Renormalization for Dyson's Hierarchical Model

(H. Koch and P. Wittwer)

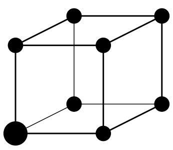
Formal definition of the RG transformation

spin config's: functions $\phi : x \in \mathbb{Z}^d \;\mapsto\; \phi_x \in \mathbb{R}$

states: finite positive measures μ on spin configurations

scaling factor: $\alpha = 2^{1/d-1/2} = 2^{-1/6} \quad (d = 3)$.

block spins: $(A\phi)_x$ = average of ϕ on the block $2x$



RG transform: the following map T on some set of states.

$$d(T\mu)(\psi) = \Big[\prod_x d\psi_x\Big] \int d\mu(\phi)\, \delta(\psi - \alpha^{-d} A\phi)\,.$$

Two properties of T are

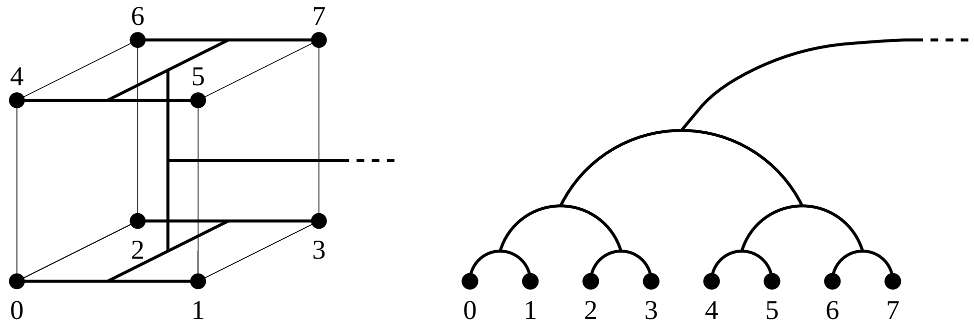
(1) $\int d(T\mu)(\psi) = \int d\mu(\phi)\,.$

(2) If C is the covariance of a Gaussian fixed point μ_C for T, then

$$C(2^n x, 2^n y) \approx C(x,y) \times \big(2^n\big)^{-d+2}\,.$$

A third root of T

Let $d = 3$. To simplify notation, label points in $\mathbb{Z}^3$ by integers:



Then $A = \mathcal{A}_+^3$ and $T = \mathcal{T}^3$, where

$$\big(\mathcal{A}_\pm \phi\big)_k = \tfrac{1}{2}\big(\phi_{2k} \pm \phi_{2k+1}\big)\,,$$
$$d(\mathcal{T}\mu)(\psi) = \Big[\prod_k d\psi_k\Big] \int d\mu(\phi)\, \delta(\psi - \alpha^{-1}\mathcal{A}_+\phi)\,.$$

The Dyson–Baker hierarchical model

Define $C_0 = \frac{1}{2}$ and

$$C_n = 4\mathcal{A}_-^*\mathcal{A}_- + 4\alpha^2\mathcal{A}_+^* C_{n-1}\mathcal{A}_+\,, \qquad n = 1, 2\ldots$$

Fact. $\mathcal{A}_+ C_n \mathcal{A}_+^* = \alpha^2 C_{n-1}$, *which "implies" that the Gaussian* μ_C *with covariance* $C = \lim\limits_{n\to\infty} C_n$ *is a fixed point of* $\mathcal{T}$ *(the UV fixed point).*

The reduced RG transformation

Let $\Lambda_n = \{0, 1, 2, \ldots, 2^n - 1\}$, and define

$$d\omega_{n,g}(\phi) = d\mu_{C_n}(\phi) \prod_{k \in \Lambda_n} g(\phi_k)\,,$$

$$d(\mathcal{T}\omega_{n,g})(\psi) = \Big[\prod_{k \in \Lambda_{n-1}} d\psi_k\Big] \int d\omega_{n,g}(\phi)\, \delta(\psi - \alpha^{-1}\mathcal{A}_+\phi)\,.$$

Fact. $\mathcal{T}\omega_{n,g} = \omega_{n-1,\mathcal{R}g}$, *where*

$$\big(\mathcal{R}g\big)(t) = \tfrac{1}{\sqrt{2\pi}} \int ds e^{-\frac{1}{2}s^2} g(\alpha t + s) g(\alpha t - s)\,, \qquad t \in \mathbb{R}.$$

RG fixed points

Two trivial fixed points of $\mathcal{R}$ are the UV fixed point $g \equiv 1$ and the HT fixed point $g = \sqrt{2\pi}\alpha\delta$. As for the IR fixed point, ...

