Modular Forms, Eisenstein Series and a short introduction to elliptic functions

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Abstract

In the first part we introduce modular forms for $\mathrm{SL}_2(\mathbb{Z})$. We will then look at few properties of such modular forms, in particular to an equivalent statment about the transformation property for modular forms.

In a second part, we give the definition of the Eisenstein series G_{2k} for $k \in \mathbb{N}$, $k > 1$ and check that they are well-defined. We then give an example of non-trivial modular forms by showing, that these Eisenstein series are modular forms of weight $2k$. In the last part of the second chapter we also give the Fourier expansion of the Eisenstein series.

In the last part we then define elliptic functions and introduce only by its defintion the Weierstrass $\wp$ function. We point out, that there is a relation between the Laurent expansion of the Weierstrass $\wp$ function and the Eisenstein series.

This notes are mostly based on Koblitz' book [1].

Contents

1 Modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and some properties

1.1 Definition of modular forms

We will introduce modular forms and give some of their properties. First recall that

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \middle| \, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let $\mathcal{H}$ be the upper half plane.

Definition 1.1. Let k be an integer. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is weakly modular of weight k if

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau) \tag{1}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

As we will see in the following modular forms are weakly modular functions that are holomorphic on $\mathcal{H}$ and holomorphic at ∞. When f satisfies the identity $f(\tau + 1) = f(\tau)$ on $\mathcal{H}$ and is holomorphic on the upper half plane $\mathcal{H}$, then f has a Fourier expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i \tau}.$$

f beeing holomorphic at ∞, thinking of ∞ lying far in the imaginary direction, means that $g(q) = \sum_{n \in \mathbb{Z}} a_n q^n$ extends holomorphically to $q = 0$, this is when $a_n = 0$ for $n < 0$. Since $q \to 0$ if and only if $\Im\tau \to \infty$, to show that a weakly modular function holomorphic on $\mathcal{H}$ is holomorphic at ∞, it is enough to show that $\lim_{\Im\tau \to \infty} f(\tau)$ exists or even just that $f(\tau)$ is bounded as $\Im\tau \to \infty$.

Next we define modular forms.

Definition 1.2. Let $k \geq 0$ be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of weight k if

1. f is holomorphic on $\mathcal{H}$
2. f satisfies $f(\gamma(\tau)) = (c\tau + d)^k f(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$
3. f is holomorphic at ∞.

The set of modular forms of weight k is denoted $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$.

1.2 Some properties of modular forms

Since $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is also in $\mathrm{SL}_2(\mathbb{Z})$, $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ has the following property: If k is an odd integer, then all modular forms of weight k are zero. This can be seen in the following way: Take $\gamma = -I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Since

$$\gamma(\tau) = \frac{-\tau + 0}{0 - 1} = \tau, \quad \tau \in \mathcal{H}$$

it follows

$$f(\tau) = f(\gamma(\tau)) = (-1)^k f(\tau) = -f(\tau)$$

and so $f(\tau) = 0$ for all $\tau \in \mathcal{H}$.

Another important property is the following

Proposition 1.3. *$f : \mathcal{H} \to \mathbb{C}$ a meromorphic function. Then*

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau), \quad \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \tag{2}$$

if and only if

$$f(\tau + 1) = f(\tau) \tag{3}$$

and

$$f(-1/\tau) = \tau^k f(\tau). \tag{4}$$

Proof. ($\Rightarrow$) This direction is trivial, choosing $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

($\Leftarrow$) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Now we have

$$\begin{aligned} \frac{d\gamma(\tau)}{d\tau} &= \frac{d}{d\tau}\left(\frac{a\tau + b}{c\tau + d}\right) \\ &= \frac{a(c\tau + d) - (a\tau + b)c}{(c\tau + d)^2} \\ &= \frac{ac\tau + ad - ac\tau - bc}{(c\tau + d)^2} \\ &= (c\tau + d)^{-2} \end{aligned}$$

since $ad - bc = 1$. So we can rewrite (2) to

$$\left(\frac{d\gamma(\tau)}{d\tau}\right)^{\frac{k}{2}} f(\gamma(\tau)) = f(\tau).$$

