Traveling wave solutions for the FPU chain: a constructive approach

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Abstract. Traveling waves for the FPU chain are constructed by solving the associated equation for the spatial profile u of the wave. We consider solutions whose derivatives u' need not be small, may change sign several times, but decrease at least exponentially. This includes multi-bump solutions. Our method of proof is computer-assisted. Unlike other methods, it does not require that the FPU potential has an attractive (positive) quadratic term. But we currently need to restrict the size of that term. In particular, our solutions in the attractive case are all supersonic.

1. Introduction

We consider a chain of interacting particles described by the equation

$$\ddot{q}_j = \phi'(q_{j+1} - q_j) - \phi'(q_j - q_{j-1}), \qquad j \in \mathbb{Z}, \tag{1.1}$$

where ϕ is a polynomial of degree at least 3. The choice

$$\phi(v) = \frac{1}{2}\phi_2 v^2 + \frac{1}{m+1}\phi_{m+1} v^{m+1} \tag{1.2}$$

corresponds to the FPU model: the α-model if $m = 2$, or the β-model if $m = 3$.

Our goal is to find traveling waves, meaning solutions of the form $q_j(t) = u(j - t/\tau)$. Substituting this ansatz into (1.1) yields the equation

$$u''(x) = \tau^2 \phi'\bigl(u(x+1) - u(x)\bigr) - \tau^2 \phi'\bigl(u(x) - u(x-1)\bigr), \qquad x \in \mathbb{R}. \tag{1.3}$$

We focus on solutions u that approach limit values at $\pm\infty$.

The first result on the existence of traveling waves for infinite chains of FPU type was obtained in [3], where solutions with prescribed energy are found as constrained minima of a suitable functional. This result has been extended to solutions with prescribed speed in [4], using the mountain pass theorem. A survey of variational results, covering both breathers and traveling waves, and including an extensive bibliography, is given in [12]. Some recent results based on variational techniques can be found e.g. in [17,19].

A perturbative approach, based on center manifold theory, has been developed in [6,7,13]. It yields small-amplitude solutions in cases where $\tau^2\phi_2 \simeq 1$. A different type of perturbative approach exploits the fact that, for slowly varying functions, the equation (1.1) is close to integrable. This property has been used e.g. in [5,8,10,20] to construct and analyze solitary waves on FPU lattices.

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[2] Supported in part by the PRIN project "Equazioni alle derivate parziali di tipo ellittico e parabolico: aspetti geometrici, disuguaglianze collegate, e applicazioni".
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These results all concern potentials that are attractive for small displacements, meaning $\phi_2 > 0$. In the approach considered here, the value of ϕ_2 is allowed to be negative or zero as well, and we find several traveling wave solutions in this case. A similar approach might apply to other equations of this type. Advance-delay equations appear e.g. in models from biology [1], economics [15], and electrodynamics [9].

The method developed in this paper is constructive but not limited to small or near-integrable solutions. Starting with an approximate numerical solution, we use computer-assisted methods to prove that there exists a true solution nearby. Our current technique requires that $|\tau^2\phi_2| < 1$, but we expect that this condition, which for $\phi_2 > 0$ characterizes supersonic [4] waves, can be relaxed in future work.

Apart from this restriction, our method applies in principle to any sufficiently analytic potential ϕ. But for simplicity, we restrict to the FPU model (1.2). Using as parameters $\mu = \tau^2\phi_2$ and $\nu = \tau^2\phi_{m+1}$, we have

$$\tau^2\phi'(v) = \mu v + \nu v^m \,. \tag{1.4}$$

Without loss of generality, we may assume that $|\nu| = 1$. In addition, we restrict to solutions u that are either even or odd. In other words, $u(-x) = (-1)^\sigma u(x)$ for all x, with $\sigma \in \{0, 1\}$. The number σ will be referred to as the parity of u.

For any function u on $\mathbb{R}$ define

$$(\mathcal{D}u)(x) = u\bigl(x + \tfrac{1}{2}\bigr) - u\bigl(x - \tfrac{1}{2}\bigr) \,. \tag{1.5}$$

Notice that $\mathcal{D}u$ has parity $1 - \sigma$, if u has parity σ.

Theorem 1.1. *Let $\nu = 1$. Consider a fixed but arbitrary row in Table 1 and the data given in that row. Then for every value of μ in some open neighborhood of $\bar{\mu}$, the equation (1.3), with the potential given by (1.4), has a real analytic solution u with parity σ. The function $v = \mathcal{D}u$ satisfies a bound $|v(x)| \le Cr^{-\kappa|x|}$ for some constant $C > 0$. Its* sup-norm *is given in column 5, where p is some positive real number. This function v has E local extrema with values $|v(t)| > {}^1\!/_{64}$. (The existence of others is not excluded.) The diagram in column 7 specifies the sequence and nature of these extrema, as described below.*

Each diagram in Table 1 represent the graph of the function $v = \mathcal{D}u$ associated with the solution u, where the endpoints correspond to $x = \pm\infty$. The vertices "∧", "‸", "⌄", and "⌵" represent positive local maxima, negative local maxima, positive local minima, and negative local minima, respectively; and these extrema appear in the indicated order. More detailed (but purely numerical) graphs are shown in Figures 1–8.

The parameters r, k, and κ that are listed in Table 1 are used during our construction of the solution, as will be explained later. Concerning our choice for the values of μ, we note that similar solutions exist for many (if not all) other values in the interval $(-1, 0)$ or $(-1, 1)$, depending on whether μ is negative or nonnegative, respectively. Further details will be given at the end of this section.

[4] The equation (1.1) with $\phi'(v) = \phi_2 v$ admits traveling waves $q_j(t) = \cos(\varepsilon(j - t/\tau))$, with wavelength $2\pi/\varepsilon$ and velocity $1/\tau$ related by $\tau^2\phi_2 = \varepsilon^2/(2 - 2\cos\varepsilon)$. Sonic waves are characterized by wavelengths that are much larger than the lattice spacing. In the units used here, this means $\varepsilon^{-1} \gg 1$ and $\tau^2\phi_2 \sim 1$.

If u is a solution of the equation (1.3), then the function $v = \mathcal{D}u$ satisfies

$$v'' = \tau^2 \mathcal{D}^2 \phi'(v)\,. \tag{1.6}$$

We prove Theorem 1.1 by first solving this equation and verifying that the solution v has the indicated properties. This involves estimates that are verified with the aid of a computer. Then we define two function u_L and u_R by setting

$$u_L(x) = \sum_{j=0}^{\infty} v\bigl(x - j - \tfrac{1}{2}\bigr)\,, \qquad u_R(x) = -\sum_{j=0}^{\infty} v\bigl(x + j + \tfrac{1}{2}\bigr)\,, \tag{1.7}$$

for all $x \in \mathbb{R}$. It is not hard to see that both $u = u_L$ and $u = u_R$ satisfy the equation (1.3), and that the difference $u_R - u_L$ is constant. By construction, both satisfy $\mathcal{D}u = v$. The solution with the proper parity is $u = {}^1\!/_2 u_L + {}^1\!/_2 u_R$.

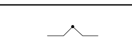
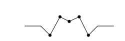
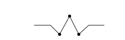
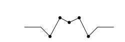
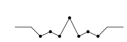
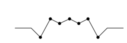
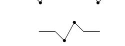
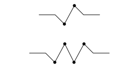
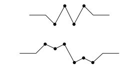
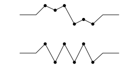
label	m	$\bar\mu$	σ	$\|v\|_\infty$	E	diagram	r	k	κ
1	2	1/4	1	$1.0 + p$	1		4	1	1
2	2	−1/4	1	$1.7 + p$	3		4	1	1
3	3	0	1	$1.3 + p$	1		4	1	1
4	3	1/2	1	$0.9 + p$	1		2	3	1
5	3	3/4	1	$0.6 + p$	1		3/2	8	1
6	3	−1/4	1	$1.4 + p$	1		4	2	1
7	3	−1/2	1	$1.5 + p$	1		2	2	2
8	3	−1/4	1	$1.4 + p$	3		4	2	1
9	3	−1/256	1	$1.3 + p$	3		3/2	1	1
10	3	−3/4	1	$1.6 + p$	5		3/2	8	1
11	3	−1/4	1	$1.4 + p$	5		4	2	1
12	3	−1/4	1	$1.4 + p$	3		4	2	1
13	3	−1/4	1	$1.4 + p$	5		2	2	1
14	3	−1/4	1	$1.4 + p$	7		3	2	1
15	3	−1/4	1	$1.4 + p$	5		2	2	1
16	3	−1/4	1	$1.4 + p$	7		3/2	2	1
17	3	−1/4	1	$1.4 + p$	9		2	2	1
18	3	−1/4	1	$1.4 + p$	7		2	2	1
19	3	−1/2	0	$1.5 + p$	2		2	3	2
20	3	−1/2	0	$1.5 + p$	4		3/2	3	2
21	3	−1/2	0	$1.5 + p$	6		3/2	3	2
22	3	−1/2	0	$1.5 + p$	6		9/8	3	2

Table 1. Parameter values and properties of solutions.

