{\mathbb{J} \{I, h\} \circ \Gamma}_{\sum_m \frac{1}{m!} W_m (\gamma)^m}$$

$$\mathbb{J} \mathcal{D} = \omega \cdot \nabla_1 + Q$$

Lindstedt series :

$$\gamma = \sum_m \frac{1}{m!} \varepsilon \mathcal{D}^{-1} W_m (\gamma)^m$$



Connection with FT

$$\mathcal{D}\gamma = \varepsilon(\nabla W)\circ(I+\gamma) \qquad \text{"field equation"}$$

$$\gamma(q) \text{ "=" } \lim_{\hbar\to 0} \frac{\int \varphi(q)\, e^{\frac{i}{\hbar}\mathcal{L}(\varphi)}\, d\varphi}{\int e^{\frac{i}{\hbar}\mathcal{L}(\varphi)}\, d\varphi}$$

$$\mathcal{L}(\varphi) = \int_{T^d} \left[\tfrac{1}{2}\varphi\cdot\mathcal{D}\varphi - \varepsilon V\circ(I+\varphi)\right]$$

[BGK] relate cancelation in Lindsted series to a translation symmetry of $\mathcal{L}$ (→ Ward identities)

Lindstedt Series ... Perturbative Renormalization

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- **J. Feldman, E. Trubowitz**, *Renormalization in Classical Mechanics and Many Body Quantum Field Theory.* Journal d'Analyse Mathématique, 58, 213–247 (**1992**).
- **G. Gallavotti**, *Perturbation Theory.* In "Mathematical Physics towards the XXI Century", R. Sen, A. Gersten (eds), Ben Gurion University Press, Ber Sheva, 275–294 (**1994**).
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- **J. Ecalle, B. Vallet**, *Prenormalization, correction, and linearization of resonant vector fields or diffeomorphisms.* Preprint 95–32, Université de Paris Sud, Paris (**1994**).
- **G. Gentile, V. Mastropietro**, *Tree Expansion and Multiscale Decomposition for KAM Tori.* Nonlinearity, 8, 1–20 (**1995**).

KAM … Nonperturbative Renormalization

- A.N. Kolmogorov, *On Conservation of Conditionally Periodic Motions Under Small Perturbations of the Hamiltonian.* Dokl. Akad. Nauk SSSR, 98, 527–530 (**1954**).
- **J. Moser**, *On Invariant Curves of Area–Preserving Mappings of an Annulus.* Nachr. Akad. Wiss. Gött., II. Math. Phys. Kl 1962, 1–20 (**1962**).
- **V.I. Arnold**, *Proof of A.N. Kolmogorov's Theorem on the Preservation of Quasi–Periodic Motions under Small Perturbations of the Hamiltonian.* Usp. Mat. Nauk, 18, No. 5, 13–40 (**1963**). Russ. Math. Surv., 18, No. 5, 9–36 (**1963**).
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- **R.S. MacKay**, *Renormalisation in Area Preserving Maps.* Thesis, Princeton (**1982**). World Scientific, London (**1993**).
- **K. Khanin, Ya.G. Sinai**, *The Renormalization Group Method and KAM Theory.* In "Nonlinear Phenomena in Plasma Physics and Hydrodynamics", R.Z. Sagdeev (ed), Mir, 93–118 (**1986**).
- **D. Kosygin**, *Multidimensional KAM Theory from the Renormalization Group Viewpoint.* In "Dynamical Systems and Statistical Mechanics", Ya.G. Sinai (ed), AMS, Adv. Sov. Math., 3, 99–129 (**1991**).
- **A. Stirnemann**, *Towards an Existence Proof of MacKay's Fixed Point.* Comm. Math. Phys., 188, 723–735 (**1997**).
- **C. Chandre, M. Govin, H.R. Jauslin**, *KAM–Renormalization Group Analysis of Stability in Hamiltonian Flows.* Phys. Rev. Lett., 79, 3881–3884 (**1997**).
- **H. Koch**, *A Renormalisation Group for Hamiltonians, with Applications to KAM Tori.* Preprint U. Texas, 50pp (**1996**), to appear in Erg. Theor. Dyn. Syst.
- **J.J. Abad, H. Koch, P. Wittwer**, *A renormalization group for Hamiltonians: Numerical results.* Nonlinearity, 11, 1185–1194 (**1998**).
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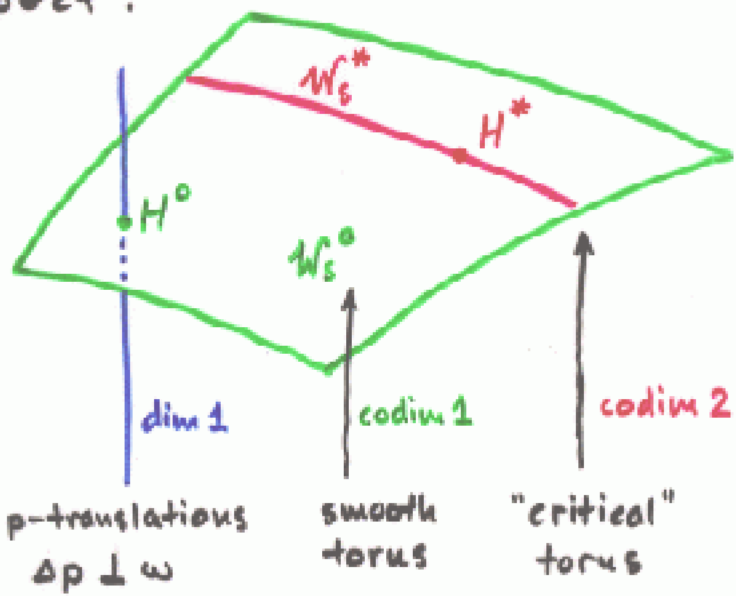
Focus on invar. ω-torus with