Let γ_1, γ_2 both hold identity (2). Then by chain rule we get

$$\begin{aligned} \left(\frac{d\,(\gamma_1\gamma_2)(\tau)}{d\tau}\right)^{\frac{k}{2}} f((\gamma_1\gamma_2)(\tau)) &= \left(\frac{d\gamma_1(\tau')}{d\tau'}\bigg|_{\tau'=\gamma_2(\tau)}\right)^{\frac{k}{2}} \left(\frac{d\gamma_2(\tau)}{d\tau}\right)^{\frac{k}{2}} f(\gamma_1\,(\gamma_2(\tau))) \\ &= \left(\frac{d\gamma_2(\tau)}{d\tau}\right)^{\frac{k}{2}} f(\gamma_2(\tau)) \\ &= f(\tau)\,. \end{aligned}$$

So we see if (2) holds for γ_1 and γ_2, it holds also for $\gamma_1\gamma_2$. We previously have seen that T and S are generators for the modular group, and since by (3) and (4) the property (2) is satisfied using T and S, the result follows. □

Proposition 1.4. $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ *forms a vectorspace over* $\mathbb{C}$.

Proof. Let f and g be in $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$ and let $\alpha \in \mathbb{C}$. Then we have to prove that $f + \alpha g$ is also in $\mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$. It is clear that $f + \alpha g$ ist holomorphic in $\mathcal{H}$ and at ∞, as well as satisfying the transformation property. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then we see by calculation that

$$\begin{aligned} (f + \alpha g)(\gamma(\tau)) &= f(\gamma\,(\tau)) + \alpha g(\gamma(\tau)) \\ &= (c\tau + d)^k f(\tau) + \alpha\,(c\tau + d)^k g(\tau) \\ &= (c\tau + d)^k\,(f(\tau) + \alpha g(\tau)) \\ &= (c\tau + d)^k\,(f + \alpha g)(\tau)\,. \end{aligned}$$

Therefore $f + \alpha g \in \mathcal{M}_k(\mathrm{SL}_2(\mathbb{Z}))$. □

Last we observe, that for $k = 0$, applying (1) we get

$$f(\gamma(\tau)) = f(\tau)\,, \quad \tau \in \mathcal{H}.$$

So f is then invariant under $\mathrm{SL}_2(\mathbb{Z})$ and f can be considered as a function on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}$. One can show that the only modular forms of weight 0 are constant functions.

2 Eisenstein series

2.1 Defintion of Eisenstein series

Let us give an explicit example of a modular form of weight $2k$ for an integer k greater than one. We first give a defintion and then show that it is well-defined. We only look at such one with even weight, since the others of odd weight are all zero.

Definition 2.1. Let k be an integer greater than 1. For $\tau \in \mathcal{H}$ we define the Eisenstein series of weight $2k$ by

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau + n)^{2k}}. \tag{5}$$

Remark 2.2. If we hold $\tau \in \mathcal{H}$ fixed, then the summation is "in fact" over all nonzero elements of the lattice $\Lambda = \Lambda(1, \tau) = \mathbb{Z} + \mathbb{Z}\tau$.

To show later on that G_{2k} is well-defined we need to show that for $\tau \in \mathcal{H}$ the series is absolutely convergent. So we need a lemma. Let in the following denote $\Lambda = \Lambda(1, \tau) = \mathbb{Z} + \mathbb{Z}\tau$.

Lemma 2.3. *If α is real the infinite series*

$$\sum_{\omega\in\Lambda}{}' \frac{1}{\omega^\alpha}$$

converges absolutely, if and only if $\alpha > 2$.

Proof. Look at the parallelogram P with boundary $-1-\tau$, $-\tau$, $1-\tau$, 1, $1+\tau$, τ, $-1+\tau$ and -1 (see figure 1). Let r and R denote the minimum respectively the maximum distances from 0 to the parallelogram P. Then if ω is any of the

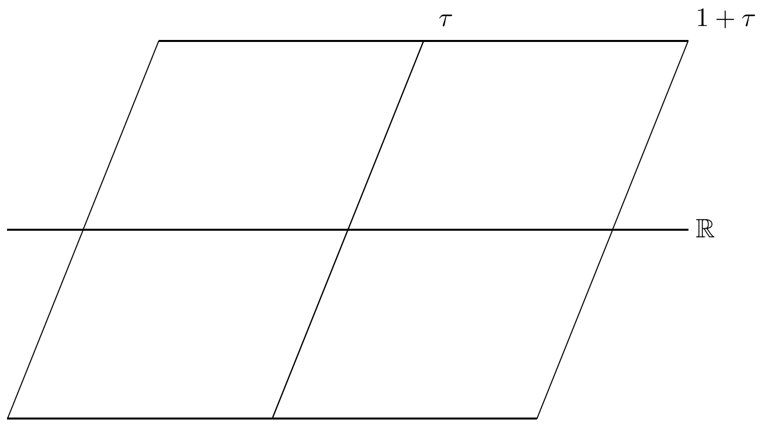

Figure 1: First layer of the eight nonzero periods ω.

