# Math 328K. Fall 2025

## Some solutions to Homework # 5

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**§3.5. Exercise 36. Solution 1.** We define $e = (a, b)$ and $u = a/e$, $v = b/e$. We know that $(u, v) = 1$. This implies that the numbers $u$ and $v$ do not share prime factors. First we assume that $u, v > 1$. This implies that $u$ and $v$ are of the form

$$u = p_1^{x_1} \cdots p_{n_1}^{x_{n_1}},$$
$$v = q_1^{y_1} \cdots q_{n_2}^{y_{n_2}},$$

where all the primes $\{p_1, \ldots, p_{n_1}\}$ are different from the primes $\{q_1, \ldots, q_{n_2}\}$, and –in addition– the exponents $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$ are positive. The number $e$ has the form

$$e = \left(p_1^{z_1} \cdots p_{n_1}^{z_{n_1}}\right)\left(q_1^{w_1} \cdots q_{n_2}^{w_{n_2}}\right) m,$$

where $z_1, \ldots, z_{n_1}, w_1, \ldots, w_{n_2}$ are non-negative, and $m$ is a number that is not divisible by any of the primes $\{p_1, \ldots, p_{n_1}, q_1, \ldots, q_{n_2}\}$.

We define

$$c = \left(p_1^{x_1+z_1} \cdots p_{n_1}^{x_{n_1}+z_{n_1}}\right), \qquad d = \left(q_1^{y_1+w_1} \cdots q_{n_2}^{y_{n_2}+w_{n_2}}\right) m.$$

Clearly, $c$ and $d$ do not share any prime factors, so $(c, d) = 1$. Also, by construction we have that $cd = uve$. This implies that

$$[a, b] = \frac{ab}{(a, b)} = \frac{(eu)(ev)}{(eu, ev)} = \frac{uve}{(u, v)} = cd.$$

It is easy to see that $c|a$ and $d|b$. For the case $u = 1$, we can take $c = 1$ and $d = b$. In this case, $a = e$ and hence $[a, b] = ab/e = b = cd$. In the same way, if $v = 1$, we can take $c = a$ and $d = 1$.

**§3.5. Exercise 36. Solution 2.** We can follow a more direct route. Using the FTA, we can find a finite number of primes $p_1, \ldots, p_n$ such that

$$a = p_1^{a_1} \cdots p_n^{a_n}, \qquad \text{and} \qquad b = p_1^{b_1} \cdots p_n^{b_n},$$

where all the exponents are non-negative.

We know that $[a, b] = p_1^{z_1} \cdots p_n^{z_n}$, where $z_k = \max\{a_k, b_k\}$, for all $k = 1, \ldots, n$. For each value of $k$, we define

$$c = p_1^{x_1} \cdots p_n^{x_n}, \qquad d = p_1^{y_1} \cdots p_n^{y_n}$$

where

$$x_k = \begin{cases} a_k, & \text{if } a_k > b_k, \\ 0, & \text{if } a_k < b_k, \\ a_k, & \text{if } a_k = b_k. \end{cases}$$

and

$$y_k = \begin{cases} 0, & \text{if } a_k > b_k, \\ b_k, & \text{if } a_k < b_k, \\ 0, & \text{if } a_k = b_k. \end{cases}$$

Clearly, for each $k = 1, \ldots, n$, we have that $x_k + y_k = \max\{a_k, b_k\} = z_k$. This implies that $cd = [a, b]$. In addition, $a_k \geq x_k$ and $b_k \geq y_k$ and this implies that $c|a$, and $d|b$.

Finally, by construction $c, d$ have no prime factors in common. In this way, we conclude that $(c, d) = 1$.

**§3.5. Exercise 40. Solution 1.** Let $d_0 = (a, c)$ and $d_1 = (b, c)$. By Bezout's theorem, we can find integers $m_1, n_1, m_2, n_2$ such that

$$d_0 = m_1\, a + n_1\, c, \qquad d_1 = m_2\, b + n_2\, c.$$

This implies that

$$d_0 d_1 = (m_1\, a + n_1\, c)(m_2\, b + n_2\, c) = (m_1 m_2)ab + (m_1 n_2\, a + m_2 n_1\, b + n_1 n_2\, c)\, c.$$

Given that $c|ab$, we conclude that $c|d_0 d_1$.

**§3.5. Exercise 40. Solution 2.** Let $d_0 = (a, c)$. We define $p = a/d_0$ and $q = c/d_0$. Clearly, $(p, q) = 1$.

If $c \mid ab$, then $a\, b = c\, k$, for some integer $k$. This implies that $p\, b = q\, k$ and therefore $q \mid pb$.

Given that $(p, q) = 1$, we conclude that $q \mid b$. We also have $q \mid c$, so $q \mid (b, c)$. Finally, this implies that $d_0\, q \mid d_0\, (b, c)$ and this can be written as $c \mid (a, c)(b, c)$.

**§3.5. Exercise 46 a).** We know that if $(x, y) = 1$ and $(x, z) = 1$ then $(x, yz) = 1$.

Let $d = (a, b)$ and define $p = a/d$, $q = b/d$. We know that $(p, q) = 1$. This implies that $(p + q, p) = 1$ and $(p + q, q) = 1$. Using the result mentioned above, we conclude that $(p + q, pq) = 1$. This implies that

$$d = d(p + q, pq) = (dp + dq, dpq) = (a + b, ab/d).$$

We have that $d = (a, b)$ and $[a, b] = ab/(a, b) = ab/d$. We conclude that $(a, b) = (a + b, [a, b])$.