Theorem 1. *$\mathcal{R}$ has a fixed point g^* which extends to an entire analytic function, and which satisfies for some $a, b > 0$ the bound*

$$0 < g^*(t) < a\, e^{-bt^6}\,, \qquad t \in \mathbb{R}\,.$$

The linearized RG transformation

Define $\mathcal{G}$ to be the Banach space of all continuous functions g on $\mathbb{R}$ such that $\exp(t^2)g(t) \to 0$ as $|t| \to \infty$, equipped with the norm

$$\|g\| = \sup_t e^{t^2} |g(t)| \,.$$

Theorem 2.
(i) $\mathcal{R}$ is C^∞ on $\mathcal{G}$, and $D\mathcal{R}(g)$ is compact for all $g \in \mathcal{G}$.
(ii) All eigenvalues of $D\mathcal{R}(g^*)$ are real and positive.

Numerically: Besides the trivial eigenvalue $\lambda_0 = 2$, $D\mathcal{R}(g^*)$ has one other eigenvalue λ_2 of modulus ≥ 1 in the even subspace of $\mathcal{G}$. The corresponding critical index is

$$\nu = \frac{\ln 2^{1/3}}{\ln \lambda_2} = 0.6495704\ldots \qquad (\nu_{\text{Ising}} \approx 0.63)\,.$$

Partition functions

Consider

$$\bigl(Z_n g\bigr)(h) = \int d\omega_{n,g}(\phi) \exp\Bigl(h \sum_{k\in\Lambda_n} \phi_k\Bigr)\,.$$

In particular,

$$\bigl(Z_0 g\bigr)(h) = \tfrac{1}{\sqrt{4\pi}} \int ds\, e^{-\frac{1}{4}s^2} g(s)\, e^{hs}\,.$$

Fact. $\bigl(Z_n g\bigr)(h) = \bigl(Z_0 \mathcal{T}^n g\bigr)\bigl((2\alpha)^n h\bigr)$ *for all n.*

Dependence on the magnetic field

Let $f^* = Z_0 g^*$.

Theorem 3. *$f^*(i.)$ is the Fourier transform of a positive measure, and f^* has a representation of the form*

$$f^*(h) = A \prod_{k=0}^{\infty} \left[1 + \left(\frac{h}{\nu_k}\right)^2\right], \qquad h \in \mathbb{C},$$

with $A > 0$ and $0 < \nu_0 < \nu_1 < \nu_2 < \dots$. Given any $r < 1/6$, there are positive constants a and b such that for $k = 1, 2, \dots$

$$\nu_k > a k^{5/6}, \qquad \nu_k - \nu_{k-1} < b k^{-r}.$$

Note: The zeros of $Z_n g^*$ are $\pm i (2\alpha)^{-n} \nu_k$.

Theorem 4. *The limit*

$$p^*(h) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \ln\big(Z_n g^*\big)(h) = \lim_{n\to\infty} 2^{-n} \ln f^*\big((2\alpha)^n h\big)$$

exists and is analytic for $\mathrm{Re} h \neq 0$. *Furthermore, there are $a, b > 0$ such that for $\delta = 5$*

$$a h^{1/\delta} < \frac{d}{dh} p^*(h) < b h^{1/\delta}, \qquad h > 0.$$

RG transformation for partition functions

Fact. *$Z_0 \mathcal{T} g = \mathcal{N} Z_0 g$, where*

$$\big(\mathcal{N} f\big)(h) = \tfrac{K}{\sqrt{4\pi\lambda}} \int ds\, e^{-\frac{1}{4\lambda}s^2} f\big(\beta(h+s)\big)^2 ,$$

for some $K, \lambda > 0$. Here, $\beta = \frac{1}{2\alpha} = 2^{-5/6}$.

Note: The function f^* described in Theorems 3,4 is obtained not by solving first $\mathcal{T} g^* = g^*$ and then setting $f^* = Z_0 g^*$, but by solving $\mathcal{N} f^* = f^*$. The Theorems 1,2 are proved afterwards for the function g^* <u>defined</u> by

$$g^*(t) = \big(Z_0^{-1} f^*\big)(t) = e^{\frac{1}{4}t^2} \tfrac{1}{\sqrt{\pi}} \int dh\, e^{-iht} f^*(ih) .$$

In particular, the analyticity of g^* follows from a corollary of Theorem 3.