8 nonzero lattice points it holds

$$r \leq |\omega| \leq R \quad \text{for } 1 \cdot 8 \text{ lattice points } \omega.$$

In the next layer of lattice points surrounding these 8 we have $2 \cdot 8 = 16$ new lattice points satisfying the inequalities

$$2r \leq |\omega| \leq 2R \quad \text{for } 2 \cdot 8 \text{ lattice points } \omega.$$

In general in the kth layer there are $8k$ new lattice points satisfying the inequalities

$$kr \leq |\omega| \leq kR \quad \text{for } 8k \text{ lattice points } \omega.$$

Therefore we have

$$\frac{1}{(kR)^\alpha} \leq \frac{1}{|\omega|^\alpha} \leq \frac{1}{(kr)^\alpha} \quad \text{for the } k\text{th } 8k \text{ lattice points } \omega.$$

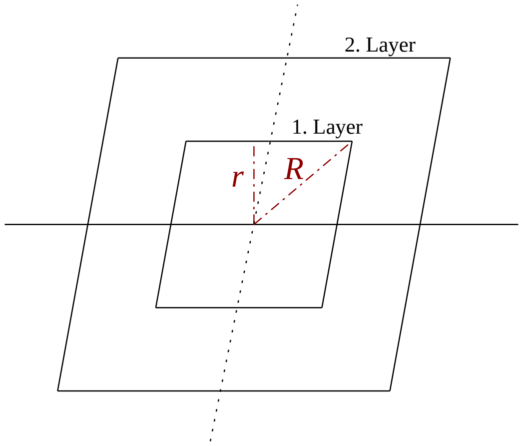

Figure 2: $S(n)$ beeing the sum over the first n layers of the terms $\frac{1}{|\omega|^\alpha}$.

Let $S(n)$ denote the sum of $\frac{1}{|\omega|^\alpha}$ over the first n layers around the origin, which means that we we take the sum over the $8\sum_{l=1}^n l$ nonzero lattice points nearest the origin (see figure 2). Then

$$\sum_{l=1}^n \frac{8l}{(Rl)^\alpha} = \frac{8}{R^\alpha}\sum_{l=1}^n \frac{1}{l^{\alpha-1}} \leq S(n) \leq \sum_{l=1}^n \frac{8l}{(rl)^\alpha} = \frac{8}{r^\alpha}\sum_{l=1}^n \frac{1}{l^{\alpha-1}}.$$

$S(n)$ is therefore bounded above by $\frac{8}{r^\alpha}\zeta(\alpha-1)$ if $\alpha > 2$. The upper bound for $S(n)$ is independent of n, therefore $\lim_{n\to\infty} S(n) \leq \frac{8}{r^\alpha}\zeta(\alpha-1)$ and hence the series $\sum' \frac{1}{|\omega|^\alpha}$ converges if $\alpha > 2$.

The lower bound for $S(n)$ also implies that for $\alpha \leq 2$ the series diverges. $\square$

2.2 Eisenstein series and modular forms

Now with the pervious result we can show, that the Eisenstein series G_{2k} are modular forms of weight $2k$ for integers $k > 1$.

Proposition 2.4. *Let $k > 1$ be an integer. Then*

$$G_{2k} \in \mathcal{M}_{2k}(\mathrm{SL}_2(\mathbb{Z}))\,.$$

Proof. First we show that G_{2k} is holomorphic on $\mathcal{H}$. Since $m\tau+n \neq 0$ for $\tau \in \mathcal{H}$ the functions $\tau \mapsto (m\tau+n)^{-2k}$ are holomorphic on $\mathcal{H}$. We now have $2k > 2$, so by Lemma 2.3 the double sum

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m\tau+n)^{2k}}, \quad \tau \in \mathcal{H}$$

is absolutely convergent and so uniformly convergent in any compact subset of $\mathcal{H}$. Hence G_{2k} is well-defined and G_{2k} is holomorphic on $\mathcal{H}$.