**§3.5. Exercise 46 b).** We want to find two numbers, $a$ and $b$, such that $a+b = 798$ and $[a, b] = 10780$. Using the formula that we just proved above, we find that $(a, b) = (a + b, [a, b]) = (798, 10780)$. This implies that $(a, b) = 14$ and hence $ab = [a, b](a, b) = 150920$. In order to find these numbers, we solve the quadratic equation

$$(x - a)(x - b) = x^2 - (a + b)x + ab = x^2 - 798x + 150920 = 0.$$

By construction, the roots of the previous equation are $a$ and $b$. Solving with the quadratic formula, we get that

$$x = 399 \pm \sqrt{399^2 - 150920} = 399 \pm \sqrt{8281} = 399 \pm 91.$$

We conclude that the two numbers we want are: $a = 490$ and $b = 308$. We can verify all the relevant conditions: $(a, b) = 14$, $ab = 150920$, $a + b = 798$ and $[a, b] = 10780$.

**5. a).** Let $p$ be a prime, $p \neq 2$ that divides $\beta = \big((a - b)(a + b), 2ab\big)$. Then $p \mid 2ab$ and $p \mid (a - b)(a + b)$. We know that $p$ does not divide 2 so, using Euclid's theorem, we get that $p \mid a$ or $p \mid b$. Suppose that $p \mid a$. We also have that $p \mid (a - b)$ or $p \mid (a + b)$. We suppose that $p \mid (a + b)$. This implies that $p \mid (a, a+b) = (a, b)$. We know that $(a, a+b) = (a, a-b) = (b, a+b) = (b, a-b) = (a, b)$. Therefore, all the other cases also imply that $p \mid (a, b)$.

**5. b).** Using part **a)**, we know that the only prime that can divide $\beta$ is 2. Then $\beta$ has to be of the form

$$\beta = 2^k,$$

for some $k \geq 0$. By assumption, $k > 0$. We will show that $4 \mid \beta$ is impossible, so we must have $k = 1$.

If $(a, b) = 1$, then $(a - b, b) = (a + b, b) = (a - b, a) = (a + b, a) = (a, b) = 1$. From these, we get that

$$((a - b)(a + b), b) = ((a - b)(a + b), a)) = 1.$$

Finally, these imply that $((a - b)(a + b), ab) = 1$.

BWOC, assume that $4 \mid \beta$. Then $4 \mid 2ab$ and hence $2 \mid ab$. We also have that $4 \mid (a - b)(a + b)$, so $2 \mid (a - b)(a + b)$. This is a contradiction, because $((a - b)(a + b), ab) = 1$. We conclude that $\beta = 2$.

**5. c).** We have two cases: $2 \mid \beta$ and $2 \nmid \beta$. If $2 \mid \beta$, then part **b)** implies that $\beta = 2$. If $2 \nmid \beta$, then part **a)** implies that no prime number divides $\beta$, so $\beta = 1$.

**5. d).** By way of contradiction, suppose that $(a-b)(a+b) = 2ab$. Let $d = (a,b)$. We define $p = a/d$, and $q = b/d$. Then $(p,q) = 1$. We also have that

$$(p-q)(p+q) = (a-b)(a+b)/d^2 = 2ab/d^2 = 2pq.$$

This implies that

$$((p-q)(p+q), 2pq) = 1 = (2pq, 2pq) = 2|pq|.$$

Given that $(p,q) = 1$, we conclude that $2|pq| = 2$ and hence $p = \pm 1$ and $q = \pm 1$. However,

$$(p-q)(p+q) = p^2 - q^2 = 1 - 1 = 0,$$

and this contradicts that $pq \neq 0$. We conclude that $(a-b)(a+b) \neq 2ab$.

**6.** By way of contradiction, suppose that $\gamma$ is rational. Clearly, $\gamma \neq 0$, so that we would have that $\gamma$ can be written as a quotient of two integers, both different from zero. This is, there exist integers $a, b$ such that $\gamma = a/b$ and $ab \neq 0$. We can write the equation as

$$\gamma^2 - 1 = 2\gamma.$$

After substitution, we find,

$$\frac{a^2}{b^2} - 1 = 2\left(\frac{a}{b}\right),$$

and therefore

$$a^2 - b^2 = 2ab.$$

Using the previous exercise, we know those integers do not exist, which leads to a contradiction. We conclude that $\gamma$ is irrational.

**§3.7. Exercise 2 a).** $x = -11 + 4n$, $y = 11 - 3n$.

**§3.7. Exercise 2 b).** Not solvable.

**§3.7. Exercise 2 c).** $x = -121 - 47n$, $y = 77 + 30n$.

**§3.7. Exercise 2 d).** $x = 776 - 19n$, $y = -194 + 5n$.

**§3.7. Exercise 2 e).** $x = 422 - 1001n$, $y = -43 + 102n$.

**§3.7. Exercise 6.** Let $x$ be the number of plantains in each one of the 63 piles. Let $y$ be the plantains that each traveler got. We want to solve the following diophantine equation.

$$63x + 7 = 23y.$$

This can be written as $63(-x) + 23y = 7$. We find

$$x = 28 - 23n, \qquad y = 77 - 63n.$$

We need to consider all solutions for which $x, y > 0$. This happens whenever $n \leq 1$. The smallest solution is $x = 5$ and $y = 14$. We can rewrite all the possible solutions as

$$x = 5 + 23t, \qquad y = 14 + 63t,$$

where $t \geq 0$.

**§3.7. Exercise 8.** Let $x$ be the number of grapefruit and $y$ the number of oranges. We have

$$33x + 18y = 549.$$

We find

$$x = -183 + 6n, \qquad y = 366 - 11n.$$

We need to consider all solutions for which $x, y > 0$. This happens whenever $n \in \{31, 32, 33\}$. We notice that $x + y = 183 - 11n$. Choosing $n = 33$, we get the minimum number of pieces of fruit. The solution is $x = 15$ and $y = 3$.