Corollary. *Given any $q < 6/5$, there are $a, b > 0$ such that*

$$|f^*(ih)| < a\, e^{-b|h|^q} , \qquad h \in \mathbb{R} .$$

RG transformation for zeros

Consider the set $\mathcal{F}$ of functions f for which a sequence

$$0 < \nu_0(f) \leq \nu_1(f) \leq \nu_2(f) \leq \ldots \leq \infty$$

exists such that

$$f(h) = f(0) \prod_{k=0}^{\infty} \Big[1 + \Big(\frac{h}{\nu_k(f)}\Big)^2\Big], \qquad h \in \mathbb{C}.$$

Set $\nu_{-k}(f) = -\nu_{k-1}(f)$, and define

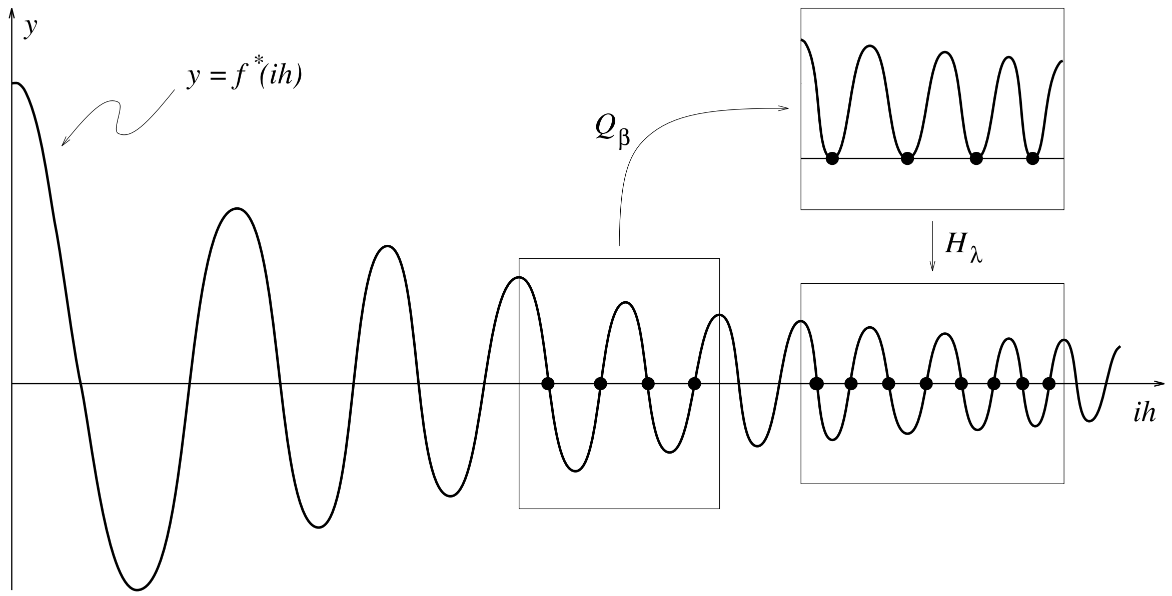
$$\begin{aligned} \big(Q_\beta f\big)(h) &= K f(\beta h)^2, \\ \big(H_\vartheta f\big)(h) &= \big(e^{\vartheta D^2} f\big)(h) = \tfrac{1}{\sqrt{4\pi\vartheta}} \int ds\, e^{-\frac{1}{4\vartheta}s^2} f(h-s). \end{aligned}$$

Fact. $\mathcal{N} = H_\lambda \circ Q_\beta$. *Both* Q_β *and* H_λ *map* $\mathcal{F}$ *into itself. Under the action of* Q_β, *the zeros double and scale:*

$$\nu_{2k}(Q_\beta f) = \nu_{2k+1}(Q_\beta f) = \beta^{-1}\nu_k(f).$$

Under the heat flow H_λ, *the zeros repel each other:*

$$\frac{d}{d\vartheta}\nu_k(H_\vartheta f) = \sum_{\substack{j\in\mathbb{Z} \\ j\neq k}} \frac{1}{\nu_k(H_\vartheta f) - \nu_j(H_\vartheta f)}.$$



Define the gaps for $f \in \mathcal{F}$ by

$$\gamma_k(f) = \nu_k(f) - \nu_{k-1}(f)\,, \qquad\qquad k \in \mathbb{Z}\,.$$

Note: After the first step Q_β of $\mathcal{N}$, one has

$$\text{average}\big\{\, \gamma_{2k}(Q_\beta f)\,,\; \gamma_{2k+1}(Q_\beta f)\,\big\} = \alpha\,\gamma_k(f)\,,$$

with $\alpha < 1$. But the gaps are irregular: $\gamma_{2k}(Q_\beta f) = 2\alpha\gamma_k(f)$, and $2\alpha > 1$. The step H_λ tends to eliminate local irregularities.