Next we show, that G_{2k} satisfies the transformation property of modular forms using Proposition 1.3.

$$\begin{aligned} G_{2k}(\tau+1) &= \sum_{(m,n)}{}' (m(\tau+1)+n)^{-2k} \\ &= \sum_{(m,n)}{}' (m\tau+(m+n))^{-2k} \\ &= \sum_{(m,l)}{}' (m\tau+l)^{-2k} \\ &= G_{2k}(\tau). \end{aligned}$$

And, again because of absolute convergence of the series, we also get

$$\begin{aligned} \tau^{-2k}G_{2k}(-1/\tau) &= \tau^{-2k}\sum_{(m,n)}{}' \left(-m\tau^{-1}+n\right)^{-2k} \\ &= \sum_{(m,n)}{}' (-m+n\tau)^{-2k} \\ &= G_{2k}(\tau), \end{aligned}$$

so $G_{2k}(-1/\tau) = \tau^{2k}G_{2k}(\tau)$. With Proposition 1.3 G_{2k} satisfies the transformation property for modular forms of weight $2k$.

Finally we have to show, that G_{2k} is holomorphic at infinity. We know that it is sufficient for that, to show that $\lim_{\Im\tau\to\infty} f(\tau)$ exists. By absolute convergence of the series we can reorder the summation and also interchange limits with summation, hence for $\Im\tau \to \infty$ we calculate

$$\begin{aligned} \lim_{\Im\tau\to\infty} \sum{}' (m\tau+n)^{-2k} &= \lim_{\Im\tau\to\infty} \sum{}' n^{-2k} + \lim_{\Im\tau\to\infty} \sum_{m}{}' \sum_{n\in\mathbb{Z}} (m\tau+n)^{-2k} \\ &= 2\zeta(2k) + \sum_{m}{}' \sum_{n\in\mathbb{Z}} \lim_{\Im\tau\to\infty} (m\tau+n)^{-2k} \\ &= 2\zeta(2k). \end{aligned}$$

So the limit $\lim_{\Im\tau\to\infty} G_{2k}(\tau) = 2\zeta(2k)$ exists and hence G_{2k} is holomorphic at ∞.

In conclusion we have that $G_{2k}(\tau)$ is a modular form of weight $2k$. $\square$

2.3 Fourier expansion of Eisenstein series

Now we compute the Fourier expansion for G_{2k}. The coefficients of the expansion are essentially the following arithmetic functions of n:

$$\sigma_{2k-1}(n) = \sum_{l|n} l^{2k-1}.$$

Proposition 2.5. *Let $k > 1$ be an integer an let $\tau \in \mathcal{H}$. Then the modular form $G_{2k}(\tau)$ has Fourier expansion given by*

$$G_{2k}(\tau) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n \tag{6}$$

where $q = e^{2\pi i \tau}$.

Proof. Let us start with the following identity

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right), \quad z \in \mathcal{H}. \tag{7}$$

The left hand side can be written as

$$\begin{aligned}
\pi \cot \pi z &= \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} \\
&= \pi i \frac{1 + e^{2\pi i z}}{1 - e^{2\pi i z}} \\
&= \pi i \left(1 - \frac{2}{1 - e^{2\pi i z}}\right) \\
&= \pi i - 2\pi i \sum_{l=0}^{\infty} e^{2\pi i z l}.
\end{aligned}$$

Next we differentiate successively both sides of (7) $(2k-1)$ times with respect to z.

$$\begin{aligned}
\frac{d^{2k-1}}{dz^{2k-1}}\left(\pi i - 2\pi i \sum_{l=0}^{\infty} e^{2\pi i z l}\right) &= -2\pi i \sum_{l=1}^{\infty} (2\pi i l)^{2k-1} e^{2\pi i z l} \\
&= -(2\pi i)^{2k} \sum_{l=1}^{\infty} l^{2k-1} e^{2\pi i l z}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d^{2k-1}}{dz^{2k-1}}\left(\frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{z+n} + \frac{1}{z-n}\right)\right) &= (-1)^{2k-1}(2k-1)!\frac{1}{z^{2k}} \\
&\quad + (-1)^{2k-1}(2k-1)!\sum_{n=1}^{\infty}\frac{1}{(z+n)^{2k}} \\
&\quad + (-1)^{2k-1}(2k-1)!\sum_{n=1}^{\infty}\frac{1}{(z-n)^{2k}} \\
&= -(2k-1)!\sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^{2k}}.
\end{aligned}$$