Our approach to solving the equation (1.6) is to turn it into a suitable fixed point problem. This is a common strategy in computer-assisted proofs. Alternatively, one could try a dynamical systems approach. Traveling waves can often be viewed as homoclinic (or heteroclinic) orbits of a dynamical system. This approach has been successful with systems of ordinary differential equations [14,21,22,26]. Integration methods have been applied also to dissipative systems in infinite dimensions [11,18,29] and to some delay equations [16,23,24,25]. But for an advance-delay equation like (1.3), this does not seem to be a workable approach. In this context, we should mention that the system (1.1) is Hamiltonian.

A fixed point equation $G(v) = v$ for the function $v = \mathcal{D}u$ can be obtained by integrating both sides of (1.6) twice. The transformation G improves regularity. But due to the non-compactness of the domain $\mathbb{R}$, we found it difficult to come up with an expansion for v that allows for accurate approximations and is suitable for a rigorous computer-assisted analysis. The following turns out to work extremely well. A function v in one of our spaces $\mathcal{B}_{\rho,r}$ is given by a sequence of "arcs" $v_j = v(\boldsymbol{.} - j)$, indexed by integers j and defined on the interval $[-{}^1\!/_2, {}^1\!/_2]$. Each arc v_j is real analytic on this interval and represented by a rapidly converging Legendre series. So v is real analytic outside the set $\mathbb{Z} + {}^1\!/_2$. Real analyticity on all of $\mathbb{R}$ is obtained if v is a fixed point of G, due to the regularity-improving property of the transformation G.

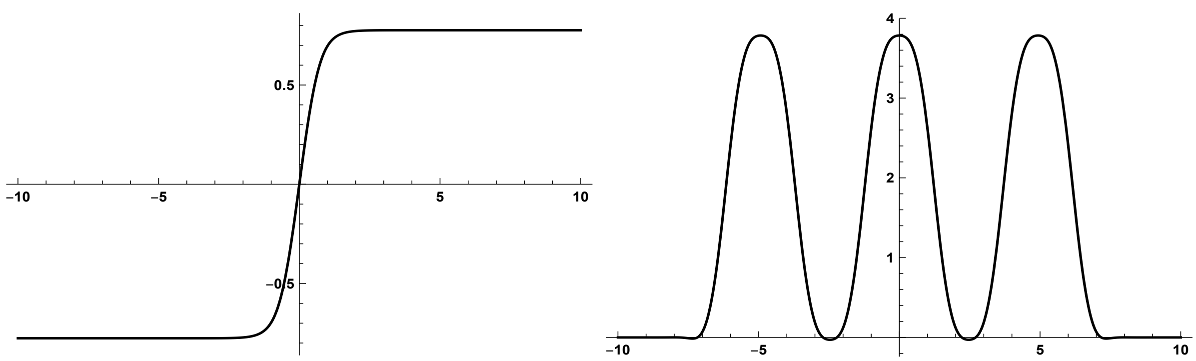


Figure 1. Profile u for the solutions 4 (left) and 22 (right).

1.1. Numerical observations

As mentioned above, we solve the equation (1.6) by converting it to a suitable fixed point equation $G(v) = v$. Then, starting with an approximate numerical solution $\bar{v}$, we prove that G has a true fixed point v nearby.

To find approximate solutions we use a Newton-type iteration. Some approximate solutions were found by starting the iteration with a randomly generated function v. For others, we started with an initial guess, produced by combining one-bump or two-bump solutions into functions with multiple bumps. (In this discussion, a "bump" is a local maximum above ${}^1\!/_8$ or local minimum below $-{}^1\!/_8$.) Figures 4 and 6 suggest that solutions with an arbitrary large number of bumps can be obtained this way. To be more precise, multi-bump solutions were found only for negative value of μ. Furthermore, the precision of our method deteriorates as $|\mu|$ approaches 1. But for each of the numerical solutions that we found, it was possible to "continue" the solution to other values of μ of the same sign, without encountering any bifurcations. So the choice of μ-values in Table 1 is somewhat random, except for the sign.

Let us call two functions v and w "independent" if they have disjoint supports, separated by an interval of length 1. In this case $\mathcal{D}^2\phi'(v+w) = \mathcal{D}^2\phi'(v) + \mathcal{D}^2\phi'(w)$. So if an approximate solution $\bar{v}$ is small (in modulus) on an interval of length 1 but has bumps on both sides of this interval, then it is a sum of two nearly independent approximate solutions. In such cases, our Newton iteration is pushing the two parts farther and farther apart. This suggests that there exists no true solution with nearly independent parts. (Near-independence is accompanied by eigenvalues close to 1 for the linearized problem, so it was necessary to increase the numerical precision in such cases.)

For negative values of μ, it appears that multi-bump solutions exist only for certain specific arrangements of the bumps. And for nonnegative values of μ, no multi-bump solutions appear to exist. As μ approaches zero from below, the bumps approach near-independence in the sense described above; see also Figure 8 (left). In the case of solutions that have negative local maxima or positive local minima, these extreme values approach zero as $\mu \to 0$. This behavior is illustrated in Figure 8 (right).

Solutions for $\mu < 0$ appear not to change significantly as μ approaches the value -1. But the one-bump solution for $\mu > 0$ widens, and its amplitude decreases, as μ approaches 1. This behavior is illustrated in Figure 7. Presumably, the function q associated with our one-bump solution v approaches a small traveling wave of the type considered in [6,7] for $\mu \simeq 1$.

2. Arcs and Legendre series

In this section we describe our decomposition of a function $w \in \mathrm{L}^2(\mathbb{R})$ into local "modes". In order to motivate our choices, let us write (1.6) as the fixed point equation

$$v = \tau^2 A^2 \phi'(v)\,, \qquad A = \mathcal{D}D^{-1}\,. \tag{2.1}$$

Here D^{-1} is defined via integration from some arbitrary point $x_0 \in \mathbb{R}$. The choice of x_0 does not matter, since for any antiderivative V of v,

$$Av = \mathcal{D}V = V\bigl(\,.\,+\tfrac{1}{2}\bigr) - V\bigl(\,.\,-\tfrac{1}{2}\bigr) = \int_{-1/2}^{1/2} v(\,.\,+s)\,ds\,. \tag{2.2}$$

Notice that A is a convolution operator: If χ denotes the indicator function of the interval $[-1/2, 1/2]$, then $Av = \chi * v$. Thus, the operator A^2 that appears in (2.1) is convolution with the function $\chi * \chi$, also known as the cardinal b-spline of order 2, with separation 1 between the knots. This suggests that we decompose a function w on $\mathbb{R}$ as follows:

$$w = \sum_{j\in\mathbb{Z}} \mathbb{P}_j w\,, \qquad (\mathbb{P}_j w)(x) = \begin{cases} w(x), & \text{if } x \in I_j; \\ 0, & \text{otherwise;} \end{cases} \tag{2.3}$$

where $I_j = [j - 1/2, j + 1/2]$ for every integer j.

Our goal is to compute $A^2 v$ via antiderivatives, but to keep the computation as local as possible. To see how this can be achieved, consider a function v on $\mathbb{R}$ that is supported on I_0. If v is orthogonal to the constant function on I_0, then v has an antiderivative $D^{-1}v$ on $\mathbb{R}$ that is again supported in I_0. More generally, if v is orthogonal to all polynomials of degree less than n, then v possesses antiderivatives $D^{-m}v$ of order $m \le n$ that are all supported on I_0. This motivates the following choices.