Lemma 5. *For every $\varepsilon > 0$ and for $\Delta > 0$ sufficiently small (depending on ε) there exists $\ell \in \mathbb{N}$ such that*

$$\gamma_{k-\ell}(f)\,, \ldots\,, \gamma_{k+\ell}(f) \,\le \Delta$$

implies

$$\gamma_{2k}(\mathcal{N}f)\,,\, \gamma_{2k+1}(\mathcal{N}f) \,\le (\alpha+\varepsilon)\Delta\,.$$

The gaps for the RG fixed point

An improved version of Lemma 5 is used to find a "realistic" $\hat{\Delta} > 0$ such that the properties

(a) $f^* \in \mathcal{F}$,
(b) $\gamma_k(f^*) < \hat{\Delta}$, *for* $k = 5 \ldots 79$,
(c) f^* *is a fixed point of* $\mathcal{N}$,

imply that the gaps $\gamma_k(f^*)$ decrease as claimed in Theorem 3.

The first condition may be weakened to $f^* \in \mathcal{F}_\varepsilon$, meaning that $f^*(h)$ is a canonical product in $\mathcal{F}$, multiplied by a factor $\exp(rh^2)$, with $|r| \le \varepsilon$. Here, ε is some small positive number.

Our proof for the existence of a fixed point f^* of $\mathcal{N}$ with the properties (a) and (b) is computer–assisted.
It uses two modified RG transformations $\mathcal{M}$ and $\mathcal{K}$ which

- have the same fixed point(s) as $\mathcal{N}$.
- are contractions on neighborhoods $U_{\mathcal{M}}$ and $U_{\mathcal{K}}$ of a given approximate fixed point for $\mathcal{N}$, with $U_{\mathcal{M}} \subset U_{\mathcal{K}}$.

Some other features of these transformations are:

$\mathcal{M}$ is simple and can be implemented in very high precision. It does not map $\mathcal{F}_\varepsilon$ to itself, but all functions in $U_{\mathcal{M}}$ have ≥ 160 zeros in $i\mathbb{R}$ that are consistent with (b).

$\mathcal{K}$ maps $\mathcal{F}_\varepsilon$ to itself, and it can be iterated on a polynomial $f \in \mathcal{F}$ lying "far" away from f^*, until $\mathcal{K}^n f$ ends up in $U_{\mathcal{K}}$. This is used to prove (a).

An order–preserving property of $\mathcal{N}$

The proof of Lemma 5 uses a generalization of

Theorem 6. *Let $f_1, f_2 \in \mathcal{F}$, and let $\vartheta > 0$. If $\gamma_k(f_1) \leq \gamma_k(f_2)$ for all k, then $\gamma_k(H_\vartheta f_1) \leq \gamma_k(H_\vartheta f_2)$ for all k.*

Note: An analogous order–preserving property holds for Q_β, and thus also for $\mathcal{N} = H_\lambda \circ Q_\beta$.

The generalized version, Theorem 6^+, applies to entire analytic functions f which satisfy

$$\overline{\lim_{|z|\to\infty}} |z|^{-2} \ln |f(z)| = 0 \,,$$

and whose zeros all lie on a line parallel to the imaginary axis.

To prove Lemma 5, apply Theorem 6^+ with a polynomial f_2 whose gaps are

$$\gamma_j(f_2) = \begin{cases} \Delta\,, & \text{if } |j-k| \leq \ell \;; \\ +\infty\,, & \text{otherwise} \;; \end{cases}$$

and then estimate the gaps $\gamma_j(\mathcal{N} f_2)$.

Some references

- F.J. Dyson, *Existence of a Phase Transition in a One-Dimensional Ising Ferromagnet.* Commun. Math. Phys. **12**, 91 (1969).
- G.A. Baker, *Ising Model with a Scaling Interaction.* Phys. Rev. **B5**, 2622 (1972).

Construction of a nontrivial RG fixed point etc. in $d = 4 - \varepsilon$

- P.M. Bleher, Ya.G. Sinai, *Critical Indices for Dyson's Asymptotically–Hierarchical Models.* Commun. Math. Phys. **45**, 247 (1975).
- P. Collet, J.-P. Eckmann, *A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics.* Lecture Notes in Physics **74**. Berlin: Springer 1978.

The results for $d = 3$ presented here are (or follow from results) proved in

- H. Koch, P. Wittwer, *On the Renormalization Group Transformation for Scalar Hierarchical Models.* Commun. Math. Phys. **138**, 537 (1991).
- ——— , *A Nontrivial Renormalization Group Fixed Point for the Dyson–Baker Hierarchical Model.* Commun. Math. Phys., to appear (also available from *mp_arc*).
- ——— , *Bounds on the Zeros of a Renormalization Group Fixed Point.* In preparation (programs available by *ftp*).