Therefore both together and substituting z with $m\tau$ we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^{2k}} = \frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{l=1}^{\infty} l^{2k-1} e^{2\pi i l m \tau}.$$

Thus, since Eisenstein series is absolut convergent, reordering terms, leads us to

$$\begin{aligned}
G_{2k}(\tau) &= \sum_{(m,n)}{}' \frac{1}{(m\tau+n)^{2k}} \\
&= \sum_{n\in\mathbb{Z}}{}' \frac{1}{n^{2k}} + \sum_{m}{}' \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^{2k}} \\
&= 2\zeta(2k) + 2\sum_{m=1}^{\infty} \left(\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{l=1}^{\infty} l^{2k-1} e^{2\pi i l m \tau} \right) \\
&= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} l^{2k-1} q^{lm}
\end{aligned}$$

and changing summation order

$$\begin{aligned}
&= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} l^{2k-1} q^{lm} \\
&= 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^{n},
\end{aligned}$$

where in the last step we collected all coefficients of a fixed power q^n obtaining $\sum_{l|n} l^{2k-1}$. □

3 Elliptic functions

Next we introduce elliptic functions. For example, Eisenstein series can be aproached from a different point of view: they reappear as coefficients in the Laurent expansion of the so called Weierstrass $\wp$ function.

Definition 3.1. A function f of a complex variable is called periodic with period ω if

$$f(z+\omega) = f(z),$$

whenever $z+\omega$ and z are in the domain of f. A function f is called doubly periodic if it has two periods ω_1 and ω_2 whose ratio $\frac{\omega_2}{\omega_1}$ is not real.

The reason to restrict $\frac{\omega_2}{\omega_1}$ to non-real values is to avoid looking at degenerated cases. Moreover for $\frac{\omega_2}{\omega_1}$ not real one can find a $\tau \in \mathcal{H}$ such that the two lattices $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\mathbb{Z} + \mathbb{Z}\tau$ are equivalent.

Indeed we see that the Eisenstein series $G_{2k}(\tau)$ are periodic since they satisfy the identity $G_{2k}(\tau+1) = G_{2k}(\tau)$.

Definition 3.2. A function f is called elliptic if it has the following two properties:

1. f is doubly periodic,

2. f is meromorphic (its only singularities in the finite plane are poles).

A known example for ellitpic functions is the Weierstrass $\wp$ function with respect to a lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Without loss of generality we restrict to $\frac{\omega_2}{\omega_1}$ to non-real values and $\frac{\omega_2}{\omega_1} \in \mathcal{H}$ to avoid working with degenerated cases.

Definition 3.3. The Weierstrass $\wp$ function with respect to Λ is defined by the series

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda}{}' \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right), \tag{8}$$

where $z \in \mathbb{C} \setminus \Lambda$.

The following proposition states that it really is an elliptic function.

Proposition 3.4. *The function $\wp$ has periods ω_1 and ω_2. It is analytic except for a double pole at each period $\omega \in \Lambda$. Moreover $\wp$ is an even function of z.*

If we calculate the Laurent expansion of $\wp$ near the origin, we see that the Eisenstein series will reappear in the coefficients of this expansion.

Proposition 3.5. *Let $r = \min_{\omega \in \Lambda \setminus \{0\}} |\omega|$. Then for $0 < |z| < r$ we have*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}, \tag{9}$$

where

$$G_n = \sum_{\omega \in \Lambda}{}' \frac{1}{\omega^n}, \quad n \geq 3. \tag{10}$$

Proof. If $0 < |z| < r$, then $\left|\frac{z}{\omega}\right| < 1$ and we have

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2 \left(1 - \frac{z}{\omega}\right)^2} = \frac{1}{\omega^2} \left(1 + \sum_{n=1}^{\infty} (n+1) \left(\frac{z}{\omega}\right)^n \right),$$

hence

$$\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} = \sum_{n=1}^{\infty} \frac{n+1}{\omega^{n+2}} z^n.$$

Taking the sum over all $\omega \in \Lambda \setminus \{0\}$ and considering absolute convergence gives

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) \sum_{\omega \in \Lambda}{}' \frac{1}{\omega^{n+2}} z^n = \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) G_{n+2} z^n.$$

Since the Eisenstein series of odd weight are zero, the coefficients G_{2k+1} are zero and we obtain

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}.$$

□

References

[1] N. Koblitz, *Introduction to Ellpitic Curves and Modular Forms*, GTM 97, Springer-Verlag.