Denote by T translation by 1; that is, $(Tw)(x) = w(x-1)$. We will represent a function $w \in \mathrm{L}^2(\mathbb{R})$ by the sequence of "arcs" $w_j = T^{-j}\mathbb{P}_j w$ indexed by integers j. Each arc w_j is supported in I_0, and when regarded as a function in $\mathcal{H} = \mathrm{L}^2(I_0)$, it admits a unique expansion

$$w_j = \sum_{n\in\mathbb{N}} w_{j,n}\mathcal{P}_n\,, \qquad \mathcal{P}_n(x) = P_n(2x)\,, \qquad x \in I_0\,. \tag{2.4}$$

Here P_n denotes the Legendre polynomial of degree n. The sequence of polynomials $n \mapsto \mathcal{P}_n$ can be obtained from the sequence of monomials $n \mapsto (x \mapsto x^n)$ via the Gram-Schmidt orthogonalization process, and then normalizing $\mathcal{P}_n(1/2) = 1$. So the scaled Legendre polynomials $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots$ constitute a complete orthogonal set in $\mathcal{H}$. More specifically, we have

$$\langle \mathcal{P}_m, \mathcal{P}_n\rangle_{\mathcal{H}} = \frac{2}{2n+1}\delta_{m,n}\,, \qquad \langle g, f\rangle_{\mathcal{H}} = 2\int_{-1/2}^{1/2} \overline{g(x)}f(x)\,dx\,. \tag{2.5}$$

Two other well-known facts are the following. The values of $\mathcal{P}_n$ on the interval I_0 are bounded in modulus by 1. Furthermore, $(4n+2)\mathcal{P}_n = \mathcal{P}'_{n+1} - \mathcal{P}'_{n-1}$ for all positive integers n. Integrating both sides of this identity yields

$$D^{-1}\mathcal{P}_n = \frac{1}{4n+2}\bigl(\mathcal{P}_{n+1} - \mathcal{P}_{n-1}\bigr)\,, \qquad n \ge 1\,, \tag{2.6}$$

up to an additive constant. Notice that $\mathcal{P}_{n+1}$ and $\mathcal{P}_{n-1}$ agree at $\pm 1/2$. Thus, the right hand side of (2.6) vanishes at $\pm 1/2$. So we define $D^{-1}\mathcal{P}_n$ for $n \ge 1$ via integration from $1/2$ or $-1/2$.

Applying the identity (2.6) twice yields

$$D^{-2}\mathcal{P}_n = C_n^+\mathcal{P}_{n+2} + C_n^-\mathcal{P}_{n-2} - (C_n^+ + C_n^-)\mathcal{P}_n\,, \qquad n \ge 2\,, \tag{2.7}$$

where

$$C_n^+ = \frac{1/4}{(2n+1)(2n+3)}\,, \qquad C_n^- = \frac{1/4}{(2n+1)(2n-1)}\,. \tag{2.8}$$

Notice that the function $D^{-2}\mathcal{P}_n$ and its first derivative vanish on the boundary of the interval I_0, for all $n \ge 2$. As mentioned above, this is one of our main reasons for having chosen scaled Legendre polynomials for our expansion of arcs.

3. The operator A^2 in more detail

The goal here is to give an explicit description of how the operator A^2 acts on each term in the decomposition (2.4). To simplify notation, we define $\mathcal{P}_n(x) = 0$ for $|x| > 1/2$.

But first, let us consider a slight generalization of the approach described in the previous subsection. For a traveling wave v that varies rapidly, it can be advantageous to partition $\mathbb{R}$ into subintervals of length $1/\kappa$, for some integer $\kappa > 1$. Equivalently, we can reformulate our fixed point problem in terms of the scaled function $w = v(./\kappa)$. The resulting equation for w is $w = \tau^2 A_\kappa^2 \phi'(w)$, where

$$A_\kappa = \kappa^{-1}\big[T^{-\kappa/2} - T^{\kappa/2}\big]D^{-1}\,, \qquad A_\kappa^2 = \kappa^{-2}\big[T^{-\kappa} + T^{\kappa} - 2\mathrm{I}\big]D^{-2}\,. \tag{3.1}$$

Here κ can be any positive integer. Notice that $A_1 = A$.

By translation invariance and linearity, it suffices to compute $A_\kappa^2 w$ for a function w of the form

$$w(x) = \begin{cases} (2x)^n, & \text{if } -\frac{1}{2} < x < \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

We are interested mainly in the cases $n = 0$ and $n = 1$, where the identity (2.7) does not apply. Let $D^{-1}w$ be the antiderivative of w that vanishes at the origin, and let $D^{-2}w$ be the antiderivative of $D^{-1}w$ that vanishes at the origin. Then

$$\big(D^{-2}w\big)(x) = \begin{cases} \frac{1}{4(n+1)(n+2)}(2x)^{n+2}, & \text{if } -\frac{1}{2} < x < \frac{1}{2}; \\ \frac{1}{4(n+1)(n+2)} + \frac{1}{2(n+1)}\big(x - \frac{1}{2}\big), & \text{if } x \ge \frac{1}{2}; \\ (-1)^n \frac{1}{4(n+1)(n+2)} + (-1)^n \frac{-1}{2(n+1)}\big(x + \frac{1}{2}\big), & \text{if } x \le -\frac{1}{2}. \end{cases} \tag{3.3}$$

Notice that the function $D^{-2}w$ is affine to the left and to the right of I_0. When we apply $T^{-\kappa} + T^{\kappa} - 2\mathrm{I}$, these affine parts of $D^{-2}w$ cancels at a distance κ or larger from I_0. Applying $T^{-\kappa} + T^{\kappa} - 2\mathrm{I}$ to the part of $D^{-2}w$ that is supported in I_0 results in three copies: one in I_0, with a factor -2, and a translated copy in each of the intervals $I_{\pm\kappa}$.

In what follows, we restrict to $\kappa = 1$ or $\kappa = 2$. After a trivial but tedious computation, we end up with the following expressions. For $n < 2$ and $\kappa = 1$ we have

$$\begin{aligned} A_1^2\mathcal{P}_0 &= T^{-1}\big[\tfrac{1}{12}\mathcal{P}_2 + \tfrac{1}{4}\mathcal{P}_1 + \tfrac{1}{6}\mathcal{P}_0\big] + \big[-\tfrac{1}{6}\mathcal{P}_2 + \tfrac{2}{3}\mathcal{P}_0\big] \\ &\quad + T\big[\tfrac{1}{12}\mathcal{P}_2 - \tfrac{1}{4}\mathcal{P}_1 + \tfrac{1}{6}\mathcal{P}_0\big]\,, \\ A_1^2\mathcal{P}_1 &= T^{-1}\big[\tfrac{1}{60}\mathcal{P}_3 - \tfrac{1}{10}\mathcal{P}_1 - \tfrac{1}{12}\mathcal{P}_0\big] + \big[-\tfrac{1}{30}\mathcal{P}_3 + \tfrac{1}{5}\mathcal{P}_1\big] \\ &\quad + T\big[\tfrac{1}{60}\mathcal{P}_3 - \tfrac{1}{10}\mathcal{P}_1 + \tfrac{1}{12}\mathcal{P}_0\big]\,. \end{aligned} \tag{3.4}$$

For $n < 2$ and $\kappa = 2$ we obtain

$$\begin{aligned} 4A_2^2\mathcal{P}_0 &= T^{-2}\Big[\tfrac{1}{12}\mathcal{P}_2 + \tfrac{1}{4}\mathcal{P}_1 + \tfrac{1}{6}\mathcal{P}_0\Big] + T^{-1}\Big[\tfrac{1}{2}\mathcal{P}_1 + \mathcal{P}_0\Big] \\ &\quad + \Big[-\tfrac{1}{6}\mathcal{P}_2 + \tfrac{5}{3}\mathcal{P}_0\Big] + T\Big[-\tfrac{1}{2}\mathcal{P}_1 + \mathcal{P}_0\Big] + T^2\Big[\tfrac{1}{12}\mathcal{P}_2 - \tfrac{1}{4}\mathcal{P}_1 + \tfrac{1}{6}\mathcal{P}_0\Big]\,, \\ 4A_2^2\mathcal{P}_1 &= T^{-2}\Big[\tfrac{1}{60}\mathcal{P}_3 - \tfrac{1}{10}\mathcal{P}_1 - \tfrac{1}{12}\mathcal{P}_0\Big] + T^{-1}\Big[-\tfrac{1}{6}\mathcal{P}_0\Big] \\ &\quad + \Big[-\tfrac{1}{30}\mathcal{P}_3 + \tfrac{1}{5}\mathcal{P}_1\Big] + T\Big[\tfrac{1}{6}\mathcal{P}_0\Big] + T^2\Big[\tfrac{1}{60}\mathcal{P}_3 - \tfrac{1}{10}\mathcal{P}_1 + \tfrac{1}{12}\mathcal{P}_0\Big]\,. \end{aligned} \tag{3.5}$$

For $n \geq 2$ it suffices to combine the identities (3.1) and (2.7).