$$\oint_{C_j} \sum_{i=1}^{d} p_i \, dq_i = 0 \quad \forall j$$

Consider e.g. $d=2$ and

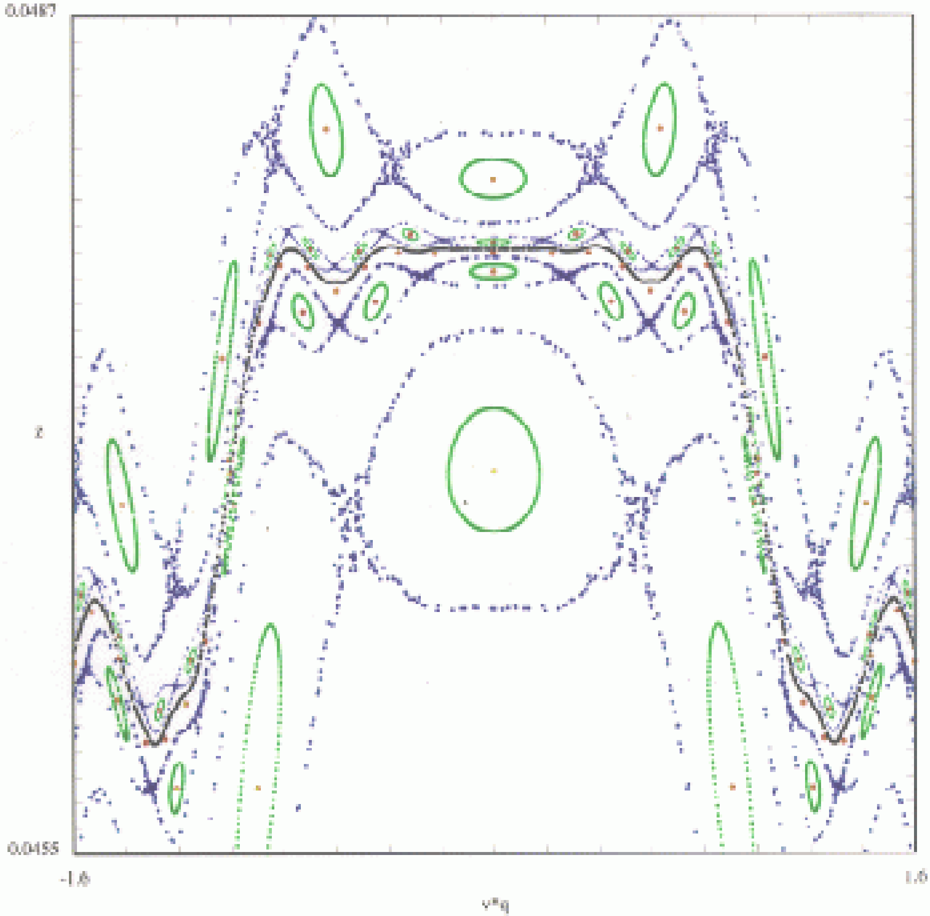
$$\omega = (1, \vartheta), \quad \vartheta = N + \cfrac{1}{N + \cfrac{1}{N + \ddots}} = \frac{N + \sqrt{N^2+4}}{2}$$

