Math 328K. Fall 2025

Some solutions to Homework # 1

Prof. Hector E. Lomeli

§1.1. Exercise 12. Let $x \in \mathbb{R}$. We have to consider two cases.

a) $0 \leq \{x\} < 1/2$,

b) $1/2 \leq \{x\} < 1$.

In each case, we will compute $[x]$, $[x + 1/2]$, $[2x]$ and show that $[x] + [x + 1/2] = [2x]$.

a) If $0 \leq \{x\} < 1/2$, then

$$0 \leq x - [x] < \frac{1}{2},$$

and this implies that $[x] \leq x < [x] + \frac{1}{2}$. From this we get that

$$[x] < [x] + \frac{1}{2} \leq x + \frac{1}{2} < [x] + 1,$$

and

$$2[x] \leq 2x < 2[x] + 1.$$

Following the definition of greatest integer function, we find that $[x + \frac{1}{2}] = [x]$ and $[2x] = 2[x]$. From this, we conclude that $[x] + [x + 1/2] = [2x]$.

b) If $1/2 \leq \{x\} < 1$, then

$$\frac{1}{2} \leq x - [x] < 1,$$

and this implies that $[x] + \frac{1}{2} \leq x < [x] + 1$. From this we get that

$$[x] + 1 \leq x + \frac{1}{2} < [x] + \frac{3}{2} < [x] + 2,$$

and

$$2[x] + 1 \leq 2x < 2[x] + 2.$$

Following the definition of greatest integer function, we find that $[x + \frac{1}{2}] = [x] + 1$ and $[2x] = 2[x] + 1$. We conclude that $[x] + [x + 1/2] = [2x]$.

§1.1. Exercise 14. Let $x, y \in \mathbb{R}$. Using problem **§1.1. #14**, we get $[x] + [x + 1/2] = [2x]$ and $[y] + [y + 1/2] = [2y]$. Then

$$[2x] + [2y] - [x] - [y] = [x + 1/2] + [y + 1/2].$$

It is enough to show that

$$[x + 1/2] + [y + 1/2] \geq [x + y].$$

We have that $[x + y] \leq x + y$ and

$$x + y = \left(x + \frac{1}{2}\right) + \left(y + \frac{1}{2}\right) - 1 < \left[x + \frac{1}{2}\right] + 1 + \left[y + \frac{1}{2}\right] + 1 - 1$$

From this, we conclude that $[x + y] < [x + 1/2] + [y + 1/2] + 1$, and hence $[x + y] \leq [x + 1/2] + [y + 1/2]$.

§1.1. Exercise 18. Let $x \in \mathbb{R}$, $n \in \mathbb{Z}$, and $m \in \mathbb{N}$. To solve this problem, we need to have $m > 0$.

We define $z = x + n$. Given that $[z] = [x] + n$, it is enough to show the identity

$$\left[\frac{[z]}{m}\right] = \left[\frac{z}{m}\right].$$

We define the following two integers.

$$\alpha = \left[\frac{[z]}{m}\right], \qquad \beta = \left[\frac{z}{m}\right].$$

We will show two inequalities: $\alpha \leq \beta, \beta \leq \alpha$.

1) Given that $[z] \leq z$ and $m > 0$, we have

$$\alpha \leq \frac{[z]}{m} \leq \frac{z}{m} < \beta + 1.$$

Given that $\alpha, \beta \in \mathbb{Z}$ and $\alpha < \beta + 1$, this implies that $\alpha \leq \beta$.

2) Clearly, $\beta \leq \frac{z}{m}$. Using that $m > 0$, we get

$$m\beta \leq z < [z] + 1.$$

We have that $m\beta, [z] \in \mathbb{Z}$ and $m\beta < [z] + 1$. This implies that $m\beta \leq [z]$, and therefore,

$$\beta \leq \frac{[z]}{m} < \alpha + 1.$$

Given that $\alpha, \beta \in \mathbb{Z}$ and $\beta < \alpha + 1$, we conclude $\beta \leq \alpha$.

§1.2. Exercise 22. The identity $\frac{1}{k^2-1} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$ will not gives us a telescoping sum. But we can modify this identity, introducing an extra term that cancels. Notice that

$$\frac{1}{k^2-1} = \frac{1}{2}\left(\left(\frac{1}{k-1} + \frac{1}{k}\right) - \left(\frac{1}{k} + \frac{1}{k+1}\right)\right).$$

This implies that

$$\frac{1}{k^2-1} = a_k - a_{k-1},$$

where $a_k = -\frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right)$.

From this, we get

$$\sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n}(a_k - a_{k-1}) = a_n - a_1 = -\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right) + \frac{3}{4}.$$