In our computations [30] we apply these formulas to finitely many terms in the expansion $w = \sum_{j,n} w_{j,n} T^j \mathcal{P}_n$. In order to estimate truncation errors, we use (among others) the following simple facts. Consider a convolution operator of the form

$$(Bw)_{j,n} = \sum_{i,m} B^{n,m}_{j-i} w_{i,m}\,, \tag{3.6}$$

where i,j,m,n denote integers, with m,n positive. Since we are interested in upper bounds only, assume for simplicity that the numbers $B^{n,m}_k$ and $w_{i,m}$ are all nonnegative. We consider bounds of the type

$$W_{J,n}(w) = \sum_{j \geq J} r^{j-J} w_{j,n} \tag{3.7}$$

with $r > 1$, and weighted sums (over n) of such bounds.

Proposition 3.1. *$W_{J,n}(Bw) = (BW(w))_{J,n}$ for all J and n. If $w_{j,n} = 0$ for all $j < L$, then $W_{J,n}(w) \leq r^{L-J} W_{L,n}$ for all J; and equality holds for $J \leq L$.*

A proof of these properties is straightforward. Notice that $w \mapsto W(w)$ is a convolution, which explains most of why $WB = BW$.

4. Analyticity

Here we describe the function spaces that are used in our analysis of the fixed point equation (2.1). Given $\rho \geq 1$, denote by $\mathcal{A}_\rho$ the space of all functions f in $\mathcal{H} = \mathrm{L}^2(I_0)$ that have a finite norm

$$\|f\|_\rho \stackrel{\text{def}}{=} \sum_{n \in \mathbb{N}} |f_n| \rho^n\,, \qquad f = \sum_{n \in \mathbb{N}} f_n \mathcal{P}_n\,. \tag{4.1}$$

The even and odd subspaces of $\mathcal{A}_\rho$ are denoted by $\mathcal{A}^0_\rho$ and $\mathcal{A}^1_\rho$, respectively. If $\rho > 1$, then every function in $\mathcal{A}_\rho$ admits an analytic continuation to the complex open neighborhood $\mathcal{E}_\rho$ of I_0 whose boundary is the ellipse $z = \frac{1}{4}\bigl(\omega + \omega^{-1}\bigr)$ with $|\omega| = \rho$. This is a consequence of the following lemma.

Lemma 4.1. (Bernstein Lemma) *Let p be a polynomial of degree n such that $|p(x)| \leq 1$ for all $x \in [-1,1]$. If $z = \frac{\omega + \omega^{-1}}{2}$ with $|\omega| = \rho > 1$, then $|p(z)| < \rho^n$.*

Proof. The function $\omega \mapsto \omega^{-n} p\bigl(\frac{\omega+\omega^{-1}}{2}\bigr)$ is analytic at any $\omega \neq 0$, including $\omega = \infty$. By the maximum modulus theorem, its maximum modulus over $|\omega| = \rho$ is less than its maximum modulus over the circle $|\omega| = 1$. **QED**

Consider now functions $w : \mathbb{R} \to \mathbb{R}$ that admit an expansion (2.3) with $w_j \in \mathcal{A}_\rho$ for all j. Given $r \geq 1$, define $\mathcal{B}_{\rho,r}$ to be the space of all such functions w that have a finite norm

$$\|w\|_{\rho,r} \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \|w_j\|_\rho r^{|j|} = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{N}} |w_{j,n}| r^{|j|} \rho^n\,. \tag{4.2}$$

The even and odd subspaces of $\mathcal{B}_{\rho,r}$ are denoted by $\mathcal{B}^0_{\rho,r}$ and $\mathcal{B}^1_{\rho,r}$, respectively. Since our fixed point equation (2.1) involves a product of functions, we will need the following.

Proposition 4.2. *The spaces $\mathcal{A}_\rho$, $\mathcal{A}^0_\rho$, $\mathcal{B}_{\rho,r}$, and $\mathcal{B}^0_{\rho,r}$ are Banach algebras.*

Proof. The product of two function in $\mathcal{A}_\rho$ can be estimated by using the linearization formula

$$\mathcal{P}_k\mathcal{P}_l = \sum_m C_{k,l,m}\mathcal{P}_m\,, \qquad \sum_m C_{k,l,m} = 1\,. \tag{4.3}$$

The existence of such an expansion follows by completeness, and the second identity is a consequence of the fact that $\mathcal{P}_n(1/2) = 1$ for all n. Clearly $C_{k,l,m} = 0$ for $m > k + l$. And by parity, we have $C_{k,l,m} = 0$ unless $k + l + m$ is even. Below will see that the coefficients $C_{k,l,m}$ are all nonnegative. So if f and g belong to $\mathcal{A}_\rho$, then

$$\begin{aligned} \|fg\|_\rho &= \sum_{m\in\mathbb{N}} \left| \sum_{k,l\in\mathbb{N}} C_{k,l,m}\bigl(\rho^k f_k\bigr)\bigl(\rho^l g_l\bigr)\rho^{m-k-l} \right| \\ &\le \sum_{k,l\in\mathbb{N}} \rho^k|f_k|\rho^l|g_l| \sum_{m\le k+l} |C_{k,l,m}| = \|f\|_\rho\|g\|_\rho\,. \end{aligned} \tag{4.4}$$

An analogous inequality $\|vw\|_{\rho,r} \le \|v\|_{\rho,r}\|w\|_{\rho,r}$ for functions $v, w \in \mathcal{B}_{\rho,r}$ follows trivially.

The following expression for the coefficients $C_{k,l,m}$ was first found by Adams [2].

$$C_{k,l,m} = \frac{a(s-k)a(s-l)a(s-m)}{a(s)}\,\frac{2m+1}{2s+1}\,, \qquad a(n) = 2^{-n}\binom{2n}{n}\,, \tag{4.5}$$

where $s = \frac{1}{2}(k+l+m)$. Our main reason for giving this formula here is that it is being used in our programs [30]. But it also shows that the coefficients $C_{k,l,m}$ are all nonnegative, as claimed above. **QED**

Remark 1. A noteworthy consequence of (4.4) is the following. Let $f \in \mathcal{A}_\rho$ and $z \in \mathcal{E}_\rho$. Since functions in $\mathcal{A}_\rho$ define continuous functions on $\mathcal{E}_\rho$, the spectrum of the operator $F : g \mapsto fg$ on $\mathcal{A}_\rho$ includes $f(z)$. Thus, $|f(z)| \le \|F\| \le \|f\|_\rho$. This argument generalizes to other Banach algebras (under pointwise multiplication) of continuous functions.

Notice that the translation operators T and T^{-1} are bounded on $\mathcal{B}_{\rho,r}$. So the following is essentially a consequence of the identity (2.7) and (2.8).

Proposition 4.3. *The operator A_κ defined by (3.1) is bounded on $\mathcal{B}_{\rho,r}$.*

The following will be used to prove that a solution $v \in \mathcal{B}_{\rho,r}$ of the equation (1.6) is real analytic.