Expect :



Periods $\frac{13}{8}$, $\frac{21}{13}$, $\frac{34}{21}$, ... and critical circle for the return map of H^*

Figure 1

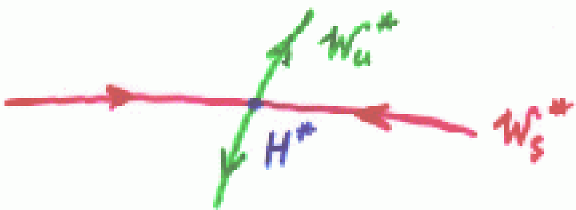


RG approach: $\mathcal{R} : \mathcal{H} \to \mathcal{H}$

expanding "relevant" ΔH's
contracting "irrelevant" ΔH's

e.g. orbits of G

Special case: self-similarity



$$
\begin{tikzcd}
H \arrow[r, "\mathcal{R}"] \arrow[d, "\text{observation}"'] & \tilde{H} \arrow[d] \\
\ell \arrow[r, "R"] & \tilde{\ell} \arrow[l, bend left, "T"]
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[r, "G"] \arrow[d] & H_2 \arrow[d] \\
\ell_1 = & \ell_2
\end{tikzcd}
$$

Statistical Mechanics

φ : field $x \in \mathbb{Z}^d \longmapsto \varphi_x \in \mathbb{R}$ (or ...)

μ : prob. measure on fields

$$\int \varphi_0 \varphi_x \, d\mu(\varphi) = \begin{cases} b\, e^{-|x|/\ell(\mu)} & \text{(HT)} \\ \dfrac{c}{|x|^{d-2+\eta}} & \text{(critical)} \end{cases}$$

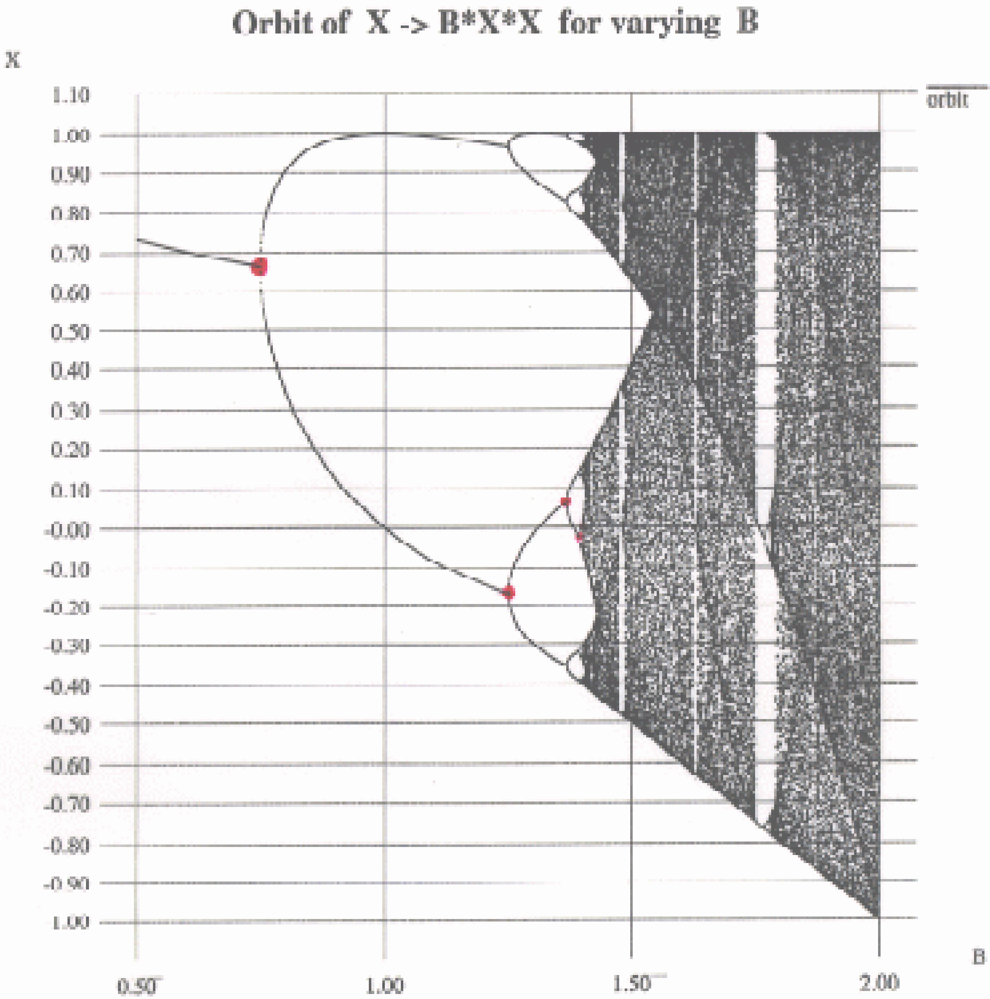
$T : \tilde{\ell} \longmapsto \ell = 2\tilde{\ell}$ (or $3\tilde{\ell}$ or ...)