Proposition 4.4. *Assume that $\rho > 1$. Then there exists $\epsilon > 0$ such that, if $w \in \mathcal{B}_{\rho,r}$ is of class C^∞, then w extends analytically to the strip $|\operatorname{Im} z| < \epsilon$.*

Proof. Let $w \in \mathcal{B}_{\rho,r}$. As described after (4.1), each arc w_j extends analytically to a domain $\mathcal{E}_\rho$ that includes an open rectangle $\mathcal{R} = (-1/2 - \epsilon, 1/2 + \epsilon) \times (-\epsilon, \epsilon)$. Thus $\mathbb{P}_j v$

extends analytically to $T^j\mathcal{R}$, for each $j \in \mathbb{Z}$. Denote this extension by $E_j w$. Then the domains of both $E_j w$ and $E_{j+1}w$ include the rectangle $\mathcal{R}_j = (j+{}^1\!/_2-\epsilon, j+{}^1\!/_2+\epsilon)\times(-\epsilon,\epsilon)$. Assume now that w belongs to $\mathrm{C}^\infty(\mathbb{R})$. Then the derivatives $D^n E_j w$ and $D^n E_{j+1}w$ agree at the point $j+{}^1\!/_2$, for all $n \geq 0$. Thus, $E_j w$ and $E_{j+1}w$ agree on $\mathcal{R}_j$. This holds for all $j \in \mathbb{Z}$, so w extends analytically to $\mathbb{R}\times(-\epsilon,\epsilon)$. **QED**

5. The fixed point equation

Consider the parameter values $(m, \bar{\mu}, \sigma, r, k, \kappa)$ associated with some fixed row in Table 1. In addition, we choose $\rho = {}^{17}\!/_{16}$. Let μ be a real number of modulus $|\mu| < 1$. Then the equation (2.1) for these parameters reads $v = \nu A^2 v^m + \mu A^2 v$. The corresponding equation for the function $w = v(\mathbf{.}/\kappa)$ is $w = \nu A_\kappa^2 w^m + \mu A_\kappa^2 w$, with A_κ as defined by (3.1). This equation for w is equivalent to the fixed point equation $G_q(w) = w$, where

$$G_q(w) = \nu A_\kappa^2 \Sigma_q w^m + \mu^k A_\kappa^{2k} w\,, \qquad \Sigma_q = \sum_{n=0}^{k-1} \mu^n A_\kappa^{2n}\,. \tag{5.1}$$

The subscript used here is $q = (m, \mu, k, \kappa)$. By Propositions 4.2 and 4.3, G_q is differentiable as a map on $\mathcal{B}_{\rho,r}$. Notice that, if w and h belong to $\mathcal{B}_{\rho,r}$, and if h is supported far away from the origin, then $DG_q(w)h$ is approximately equal to $\mu^k A_\kappa^{2k} h$. So the transformation G_q contracts tails by roughly a factor $|\mu|^k$. This should make clear why we need larger values of k when $|\mu|$ is close to 1.

Definition 5.1. *A compactly supported function $w : \mathbb{R} \to \mathbb{R}$ whose arcs w_j are all polynomials will be called a spline.*

Let $\tau = 1-\sigma$. Our choice of the exponent m in (1.4) guarantees that $G_q(w)$ has parity τ whenever w has parity τ. Given a function $\bar{w} \in \mathcal{B}_{\rho,r}^\tau$, and a bounded linear operator M on $\mathcal{B}_{\rho,r}^\tau$, define

$$\mathcal{N}_q(h) = G_q(\bar{w} + \Lambda h) - \bar{w} + Mh\,, \qquad \Lambda = \mathrm{I} - M\,, \tag{5.2}$$

for every function $h \in \mathcal{B}_{\rho,r}^\tau$. Clearly, if h is a fixed point of $\mathcal{N}_q$ then $\bar{w} + \Lambda h$ is a fixed point of G_q. For practical reasons, we choose $\bar{w}$ to be a spline that is an approximate fixed point of $G_{\bar{q}}$ with $\bar{q} = (m, \bar{\mu}, k, \kappa)$. And for M we choose a finite rank operator such that Λ is an approximate inverse of $\mathrm{I} - DG_{\bar{q}}(\bar{w})$. If μ is sufficiently close (but not necessarily equal) to $\bar{\mu}$, then we can expect $\mathcal{N}_q$ to be a contraction near the origin.

Lemma 5.2. *Consider a fixed but arbitrary row in Table 1 and the parameter values given in that row. Let $\rho = {}^{17}\!/_{16}$ and $\tau = 1-\sigma$. Then there exists a spline $\bar{w}$, a bounded linear operator M on $\mathcal{B}_{\rho,r}^\tau$, and positive constants ε, K, δ satisfying $\varepsilon + K\delta < \delta$, such that for every value of μ in some open neighborhood of $\bar{\mu}$, the transformation $\mathcal{N}_q$ defined by (5.2) satisfies*

$$\|\mathcal{N}_q(0)\|_{\rho,r} \leq \varepsilon\,, \qquad \|D\mathcal{N}_q(h)\|_{\rho,r} \leq K\,, \qquad h \in B_\delta\,, \tag{5.3}$$

where B_δ denotes the closed ball of radius δ in $\mathcal{B}^\tau_{\rho,r}$, centered at the origin. Furthermore, if $w = \bar{w} + \Lambda h$ with $h \in B_\delta$, then the function $v = w(\kappa\,\boldsymbol{.})$ has the properties listed in (the given row of) Table 1 concerning the sup*-norm and the local extrema.*

Notice that $\|w - \bar{w}\|_{\rho,r} \le \delta'$, where $\delta' = \|\Lambda\|\delta$. Our proof of Lemma 5.2 yields $\delta' < 2^{-32}$ for all solutions. This bound can be made as small as desired by running our programs (which also determine $\bar{w}$) at higher numerical precision.

Based on this lemma, we can now give a

Proof of Theorem 1.1. By the contraction mapping principle, the given bounds imply that $\mathcal{N}_q$ has a unique fixed point h in B_δ. The corresponding function $w = \bar{w} + \Lambda h$ is a fixed point of G_q. Given that A^2_κ includes two antiderivatives, the identity $w = G_q(w)$ implies that w is of class C^∞. So by Proposition 4.4, w extends to an analytic function on some strip $|\operatorname{Im} z| < \epsilon$. This extension still decreases exponentially: by Remark 1 we have a uniform bound $|w(x + iy)| \le C r^{-|x|}$ that holds for all $x, y \in \mathbb{R}$ with $|y| < \epsilon$. These properties of w imply that the function $v = w(\kappa\,\boldsymbol{.})$ extends analytically to the strip $|\operatorname{Im} z| < \epsilon/\kappa$, decreases exponentially, and satisfies the the equation (1.6).

Consider now the function $u = u_L$ defined by the equation (1.7). The above-mentioned properties of v imply that u is real analytic. Furthermore, $\mathcal{D}u = v$. The equation (1.6) implies that the function g defined by $g = u'' - \tau^2 \mathcal{D}\phi'(v)$ satisfies $\mathcal{D}g = 0$. Thus g is periodic with period 1. The function $\mathcal{D}\phi'(v)$ vanishes at $\pm\infty$, so u'' approaches the periodic function g at $\pm\infty$. But $u'' = u''_L$ vanishes at $-\infty$. Thus $g = 0$, which implies that $u'' = \tau^2 \mathcal{D}\phi'(\mathcal{D}u)$. In other words, u satisfies the equation (1.1).

The same arguments apply to the function $u = u_R$. The difference $f = u_L - u_R$ is periodic, since $\mathcal{D}f = 0$. But f'' vanishes at infinity, so f is constant. Now define $u = {}^1\!/_2 u_L + {}^1\!/_2 u_R$. Then u is real analytic, satisfies the equation (1.1), and has parity σ.

The claims in Theorem 1.1 concerning the sup-norm and the local extrema of the function $v = \mathcal{D}u$ follow from the last statement in Lemma 5.2. **QED**

6. Computer estimates

What remains to be done is to verify Lemma 5.2. This is carried out with the aid of a computer. To be more specific, consider the parameter values $(m, \bar{\mu}, \sigma, r, k, \kappa)$ from a fixed but arbitrary row in Table 1. As a first step, we determine an approximate fixed point $\bar{w}$ of G_q and an approximate inverse of $\mathrm{I} - DG_{\bar{q}}(\bar{w})$ of the form $\Lambda = \mathrm{I} - M$, with M of finite rank. The remaining steps are rigorous: We compute an upper bound ε on the norm of $\mathcal{N}_q(0)$, and an upper bound K on the operator norm of $D\mathcal{N}_q(h)$ that holds for all h of norm 4ε or less. This is done simultaneously for all values of μ in some open interval centered at $\bar{\mu}$. After verifying that $K < {}^7\!/_8$, we choose a positive $\delta < 8\varepsilon$ in such a way that $\varepsilon + K\delta < \delta$. The last statement in Lemma 5.2 is verified by estimating $v(x_i)$ at a finite number of points $x_i \in \mathbb{R}$.

The rigorous part is still numerical, but instead of truncating series and ignoring rounding errors, it produces guaranteed enclosures at every step along the computation. This part of the proof is written in the programming language Ada [31]. The following is meant to be a rough guide for the reader who wishes to check the correctness of our programs. The complete details can be found in [30].

In the present context, a “bound” on a map $f : \mathcal{X} \to \mathcal{Y}$ is a function F that assigns to a set $X \subset \mathcal{X}$ of a given type (`Xtype`) a set $Y \subset \mathcal{Y}$ of a given type (`Ytype`), in such a way that $y = f(x)$ belongs to Y for all $x \in X$. In Ada, such a bound F can be implemented by defining a `procedure F(X: in Xtype; Y: out Ytype)`.

For balls in a real Banach algebra $\mathcal{X}$ with unit $\mathbf{1}$, we use a data type `Ball`. A `Ball` consists of a pair `B=(B.C,B.R)`, where `B.C` is a representable number (`Rep`) and `B.R` a nonnegative representable number (`Radius`). The corresponding ball in $\mathcal{X}$ is the set $\mathtt{B}_{\mathcal{X}} = \{x \in \mathcal{X} : \|x - (\mathtt{B.C})\mathbf{1}\| \le \mathtt{B.R}\}$. Our bounds on some standard functions involving the type `Ball` are defined in the packages `Std_Balls`. Other basic functions are covered in the packages `Vectors` and `Matrices`. Bounds of this type have been used in many computer-assisted proofs; so we focus here on the more problem-specific aspects of our programs.

6.1. Analytic arcs

Consider the space $\mathcal{A}_\rho$ for a given `Radius` $\rho \ge 1$. Our enclosures for functions in $\mathcal{A}_\rho$ are associated with a data type `Legend`, based on scalars of type `Ball`, with $\mathcal{X} = \mathbb{R}$. Given a fixed odd integer $D > 1$, a `Legend` is in essence a pair `G=(G.C,G.E)`, where `G.C` is an `array(0..D) of Ball` and `G.E` is an `array(0..D+2) of Radius`. The corresponding set $\mathtt{G}_{\mathcal{A}} \subset \mathcal{A}_\rho$ consists of all functions g that admit a representation

$$g = \sum_{n=0}^{D} c_n \mathcal{P}_n + \sum_{m=0}^{D+2} \rho^m E_m \,, \tag{6.1}$$

with $c_n \in \mathtt{G.C}(n)_{\mathbb{R}}$ and $E_m \in \mathcal{A}_{\rho,m}$, satisfying $\|E_m\|_\rho \le \mathtt{G.E}(m)$. Here $\mathcal{A}_{\rho,m}$ denotes the subspace of $\mathcal{A}_\rho$ consisting of all functions $E \in \mathcal{A}_\rho$ that are orthogonal to all polynomials of degree less than m and have the same parity as $\mathcal{P}_m$.

The type `Legend` is defined in the package `Legends`. This package also implements basic bounds on functions to/from the space $\mathcal{A}_\rho$. This includes a bound `Prod` on the product $(f, g) \mapsto fg$, based on the identities (4.3) and (4.5). These bounds are quite straightforward, so we refer to [30] for details.

In `Legends.Chain` we use (2.7) to define a bound `DDInvHigh` on the operator D^{-2} on $\mathcal{A}_\rho$, restricted to function with $f_0 = f_1 = 0$. For compactly supported functions $w \in \mathcal{A}_\rho^{\mathbb{Z}}$ we use enclosures of a type `LVector`. This type is defined as an unconstrained `array (Integer range <>) of Legend`. Using `DDInvHigh`, as well as the identities (3.1), (3.4), and (3.5), we define a bound `AA` on the linear operator A_κ^2 for such functions.

6.2. Piecewise real analytic functions

Let L be a fixed integer larger than 1. Then a function $w \in \mathcal{B}_{\rho,r}^{\tau}$ has a unique decomposition

$$w = \sum_{|j|<L} T^j w_j + G \,, \tag{6.2}$$

with $G \in \mathcal{B}^{\tau}_{\rho,r}$ supported outside $({}^1\!/_2 - j, j - {}^1\!/_2)$. The sum in this equation will be referred to as the "center" of w, and G will be referred to as the "tail" of w.

Let $r \geq 1$ be a fixed `Radius`. An enclosure for a tail in $\mathcal{B}^{\tau}_{\rho,r}$ is defined by a `Legend G` with the property that `G.C`(n)`.C` is zero for all n. The corresponding set $\mathtt{G}_{\mathcal{B}} \subset \mathcal{B}^{\tau}_{\rho,r}$ consists of all functions G that admit a representation

$$G = r^L \sum_{|j|\geq L} T^j g_j \,, \qquad g_j = \sum_{n=0}^{D} c_{j,n}\mathcal{P}_n + \sum_{m=0}^{D+2} \rho^m E_{j,m} \,, \tag{6.3}$$

with coefficients $c_{j,m} \in \mathbb{R}$ and functions $E_{j,m} \in \mathcal{A}_{\rho,m}$ satisfying the bounds

$$\sum_{j\geq L} r^{j-L}|c_{j,n}| \leq \mathtt{G.C}(n)\mathtt{.R} \,, \qquad \sum_{j\geq L} r^{j-L}\|E_{j,m}\|_\rho \leq \mathtt{G.E}(m) \,. \tag{6.4}$$

The coefficients $c_{j,m}$ and functions $E_{j,m}$ for $j \leq -L$ are determined by the requirement that G has parity τ.

For more general subsets of $\mathcal{B}^{\tau}_{\rho,r}$ we use a data type `LChain`, which consists of a triple `W=(W.R,W.Par,W.C)`, where `W.R` $= r$, `W.Par` $= \tau$, and where `W.C` is an `LVector(0..L)`. The component `W.C(0)` must have parity τ. The corresponding set $\mathtt{W}_{\mathcal{B}}$ is defined as the set of all functions (6.2), where $w_j \in \mathtt{W.C}(j)_{\mathcal{A}}$ for $0 \leq j < L$, and $G \in \mathtt{W.C}(L)_{\mathcal{B}}$. The arcs w_j for $-L < j < 0$ are determined by the requirement that w has parity τ.

These types are defined in the package `Legends.Chain` which takes `JCMax` $= L-1$ and `Scale` $= \kappa$ as arguments. This package also implements basic bounds on functions to/from the spaces $\mathcal{B}^{\tau}_{\rho,r}$. Most are straightforward combinations of bounds defined in `Legends`, such as the bound `Norm` on $w \mapsto \|w\|_{\rho,r}$ or the bound `Prod` on $(v,w) \mapsto vw$. The representation (6.3) for the tail G has been chosen in such a way that the tail component `W.C(L)` of an `LChain W` can often be treated the same way as the other components `W.C(J)`.

6.3. Transformations and their derivatives

Consider first the operators A^{2k}_κ and $A^2_\kappa \Sigma_q$ that appear in the definition (5.1) of the transformation G_q. Our bounds on these two operators are given by the two procedures `AAPower` and `SumAAPowers`. They are more elaborate than the bounds discussed so far, due to the fact that A^2_κ is nonlocal. In particular, A^2_κ couples the center and tail of a function w.