$\mathcal{R} : \mu \longmapsto \tilde{\mu}$

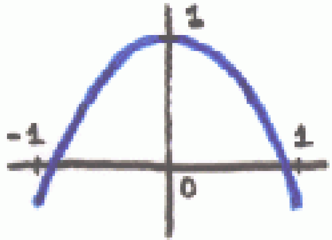
$$d\tilde{\mu}(\psi) \text{ "=" } \Big[\prod_y d\psi_y\Big] \int \delta(A\varphi - \psi)\, d\mu(\varphi)$$

$$(A\varphi)_y = \sum_x a(2y - x)\, \varphi_x$$

Period doubling cascades



Interval maps



$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 < 0$$

$\ell(f)$: period of stable orbit (or ∞)

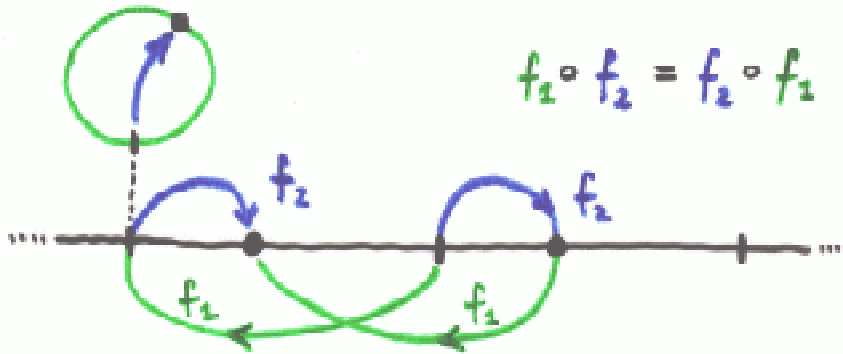
$$\leftarrow \quad n2^k \xleftarrow{T} n2^{k-1} \xleftarrow{T} n2^{k-2} \leftarrow$$

$$\rightarrow \quad f \xrightarrow{\mathcal{R}} \tilde{f} \xrightarrow{\mathcal{R}} \tilde{\tilde{f}} \quad \rightarrow$$

$$T(\tilde{\ell}) = 2\,\tilde{\ell}$$

$$\mathcal{R}(f) = \Lambda_f^{-1} \circ f^2 \circ \Lambda_f \qquad (f^2 = f \circ f)$$

Circle map $f_2 \pmod{f_1}$, monotone



$$f_1 \circ f_2 = f_2 \circ f_1$$

Frequency $\frac{\nu_1}{\nu_2}$: $\quad (f_1^{\nu_1} \circ f_2^{\nu_2})(x) = x$

$$\cdots \xleftarrow{T} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \xleftarrow{T} \begin{bmatrix} \tilde{\nu}_1 \\ \tilde{\nu}_2 \end{bmatrix} \xleftarrow{T} \cdots$$

$$\cdots \xrightarrow{\mathcal{R}} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \xrightarrow{\mathcal{R}} \begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{bmatrix} \xrightarrow{\mathcal{R}} \cdots$$

$$T = \begin{bmatrix} 0 & 1 \\ 1 & N \end{bmatrix}$$

$$\mathcal{R}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \Lambda^{-1} \circ f_1^{0} \circ f_2^{1} \circ \Lambda \\ \Lambda^{-1} \circ f_1^{1} \circ f_2^{N} \circ \Lambda \end{bmatrix}$$

Back to H's and

$$\omega = (1, \omega_2, \dots, \omega_d) \qquad \omega_j \text{ real alg.}$$

Call ω self-similar if
for some integral matrix T

$$T\omega = \vartheta\omega \;, \quad |\vartheta| > 1 \quad \text{(simple)}$$

and all other eigenvalues of T are of modulus < 1, nonzero, simple.

Lemma: ω self-similar $\Leftrightarrow$ the ω_j's span an alg. number field of degree d
$\Rightarrow$ some T's also satisfy $\det(T) = \pm 1$.

Example:

$$\omega = (1, \vartheta), \quad \vartheta = \frac{N + \sqrt{N^2+4}}{2}, \quad T = \begin{bmatrix} 0 & 1 \\ 1 & N \end{bmatrix}$$

Given T, ...

H is self-similar if

$$H(T\cdot, (T^*)^{-1}\cdot) = H \quad (\mathrm{mod}\ G)$$

converts orbits with frequency ν
to —"— $T^{-1}\nu$

G is generated by transformations

$H \mapsto H \circ U$	canonical coord. change homotopic to I
$H \mapsto \tau H - \varepsilon$	energy (or time) scaling
$H \mapsto \frac{1}{\mu} H(\cdot, \mu\cdot)$	p-scaling

Trivial self-similar H:

$$H^\circ(q,p) = \omega\cdot p + \frac{s}{2}(\Omega\cdot p)^2$$

another eigenvector of T

Let

$$\mathcal{T}_\mu(q,p) = \left(Tq,\ \mu (T^*)^{-1} p\right)$$

Then

$$\mathcal{R}_\mu(H) = \frac{\tau}{\mu} H \circ \mathcal{U}_H \circ \underbrace{\mathcal{T}_\mu}_{\substack{\text{singular on } \mathbb{P}_\omega \mathcal{H} \\ (\sim \text{ nonresonant modes})}} - E$$

$\mathcal{U}_H$ defined by $\mathbb{P}_\omega (H \circ \mathcal{U}_H) = 0$
eliminates irrelevant (~nonresonant) modes

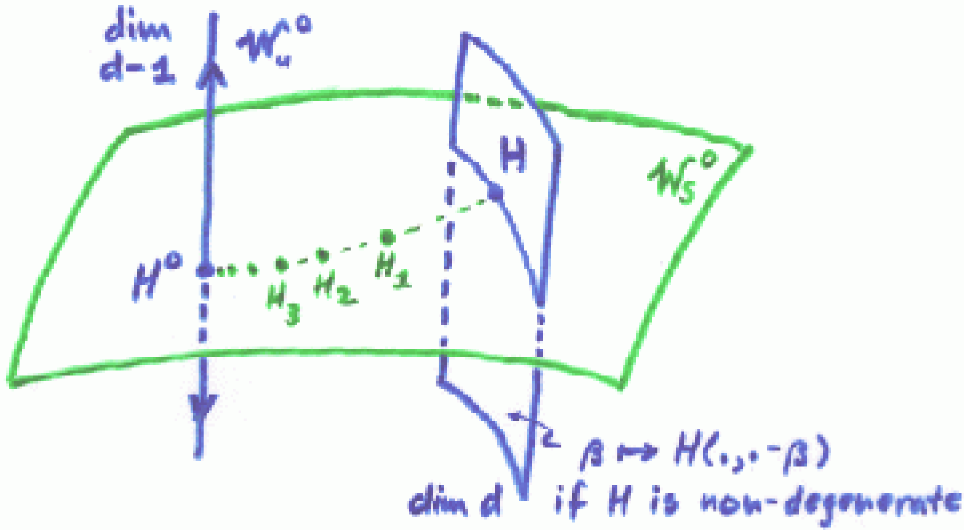
Choosing μ and $H \mapsto \tau(H)$ s.t.

$$H^\circ(q,p) = \omega \cdot p$$

is an isolated fixed point ...