Consider the task of implementing a bound on A^2_1. Given an `LChain W`, consider a fixed but arbitrary function $w \in \mathtt{W}_{\mathcal{B}}$ of the form (6.2). The goal is to find an enclosure $\mathtt{U}_{\mathcal{B}}$ for the function $u = A^2_1 w$ that only depends on `W`. By linearity, we can consider centers and tails separately. Assume first that `W` has a zero tail `W.C(L)`. Then, by using the above-mentioned bound `AA` for compactly supported chains, we obtain a `LVector`-type enclosure $\mathtt{P(0..L)}_{\mathcal{A}^{\mathbb{Z}}}$ for u. Setting `U.C(0..L-1) := P(0..L-1)` and converting `P(L)` to a tail `U.C(L)`, we obtain the desired enclosure $\mathtt{U}_{\mathcal{B}}$. This part, generalized to A^{2k}_κ, is implemented by the procedure `LA_AAPower`. Next, consider the case where `W.C(J)` is zero for all `J<L`. In order to determine the center part of `U`, the tail `W(L)` can be considered to be a bound on w_L only, since A^2_1 is a convolution with a kernel supported in $[-1,1]$. So the center components `U.C(0..L-1)` of `W` can be obtained again via the procedure `AA`. This part, generalized to

A_κ^{2k}, is implemented by `HL_AAPower`. An enclosure $\texttt{U.C(L)}_{\mathcal{B}}$ for the tail of u is constructed in the procedure `HH_AAPower`. In this part we use Proposition 3.1. The enclosure `U` for $u = A_\kappa^{2k}w$ returned by `AAPower` is the `Sum` of the enclosures returned by `LA_AAPower`, `HL_AAPower` and `HH_AAPower`. Our bound `SumAAPowers` on $A_\kappa^2\Sigma_q$ is very similar. For more details we refer to the program code [30].

Our bounds `GMap` and `DGmap` on the map G_q and its derivative, respectively, are defined in the package `Legends.Chain.Fix`. They are obtained simply by combining lower-level bounds like `Prod`, `AAPower`, and `SumAAPowers`. The construction (5.3) of a quasi-Newton map $\mathcal{N}$ from a given map G is sufficiently general and useful that it has been implemented in a generic package `Linear.Contr`. The same package has been used before in [27,28]. Our instantiation of `Linear.Contr` defines bounds `Contr` and `Contr` on the transformation $\mathcal{N}_q$ and its derivative, respectively. The type `LMode` that is used to instantiate `Linear` will be described below.

6.4. Operator norms

Consider the task of estimating the norm of a linear operator on $\mathcal{B}_{\rho,r}^\tau$. Let $\hat\tau = 1 - 2\tau$. Denote by $\mathcal{S}$ the set of all pairs $s = (j,n)$ of integers $j, n \ge 0$ with the property that $n \equiv \tau \pmod 2$ whenever $j = 0$. If we define $\varrho_{(j,n)} = r^j\rho^n$ and

$$h^{(j,n)}(t) = \begin{cases} \frac{1}{2}r^{-j}\rho^{-n}[\mathcal{P}_n(t-j) + \hat\tau\mathcal{P}_n(j-t)], & \text{if } t \in I_j \cup I_{-j};\\ 0, & \text{otherwise;}\end{cases} \tag{6.5}$$

then a function $w \in \mathcal{B}_{\rho,r}^\tau$ and its norm $\|w\| = \|w\|_{\rho,r}$ can be written as

$$w = \sum_{s\in\mathcal{S}} w_s h^s\,, \qquad \|w\| = \sum_{s\in\mathcal{S}} |w_s|\varrho_s\,. \tag{6.6}$$

A useful feature of such weighted ℓ^1 spaces is the following. Let $\mathcal{L}$ be a continuous linear operator on $\mathcal{B}_{\rho,r}^\tau$. Then the operator norm of $\mathcal{L}$ is simply $\|\mathcal{L}\| = \sup_{s\in\mathcal{S}}\|\mathcal{L}h^s\|$. In order to estimate this norm, we first choose a suitable partition $\{S_1, S_2, \ldots, S_M\}$ of $\mathcal{S}$. Then

$$\|\mathcal{L}\| = \max\{b_1, b_2, \ldots, b_M\}\,, \qquad b_m = \sup_{s\in S_m}\|\mathcal{L}h^s\|\,. \tag{6.7}$$

The sets S_m that we use in our partitions of $\mathcal{S}$ are specified by data of type `LMode`. A partition is represented by an `array (1..M) of LMode`. Such partitions are created by the procedure `Make` in `Legends.Chain`. To simplify notation, let us identify a `LMode S` with the corresponding subset $S \subset \mathcal{S}$. A `procedure Assign(S: in LMode; H: out LChain)` defines a set $\texttt{H}_{\mathcal{B}} \subset \mathcal{B}_{\rho,r}^\tau$ that contains all functions h^s with $s \in S$.

The way this is being used is as follows. Let `LinOp` be a bound on the operator $\mathcal{L}$. Then a `Ball`-type enclosure $\texttt{B}_{\mathbb{R}}$ for the constant b_m is obtained by calling `Assign(S,H)` with $S = S_m$, followed by `LinOp(H,G)` and then `Norm(G,B)`. For the operator $\mathcal{L} = D\mathcal{N}_q(w)$, this is carried out by the procedure `DContrNorm` in `Legends.Chain.Fix`. This procedure is little more than an instantiation of the procedure `Op_Norm` from the generic package `Linear`, with `LinOp` being in essence `DContr`.

For the complete details we refer to the source code of our programs [30]. For the set of representable numbers (`Rep`) we choose standard extended floating-point numbers [33] that support controlled rounding, and for bounds on non-elementary `Rep`-operations we use the open source MPFR library [34]. Our programs were run successfully on a standard desktop machine, using a public version of the gcc/gnat compiler [32].

7. Appendix

The figures below show graphs of the functions $v = \mathcal{D}u$ associated with our solutions u of the equation (1.3). Recall that v is a solution of the equation

$$v = A^2\bigl(\mu v + v^m\bigr)\,, \qquad A = \mathcal{D}D^{-1}\,. \tag{7.1}$$

Notice that odd solutions ($\sigma = 0$) of this equation require m to be odd.

All solutions depicted here are for the β-model ($m = 3$). The solutions 1 and 2 for the α-model ($m = 2$) are similar to the β-model solutions 4 and 8.

7.1. Some solutions with $\sigma = 1$

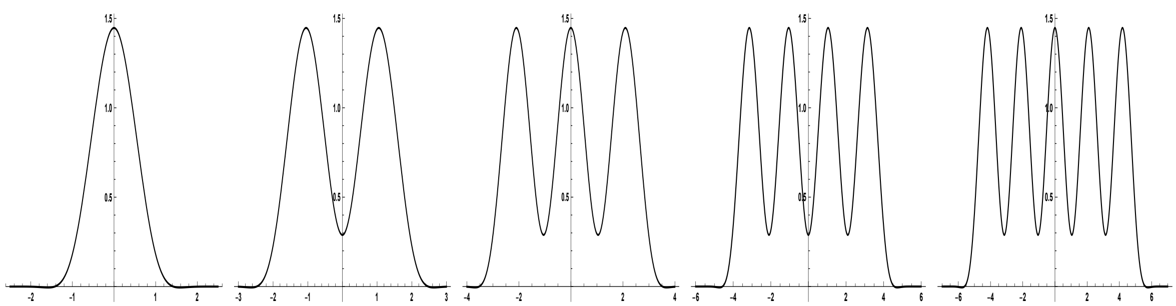


Figure 2. $v = \mathcal{D}u$ for the solutions 6, 8, 11, 14, and 17; for $m = 3$ and $\mu = -1/4$.

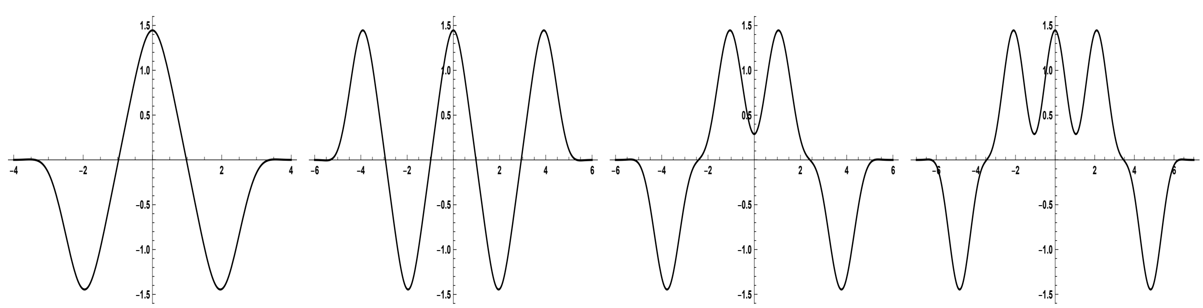


Figure 3. $v = \mathcal{D}u$ for the solutions 12, 15, 13, and 18; for $m = 3$ and $\mu = -1/4$.