Theorem: If … then near H°,

$\mathcal{R}_\mu$ is well defined and analytic and …



Invariant cw-torus:

$$\Gamma_H = \mathcal{U}_0 \circ \mathcal{U}_1 \circ \mathcal{U}_2 \circ \dots$$

$$\mathcal{U}_n = \mathcal{T}_\mu^{\,n} \circ \mathcal{U}_{H_n} \circ \mathcal{T}_\mu^{\,-n}$$

$$H_n = \mathcal{R}_\mu^n(H)$$

Consider the golden mean ϑ, and

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad \omega = (1, \vartheta), \quad \Omega = (\vartheta, -1)$$

$$H(q,p) = \omega \cdot p + \sum_{\nu,n} h_{\nu,n} \cos(\nu \cdot q)\,(\Omega \cdot p)^n$$

Non-resonant modes: $|\omega \cdot \nu| > \sigma \|\nu\| + \kappa n$

Numerical results:

$$\mathcal{R}: \; H \longmapsto \frac{\vartheta}{\mu} H \circ \mathcal{U}_H \circ T_\mu \qquad \mu = \mu(H)$$

has a nontrivial fixed point H^*.
Its critical indices

$\delta_2 = 1.6279502\ldots$ (relevant eigenv.)
$\mu_* = 0.230460196\ldots$
$\lambda_2 = -0.706795669\ldots$ (critical scaling)

agree with those obtained
from area-preserving maps (MacKay ...)

$$H^0(q,p) = \omega\cdot p + \tfrac{1}{2}(\Omega\cdot p)^2\,, \quad \mu_0 = \vartheta^{-3}$$

$\mathbb{T}^d \times \{0\}$ is an attractor for $\mathcal{T}_{\mu_0}$

$\mathcal{T}_{\mu_0}$ has a fixed point $(0,0)$ on $\mathbb{T}^d \times \{0\}$

Apparently

$\Gamma_{H^*}(\mathbb{T}^d \times \{0\})$ is an attractor for $S = \mathcal{U}_{H^*} \circ \mathcal{T}_{\mu_0}$

S has a fixed point $(0, p_*)$ on $\Gamma_{H^*}(\mathbb{T}^d \times \{0\})$

The eigenvalues λ_j of $DS(0,p_*)$

describe the accumulation rate of orbits

$$\frac{\nu_1}{\nu_2} = \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots$$

Formally

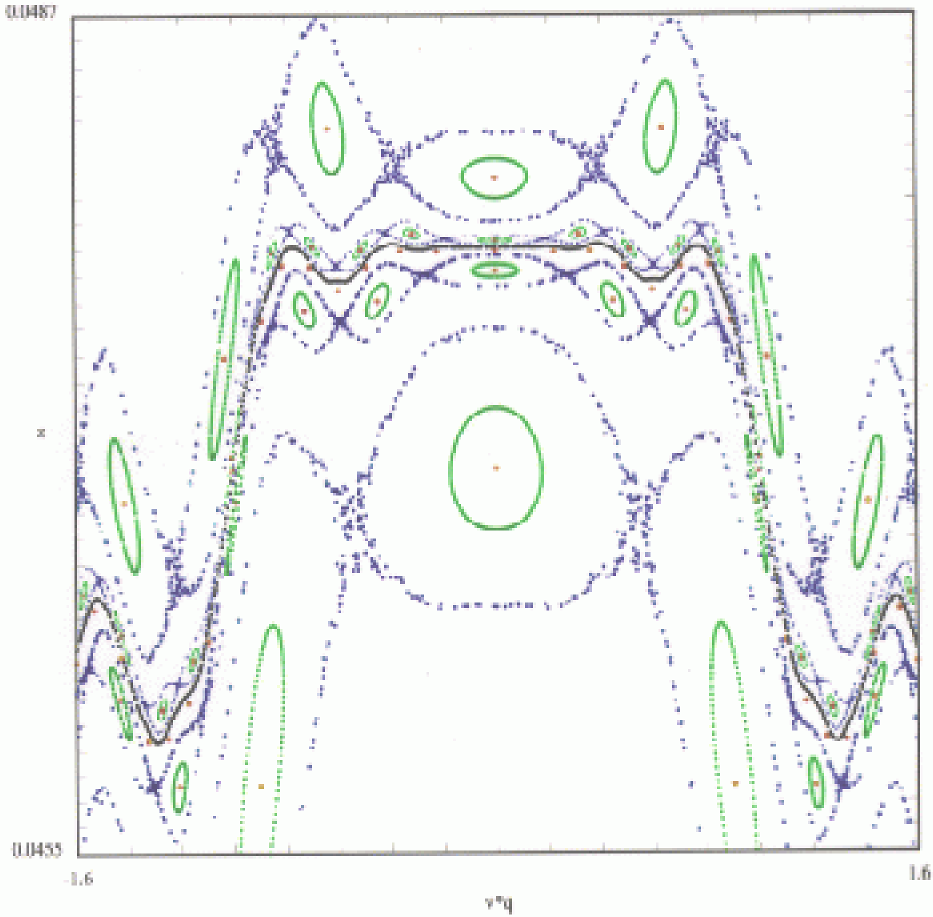
$$\Gamma_{H^*} = \lim_n S^n \circ \mathcal{T}_{\mu_0}^{-n}\,, \quad S \circ \Gamma_{H^*} = \Gamma_{H^*} \circ \mathcal{T}_{\mu_0}\,,$$