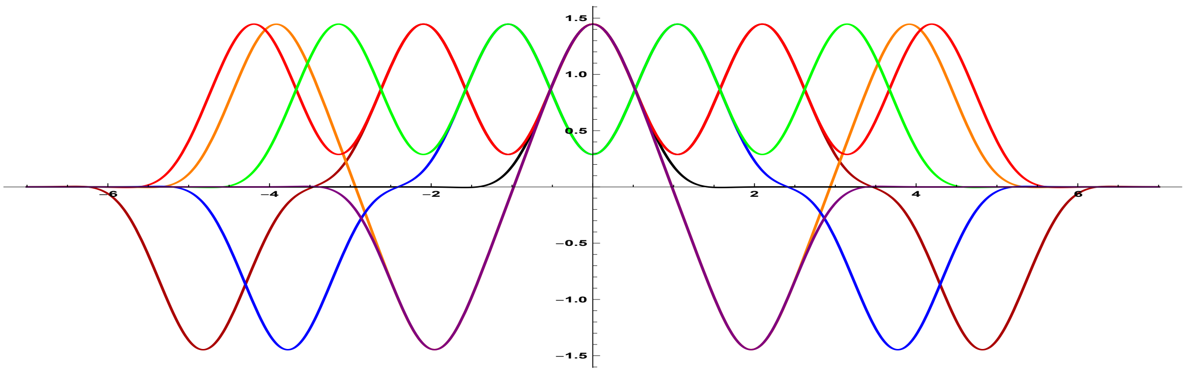


Figure 4. Comparison of graphs from Figures 2 and 3.

7.2. Some solutions with $\sigma = 0$

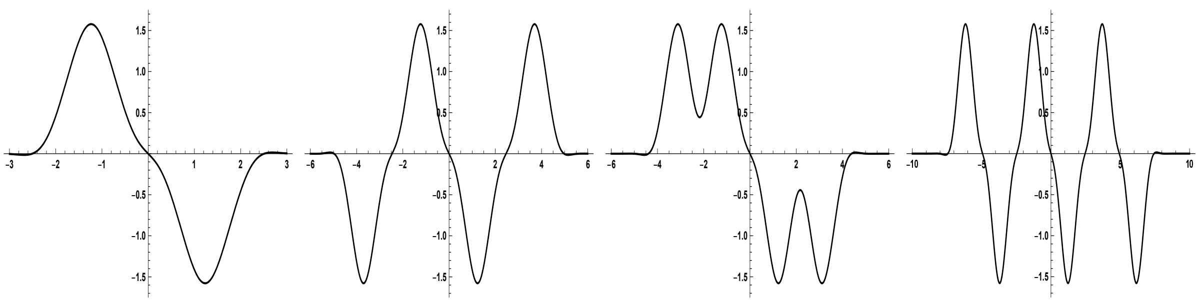


Figure 5. $v = \mathcal{D}u$ for the solutions 19, 20, 21, and 22; for $m = 3$ and $\mu = -1/2$.

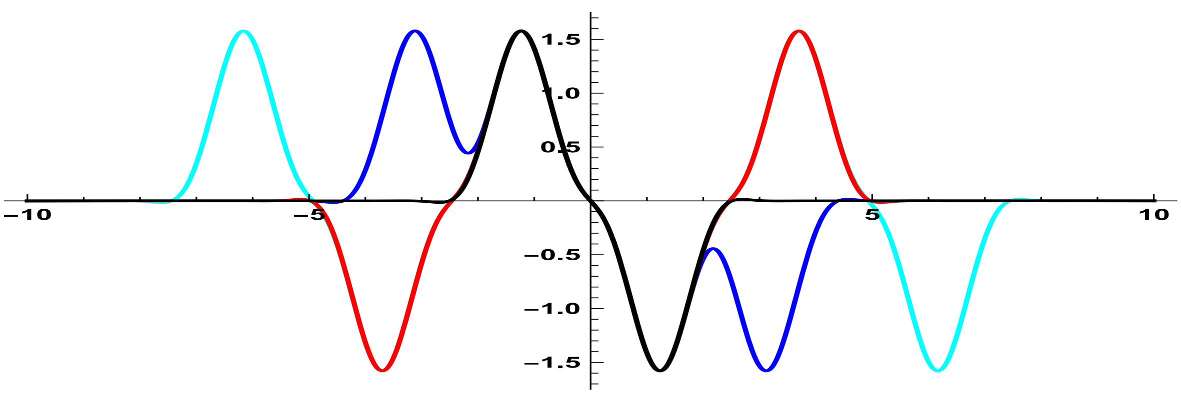


Figure 6. Comparison of graphs from Figure 5.

7.3. Limit behavior

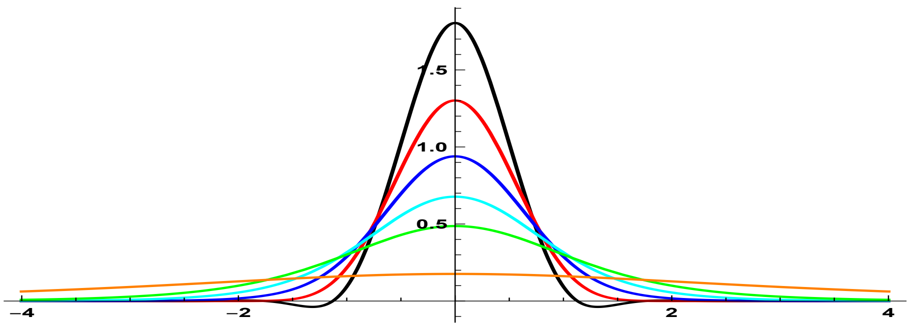


Figure 7. Behavior of $v = \mathcal{D}u$ as $\mu \nearrow 1$; $\mu = -{}^{63}/_{64}, 0, {}^{1}/_{2}, {}^{3}/_{4}, {}^{7}/_{8}, {}^{63}/_{64}$.

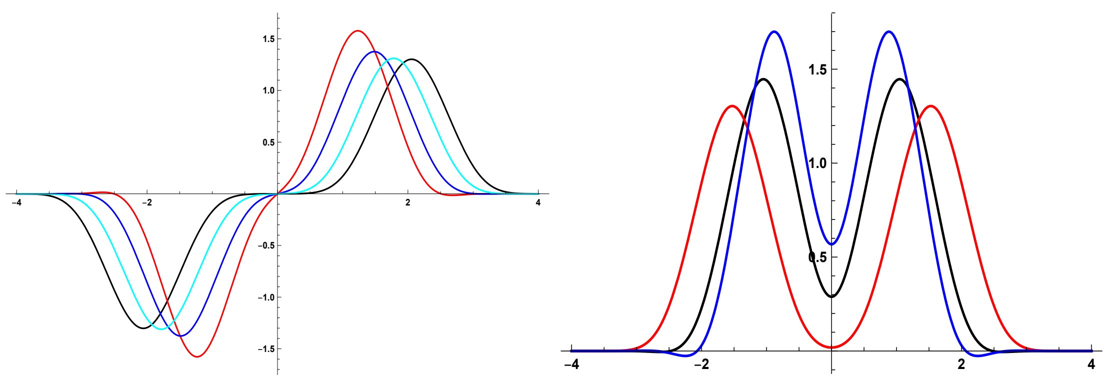


Figure 8. Behavior of $v = \mathcal{D}u$ as $\mu \nearrow 0$;
$\mu = -{}^{1}/_{2}, -{}^{1}/_{8}, -{}^{1}/_{64}, -{}^{1}/_{1024}$ (left); $\mu = -{}^{1}/_{2}, -{}^{1}/_{4}, -{}^{1}/_{256}$ (right).

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