indicating a degree of differentiability

$$\le \frac{\ln(-\lambda_2)}{\ln(1/\vartheta)} \approx 0.72$$

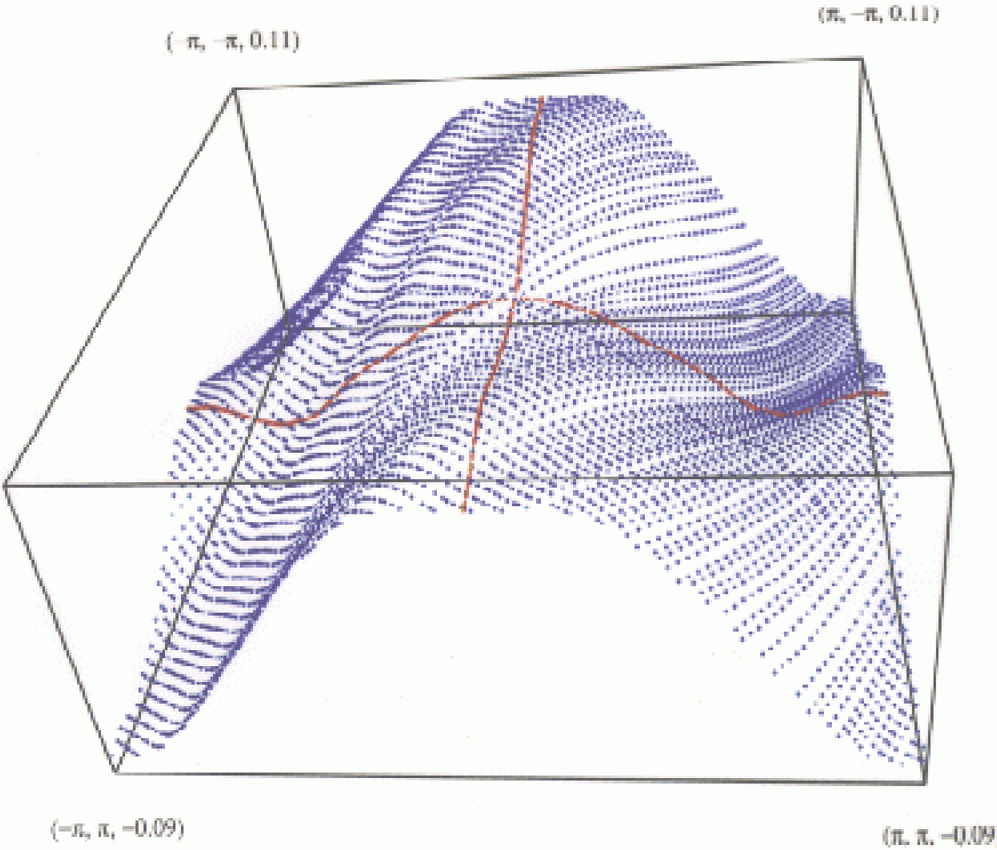
Periods $\frac{13}{8}$, $\frac{21}{13}$, $\frac{34}{21}$, ... and critical circle for the return map of H^*

Figure 1



Critical invariant torus for H^*

Figure 2



Note: $H \mapsto \Gamma_H$ is well defined near H°
(H° is iso... nondegenerate)

Some related problems

- prove the existence of H^*
- $T = \begin{bmatrix} 0 & 1 \\ 1 & N \end{bmatrix}$ with $N \to \infty$
- more general ω's in $d=2$
  (interplay of $\mathcal{R}$'s with different N)
- $d=3$ with just 1 real root,
  e.g. the spiral mean
  $$\vartheta^3 - \vartheta - 1 = 0\,, \quad T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ or } \ldots$$
- $d>3$ with special symmetries, e.g.
  H's reducing to iso... nondegenerate H's
  in $d=2,3$ with quasiperiodic t-dependence.
- ...